REGGE POLES AND RESONANCE NUCLEAR REACTIONS. II

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Formulas for the characteristic asymmetry of a resonance level arising in the angular distribution of reaction products when the resonances of a compound nucleus are described as moving Regge poles have been derived taking the Coulomb interaction into account. Results of numerical calculations of this asymmetry are presented for the case of elastic resonance scattering of α particles by C^{12} nuclei for α particles with energies lying in the range up to 5 MeV.

1. INTRODUCTION

As has been noted by Shapiro (private communication) and by Rebolia and Viano [1], when resonances of a compound nucleus are interpreted as moving Regge poles in the plane of complex angular momentum λ one can expect the appearance of a characteristic forward-backward asymmetry in the angular distribution of products of a nuclear reaction (the so-called "characteristic asymmetry" of a resonance level—CAL). The question of the possibility of observing CAL experimentally has been discussed in a preceding paper by the present authors [2] (henceforth denoted by I) within the framework of the hypothesis of smooth compensation of nonresonant phases. For reactions between spinless particles in the case of no Coulomb interaction formulas were obtained in I for the function $R_l(\cos\theta)$ which determines CAL, and an investigation was made of the problem of determining the range of the scattering angles θ for which an essential contribution to the amplitude of a nuclear reaction is made by only a single Regge pole which produces an isolated resonance, and in which it is possible to neglect the integral term in the "Reggeized" form of the amplitude.

The reduction carried out in I of experimental data [3] on the phase analysis of elastic resonance scattering of α particles by C^{12} nuclei apparently gives support to the hypothesis of a smooth compensation of non-resonance phases and indicates the possibility of experimentally observing CAL.

However, for a precise quantitative comparison of theory with experiment it is necessary to take into account the effect on CAL of the Coulomb interaction between the particles participating in the reaction. In the present paper formulas have been obtained for the function $R_l^C(\cos\theta)$ which describes the CAL for charged spinless particles. It is shown that the Coulomb interaction changes the behavior of the CAL in an essential manner, particularly in the domain of large scattering angles. Results are presented of a numerical calculation of $R_l^C(\cos\theta)$ for the elastic scattering process $C^{12}(\alpha,\alpha)C^{12}$.

2. REGGEIZED FORM OF THE AMPLITUDE IN THE PRESENCE OF COULOMB INTERACTION

The amplitude for the scattering of charged spinless particles can be conveniently represented as a sum of the Coulomb amplitude $f^C(E,z)$ and of a "nuclear" amplitude $F^N(E,z)$ [4]:

$$f(E,z)=\frac{1}{2ik}\sum_{n=0}^{\infty}(2n+1)e^{2i(\sigma_n+\delta_n)}P_n(z)$$

$$=f^c(E,z)+f^N(E,z), \tag{1}$$

$$f^c(E,z)=\frac{1}{2ik}\sum_{n=0}^{\infty}(2n+1)e^{2i\sigma_n}P_n(z)=\frac{C(\eta)}{k(1-z)^{1+i\eta}}, \tag{2}$$

$\eta = Z_1Z_2e^2/\hbar v$ is the Coulomb parameter,

$$e^{2i\sigma_n}=\Gamma(n+1+i\eta)/\Gamma(n+1-i\eta),\quad C(\eta)=-\eta 2^{i\eta}e^{2i\sigma_0},$$

$$f^N(E,z)=\frac{1}{k}\sum_{n=0}^{\infty}(2n+1)e^{2i\sigma_n}f_n^N(E)P_n(z),$$

$$f_n^N(E)=e^{i\delta_n}\sin\delta_n, \tag{3}$$

$\delta_n=\delta_n(E)$ are the so-called nuclear scattering phases. Here E, k, and θ are the kinetic energy of the colliding particles, the wave number and the scattering angle in the center-of-mass system, $z=\cos\theta$.

For a generalization of the results of I to charged particles it is necessary to write $f^N(E,z)$

in Reggeized form. As in I, we assume that the partial amplitude

$$f_\lambda(E) = -{}^1/_2\, i \exp\left[2i(\sigma_\lambda(E) + \delta_\lambda(E))\right]$$

is an analytic function of the complex angular momentum λ, which for $\mathrm{Re}\,\lambda > -{}^1/_2$ has only simple poles, and that its behavior on the circumference of a large circle allows the Watson-Sommerfeld transformation to be carried out. Since the Coulomb partial amplitude

$$f_\lambda^c(E) = -{}^1/_2\, i \exp\left[2i\sigma_\lambda(E)\right] = \Gamma(\lambda + 1 + i\eta) \,/\, [2i\Gamma(\lambda + 1 - i\eta)]$$

is bounded as $|\lambda| \to \infty$ and does not have for $E > 0$ any singularities in the half-plane $\mathrm{Re}\,\lambda > -{}^1/_2$, then all the poles with respect to λ of the functions $f_\lambda(E)$ and

$$f_\lambda^N(E) = \exp\left[i\delta_\lambda(E)\right] \sin \delta_\lambda(E)$$

for $\mathrm{Re}\,\lambda > -{}^1/_2$ and $E > 0$ coincide. This corresponds to the fact that the resonances in the scattering of charged particles are due to poles with respect to E ($E = E_0 - i\Gamma/2$) in the nuclear amplitude $f_n^N(E)$. The properties of $f_n^N(E)$ are basically determined by the nuclear interactions and, in particular, for $n > L \approx kR$ the amplitudes $f_n^N(E)$ fall off exponentially with increasing n.

In accordance with the above the function

$$\tilde{f}^N(E, z) = \sum_{n=0}^{\infty} (2n + 1) f_n^N(E) P_n(z) \tag{4}$$

can be written in Reggeized form analogous to formula (2) in I. We note that if the functions $f_i(z)$ ($i = 1, 2, 3$) can be expanded in series in terms of Legendre polynomials

$$f_i(z) = \sum_{n=0}^{\infty} (2n + 1) a_n^{(i)} P_n(z),$$

with $a_n^{(3)} = a_n^{(1)} a_n^{(2)}$, then

$$f_3(z) = f_3(\mathbf{n}_1\mathbf{n}_2) = \frac{1}{4\pi} \int f_1(\mathbf{n}_1\mathbf{n}) f_2(\mathbf{n}_2\mathbf{n})\, d\Omega_{\mathbf{n}}, \tag{5}$$

where $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}$ are unit vectors, $d\Omega_n = \sin\vartheta\, d\vartheta\, d\varphi$. Utilizing (5) we obtain from (2), (3) and formula (2) from I the desired Reggeized form for $F^N(E, z)$:

$$f^N(E, z) = -\frac{\pi}{k} \sum_i \frac{(2\alpha_i + 1) r_{\alpha_i}^N(E)}{\sin \pi\alpha_i} P_{\alpha_i}^c(-z) - \frac{1}{2ik} \int_{-1/2 - i\infty}^{-1/2 + i\infty} \frac{(2\lambda + 1)}{\sin \pi\lambda} f_\lambda^N(E) P_\lambda^c(-z)\, d\lambda, \tag{6}$$

where $\alpha_i = \alpha_i(E)$ are the poles of $f_\lambda^N(E)$ for $\mathrm{Re}\,\lambda > -{}^1/_2$, $r_{\alpha_i}^N(E)$ is the residue of $f_\lambda^N(E)$ at the pole $\lambda = \alpha_i$, while the function $P_\lambda^c(z)$ is determined by the integral representation[1)]

$$P_\lambda^c(z) = \frac{ik}{2\pi} \int P_\lambda(\mathbf{n}_1\mathbf{n}) f^c(\mathbf{n}_2\mathbf{n})\, d\Omega_{\mathbf{n}} = \frac{iC(\eta)}{2\pi} \int_{-1}^{1} dz' \int_0^{2\pi} \frac{d\varphi}{(1 - z')^{1+i\eta}} P_\lambda \times (zz' + [(1 - z^2)(1 - z'^2)]^{1/2} \cos\varphi). \tag{7}$$

The expansion of $P_\lambda^c(z)$ in terms of Legendre polynomials has the form

$$P_\lambda^c(z) = \frac{\sin \pi\lambda}{\pi} \sum_{n=0}^{\infty} (-)^n \left(\frac{1}{\lambda - n} - \frac{1}{\lambda + n + 1}\right) e^{2i\sigma_n} P_n(z). \tag{8}$$

For integral $\lambda = n$

$$P_n^c(z) = \lim_{\lambda \to n} P_\lambda^c(z) = e^{2i\sigma_n} P_n(z). \tag{9}$$

We note that usually the Reggeized form of the amplitude is obtained from the expansion of the amplitude in terms of partial waves with the aid of the Watson-Sommerfeld transformation [5]. In this case, as can be seen from (3), the problem arises of the analytic continuation into the λ-plane of the function $e^{2i\sigma_n} P_n(z)$ from the integral points $\lambda = n$. There exist at least two different analytic functions of λ which satisfy the conditions required for carrying out the Watson-Sommerfeld transformation, whose values at $\lambda = n$ coincide with $e^{2i\sigma_n} P_n(z)$: $P_\lambda^c(z)$ and $[\Gamma(\lambda + 1 + i\eta)/\Gamma(\lambda + 1 - i\eta)] \times P_\lambda(z)$. From the first of these we obtain (6), while the utilization of the second one leads to the formula

$$f^N(E, z) = -\frac{\pi}{k} \sum_i \frac{(2\alpha_i + 1) r_{\alpha_i}^N(E)}{\sin \pi\alpha_i} e^{2i\sigma_{\alpha_i}(E)} P_{\alpha_i}(-z) - \frac{1}{2ik} \int_{-1/2 - i\infty}^{-1/2 + i\infty} \frac{(2\lambda + 1)}{\sin \pi\lambda} f_\lambda^N(E) e^{2i\sigma_\lambda(E)} P_\lambda(-z)\, d\lambda. \tag{10}$$

In their exact form (6) and (10) agree identically, differing only by relative contributions made to $f^N(E, z)$ by the sum over the poles and by the integral. However, we are interested in the case when in the amplitude $f^N(E, z)$ we can retain only a single pole term, neglecting the other poles (which at a given energy lie far from the real axis) and the integral term. In formula (10) this approximation cannot be made, as can be seen from the following. We neglect in (6) and (10) all terms except for the term corresponding to the pole $\alpha(E) = l + \nu(E)$ ($|(E)| \ll 1$). In this approximation the resonance partial amplitudes $f_l^N(E)$ obtained from (6) and (10) coincide, while for small

1)The integral over z' in (7) must be interpreted in the regularized sense (cf. Appendix).

nonresonance phases (adjacent to the resonance phase) $\delta_n^N(E) \approx f_n^N(E)$, $n \neq l$ we obtain: from (6)

$$\delta_n(E) \approx r_\alpha^N(E)\left(\frac{1}{n-l}-\frac{1}{n+l+1}\right), \quad (6a)$$

from (10)

$$\delta_n(E) \approx \exp\left[2i(\sigma_l-\sigma_n)\right] r_\alpha^N(E)\left(\frac{1}{n-l}-\frac{1}{n+l+1}\right). \quad (10a)$$

Formula (6a) coincides with (10) from I and does not contradict general physical considerations, while in (10a) the Coulomb factor $\exp[2i(\sigma_l - \sigma_n)]$ makes the nuclear phase $\delta_n(E)$ complex even in the case when only a single elastic scattering channel is open. This shows that in (10) even in the case of an isolated resonance it is not possible to neglect the integral term.

3. THE CHARACTERISTIC ASYMMETRY OF A LEVEL IN THE CASE OF CHARGED PARTICLES

We assume that at an energy $E \approx E_0$ one of the poles $\alpha(E)$ passes close to a positive integral value of l:

$$\alpha(E) = l + \nu(E), \quad |\nu(E)| \ll 1. \quad (11)$$

We pick out in (6) the term $f_\alpha^N(E, z)$ corresponding to this pole:

$$f_\alpha^N(E,z) = -(2\alpha+1)\, r_\alpha^N(E)\frac{\pi}{\sin \pi\alpha} P_\alpha^c(-z)$$

$$= -(2\alpha+1)\, r_\alpha^N(E)\, e^{2i\sigma_l}\left[\frac{P_l(z)}{\nu(E)} + R_l^c(z) + O(\nu)\right], \quad (12)$$

where

$$R_l^c(z) = (-)^l e^{-2i\sigma_l}\left[\frac{\partial}{\partial\lambda} P_\lambda^c(-z)\right]_{\lambda=l}. \quad (13)$$

The first term in (12) is the usual Breit-Wigner resonance amplitude. By analogy with I we shall refer to the second term $R_l^c(z)$ as CAL.

With the aid of (12) we obtain from (6)

$$f^N(E,z) = -(2\alpha+1) r_\alpha^N(E) e^{2i\sigma l}$$
$$\times\left[P_l(z)/\nu(E) + R_l^c(z) + g_l^c(E,z)\right], \quad (14)$$

where $g_l^c(E, c)$ denotes the contribution arising from the other poles $\alpha_i \neq \alpha$ and from the integral. It can be seen from (6a) that the phases corresponding to the pole amplitude $f_\alpha^N(E, z)$ behave as $\delta_n(E) \sim 1/n^2$ for $n \to \infty$. Therefore, $g_l^c(E, z)$ in (14) guarantees the exponential falling off of δ_n for large n and cannot be neglected over the whole range of angles $0 < \theta < \pi$ (cf., analogous considerations in I). In order to estimate the relative contributions of $R_l^c(z)$ and $g_l^c(E, z)$ we write $f_n^N(E)$ in the form

$$f_n^N(E) = r_\alpha^N(E)\left(\frac{1}{n-\alpha}-\frac{1}{n+\alpha+1}\right)\xi_n, \quad (15)$$

where ξ_n is a cut-off factor analogous to the one introduced in I.

In accordance with the hypothesis introduced in I we assume that a smooth compensation of pole phases occurs, i.e., for $0 \le n \le L \sim kR$, ξ_n is a slowly varying function of n close to unity (we recall that $\xi_l \approx 1$). Utilizing for ξ_n a rough model with a sharp cut-off (cf., formula (12) from I) we obtain for $L \gg l, \eta$ [2)]

$$g_l^c(E,z) \approx \left(\frac{2}{\pi \sin\theta}\right)^{1/2}\frac{\cos(L\theta+\pi/4)}{L^{3/2}\sin(\theta/2)}\, e^{2i(\sigma_L-\sigma_l)},$$
$$0 < \varepsilon \leqslant \theta \leqslant \pi-\varepsilon, \quad \varepsilon \gg 1/L, \quad (16)$$

which up to an unessential phase factor $\exp[2i(\sigma_L - \sigma_l)]$ coincides with the estimate for $g(E, z)$ in I. For angles θ satisfying the relation

$$F_l^c(\theta) = (\sin\theta)^{1/2}\sin(\theta/2)\,|R_l^c(\cos\theta)| \geqslant 0.8/L^{3/2}\delta, \quad (17)$$

the inequality $|g_l^c(E, z)/R_l^c(z)| \le \delta$ will hold. When $\delta \ll 1$ we can neglect for these angles θ the effect of the compensation of distant pole phases [3)] and we can approximate the nuclear amplitude $f^N(E, z)$ by the pole amplitude $f_\alpha^N(E, z)$.

Utilizing the relation

$$-(2\alpha+1) r_\alpha^N(E)/\nu(E) = (2l+1)e^{i\delta_l}\sin\delta_l + O(\nu), \quad (18)$$

where $\delta_l(E)$ is the resonance phase, we obtain from (1) and (14) for angles θ satisfying (17) the formula for the differential scattering cross section

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2)In the case of scattering of α-particles with energies of several MeV by light nuclei we have $\eta \approx 1-2$.

3)We stress that the estimates obtained for the range of angles θ in which the compensation is not essential depend very little on the specific model utilized for the cut-off factor $\xi_n(E)$. For example, a model which is more realistic than the model with a sharp cut-off is the one given by $\xi_n = \exp[-(n-l)/L]$, which correctly represents the exponential character of the decrease of distant phases δ_n for $n \gg l$. With the aid of this model we obtain in the case $\eta = 0$ (absence of Coulomb interaction) and $l = 0$ the following range of angles θ in which CAL is distorted as a result of compensation by not more than 10%: $26° < \theta < 87°$ for $L = 6$ and $17° < \theta < 100°$ for $L = 10$. Comparison with the corresponding values given in I, § 2 for a model with a sharp cut-off clearly shows that in order to draw the conclusion that it is possible to observe CAL over a wide range of angles that assumption of a smooth variation of ξ_n near the resonance value $n = l$ is essential, while the details of the behavior of ξ_n for $n \gg l$ are unimportant.

$$\frac{d\sigma}{d\Omega} = \left| f^c(E, z) + \frac{2l+1}{k} e^{i(2\sigma_l + \delta_l)} \sin \delta_l \left[P_l(z) + \nu(E) R_l^c(z) \right] \right|^2. \quad (19)$$

Thus, knowing the variation with energy of the resonance phase $\delta_l(E)$ we can obtain from the angular distribution of a resonance nuclear reaction the trajectory of the Regge pole $\alpha(E) = l + \nu(E)$ in the neighborhood of the resonance due to it. In contrast to the case when Coulomb interaction is absent the function $R_l^C(z)$ is complex and, therefore, from an analysis of the angular distribution one can derive not only Re $\nu(E)$, but also Im $\nu(E)$. We note that from (13) and (16) it follows that the formulas for non-resonance nuclear phases remain unchanged in the presence of Coulomb interaction, and this was already utilized in I, Sec. 3.

The quantities characterizing resonance scattering in the theory of moving Regge poles can be easily related to parameters utilized in the formal theory of resonance nuclear reactions. In this theory the dependence of the resonance phase $\delta_l(E)$ on the energy is given by the formula[7]:

$$\delta_l(E) = \delta_l^{(r)}(E) + \Phi_l(E),$$

$$\delta_l^{(r)} = \tan^{-1} \frac{\gamma_\lambda^2 s_l}{E_\lambda + \Delta_\lambda - E}, \quad \Phi_l = -\tan^{-1} \left(\frac{F_l}{G_l} \right)_{r=R}. \quad (20)$$

Here E_λ are the formal resonance energy levels, Δ_λ is the level shift, $E_0 = E_\lambda + \Delta_\lambda$ is the energy of the resonance observed experimentally, $s_l = kRP_l$, $P_l = (G_l^2 + F_l^2)^{-1}_{r=R}$ is the penetrability coefficient, $F_l(r)$ and $G_l(r)$ are the regular and the singular Coulomb radial wave functions, R is the channel radius, $\Phi_l = \xi_l - \sigma_l$, ξ_l is the scattering phase for an impenetrable charged sphere, σ_l is the Coulomb scattering phase. For a narrow resonance ($\Gamma \ll E_0$) we obtain from (18) and (20) the relation:

$$r_\alpha^N \left(\frac{d\alpha}{dE} \right)^{-1} = \frac{\hbar^2 k}{MR} \theta_l^2 P_l \exp(2i\Phi_l), \quad (21)$$

where M is the reduced mass, θ_l^2 is the dimensionless reduced width and all the quantities which depend on the energy are taken at $E = E_0$. In the absence of Coulomb interaction (21) reduces to (8) of I.

4. RESULTS OF NUMERICAL CALCULATIONS

With the aid of formula (A4) of the Appendix the function $R_l^C(\cos\theta)$ was calculated for the case of elastic resonance scattering of α-particles by C^{12}, C^{14}, and O^{16} nuclei. Owing to lack of space we shall give results only for several resonances in the scattering process $C^{12}(\alpha, \alpha)C^{12}$. Figure 1a

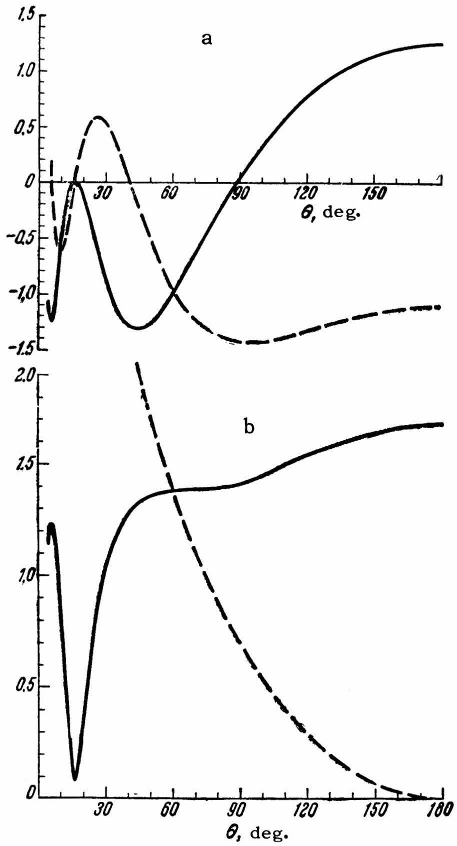

FIG. 1

shows the real (dotted curve) and the imaginary (solid curve) parts of the function $R_0^C(\cos\theta)$ for the 0^+ resonance (the resonance energy is $E_i = 5.47$ MeV, $\eta = 1.62$). Figure 1b compares the quantity $|R_0^C(\cos\theta)|$ (solid curve) for this resonance with the quantity $|R_0(\cos\theta)|$ (dotted curve) calculated for $\eta = 0$. As can be seen from the diagram, for $\eta \gtrsim 1$ the Coulomb interaction radically changes the behavior and the value of CAL, particularly in the domain of large angles.

The table gives values of the functions $R_l^C(\cos\theta)$ for the resonances 1^- ($E_r = 3.205$ MeV, $\eta = 2.10$) and 4^+ ($E_r = 4.241$ MeV, $\eta = 1.83$), for which we have constructed in I on the basis of phase analysis data the trajectories of the Regge poles $\alpha_1(E)$ and $\alpha_4(E)$. As was noted in I, a measurement of the angular distribution of the α particles for these resonances is of great interest from the point of view of checking the theory.

Figure 2 gives graphs of the functions $F_1^C(\cos\theta)$ (solid curve) and $F_4^C(\cos\theta)$ (dotted curve) for the resonances 1^- and 4^+. From Fig. 2 and formula (17) it can be seen that for $L \gtrsim 5-6$ the effect of CAL on the angular distribution should be observed over a wider energy range than in the case without Coulomb interaction (cf., Fig. 2 in I). It was shown in I that in the neighborhood of the resonance 1^- the

θ, deg	$\mathrm{Re}R_l^c$ (cos θ)		$\mathrm{Im}R_l^c$ (cos θ)		θ, deg	$\mathrm{Re}R_l^c$ (cos θ)		$\mathrm{Im}R_l^c$ (cos θ)	
	$l=1$, $\eta=2.10$	$l=4$, $\eta=1.83$	$l=1$, $\eta=2.10$	$l=4$, $\eta=1.83$		$l=1$, $\eta=2.10$	$l=4$, $\eta=1.83$	$l=1$, $\eta=2.10$	$l=4$, $\eta=1.83$
25	0.284	—0.148	—1.405	0.034	105	—0.011	0.587	—0.009	0.084
28	0.084	0.104	—1.544	0.261	107.5	0.002	0.603	—0.069	0.023
31	—0.156	0.295	—1.585	0.379	110	0.010	0.601	—0.131	—0.041
34	—0.394	0.411	—1.538	0.418	112.5	0.015	0.580	—0.195	—0.104
37	—0.605	0.450	—1.424	0.404	115	0.016	0.544	—0.259	—0.166
40	—0.775	0.422	—1.264	0.361	117.5	0.014	0.493	—0.324	—0.224
42.5	—0.883	0.357	—1.109	0.314	120	0.009	0.431	—0.389	—0.276
45	—0.959	0.262	—0.944	0.265	122.5	0.001	0.360	—0.454	—0.320
47.5	—1.007	0.149	—0.777	0.218	125	—0.009	0.283	—0.518	—0.355
50	—1.028	0.025	—0.614	0.179	127.5	—0.021	0.204	—0.581	—0.379
52.5	—1.028	—0.101	—0.458	0.149	130	—0.035	0.125	—0.644	—0.391
55	—1.008	—0.221	—0.314	0.129	132.5	—0.050	0.050	—0.705	—0.391
57.5	—0.973	—0.329	—0.183	0.120	135	—0.066	—0.019	—0.763	—0.377
60	—0.926	—0.420	—0.067	0.120	137.5	—0.084	—0.080	—0.820	—0.351
62.5	—0.870	—0.489	0.035	0.130	140	—0.101	—0.132	—0.875	—0.312
65	—0.807	—0.534	0.121	0.148	142.5	—0.119	—0.171	—0.927	—0.261
67.5	—0.740	—0.554	0.192	0.171	145	—0.138	—0.199	—0.977	—0.200
70	—0.670	—0.547	0.249	0.198	147.5	—0.155	—0.215	—1.024	—0.129
72.5	—0.600	—0.517	0.292	0.227	150	—0.173	—0.219	—1.068	—0.051
75	—0.531	—0.463	0.321	0.256	152.5	—0.190	—0.213	—1.109	0.033
77.5	—0.465	—0.390	0.339	0.282	155	—0.206	—0.197	—1.147	0.121
80	—0.401	—0.300	0.345	0.304	157.5	—0.221	—0.173	—1.182	0.209
82.5	—0.340	—0.198	0.342	0.320	160	—0.235	—0.143	—1.213	0.296
85	—0.284	—0.089	0.329	0.329	162.5	—0.248	—0.110	—1.241	0.380
87.5	—0.233	0.025	0.307	0.329	165	—0.259	—0.076	—1.265	0.457
90	—0.186	0.137	0.278	0.320	167.5	—0.269	—0.042	—1.286	0.527
92.5	—0.145	0.244	0.242	0.302	170	—0.277	—0.011	—1.303	0.587
95	—0.108	0.342	0.201	0.274	172.5	—0.284	0.015	—1.316	0.635
97.5	—0.077	0.428	0.154	0.238	175	—0.288	0.034	—1.325	0.671
100	—0.050	0.498	0.103	0.193	177.5	—0.291	0.047	—1.331	0.693
102.5	—0.028	0.552	0.049	0.141	180	—0.292	0.051	—1.333	0.700

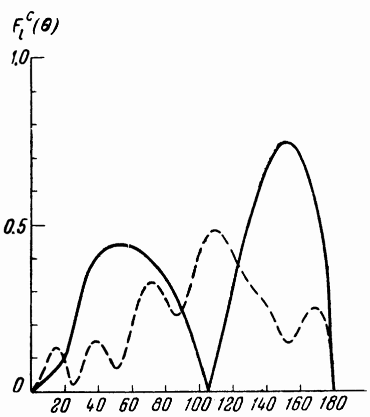

FIG. 2

only essential contribution is from the single Regge pole $\alpha_1(E) = 1 + \nu_1(E)$. Utilizing the value $\nu_1(E_r) \approx i0.06$ obtained in I and the data of the table we have calculated by means of formula (19) the differential cross section $d\sigma/d\Omega$ for $E = E_r = 3.205$ MeV. The results are shown in Fig. 3a where along the vertical axis we have plotted the logarithm of the ratio of $d\sigma/d\Omega$ to the differential cross section for Coulomb scattering $d\sigma^c/d\Omega = |f^c(E, z)|^2$. The dotted curve corresponds to the calculation without CAL ($\nu_1 = 0$), the solid curve corresponds to the calculation taking CAL into account ($\nu_1 = i0.06$).

From the diagram it can be seen that in the domain of large angles the contribution of CAL to $d\sigma/d\Omega$ attains a value of 15—20%. Figure 3b shows the ratio of the differential cross section taking CAL into account to the differential cross section without CAL for the 1^- resonance. Figures 3a and b do not show the domain of angles close to $\theta = 110°$, since at these angles there occurs a compensation of the Coulomb and the Breit-Wigner amplitudes, and for accidental reasons $R_1^c(\cos\theta)$ is also close to zero, and this makes the calculations unreliable.

From the results of numerical calculations we can draw the following conclusions: 1) the contribution of CAL to the differential cross section of resonance scattering can attain a value of 5—20% and should be easily observed experimentally; 2) allowance for the Coulomb interaction sharply increases the effect of CAL for large scattering angles.

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APPENDIX

Integration over φ in (7) with the aid of formulas (8.794) and (8.796) from [6] yields

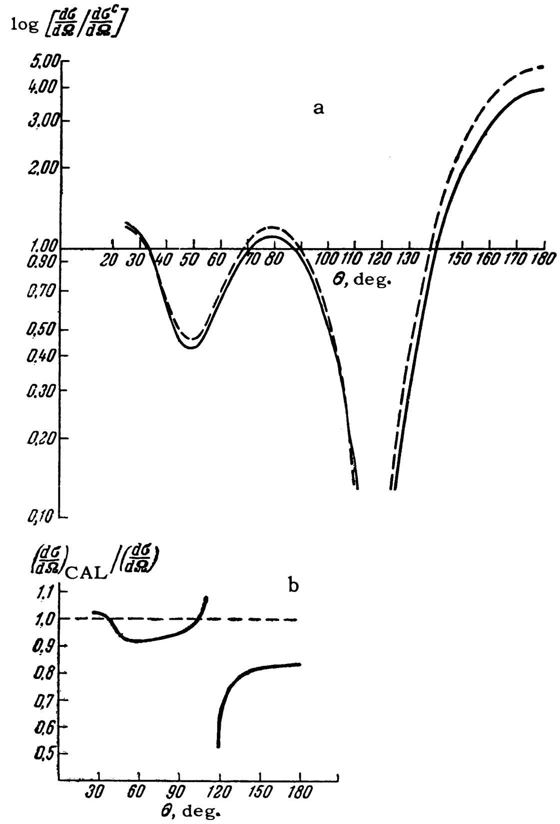

FIG. 3

$$P_\lambda^c(z) = iC(\eta)\lim_{\rho\to1+i\eta}\Big[P_\lambda(z)\int_{-z}^{1}P_\lambda(z')(1-z')^{-\rho}dz' + P_\lambda(-z)\int_{-1}^{-z}P_\lambda(-z')(1-z')^{-\rho}dz'\Big],$$

$$C(\eta) = -\eta 2^{i\eta}e^{2i\sigma_0}, \qquad \text{(A1)}$$

where prior to integration one must assume $\operatorname{Re}\rho < 1$. From this it follows that

$$P_\lambda^c(-z) = iC(\eta)\lim_{\rho\to1+i\eta}\int_{-1}^{1}P_\lambda(-z_<)P_\lambda(z_>)(1-z')^{-\rho}dz', \qquad \text{(A2)}$$

where $z_> = \max(z, z')$, $z_< = \min(z, z')$.

From (13), (A2), and formula (A4) of I for $R_l(z)$

$$R_l(z) = (-)^l\Big[\frac{\partial}{\partial\lambda}P_\lambda(-z)\Big]_{\lambda=l} = P_l(z)\ln\frac{1-z}{2} + V_l(z)$$

it follows that

$$R_l^c(z) = iC(\eta)\,e^{-2i\sigma_l}$$

$$\times\lim_{\rho\to1+i\eta}\int_{-1}^{1}\Big[P_l(z)P_l(z')\ln\frac{(1+z_>)(1-z_<)}{4} + P_l(z_>)V_l(z_<) + P_l(-z_<)V_l(-z_>)\Big](1-z')^{-\rho}dz'. \qquad \text{(A3)}$$

For the evaluation of the integrals in (A3) we must introduce $\xi = (1-z)/2$, $\xi' = (1-z')/2$ and utilize the relations

$$P_l(1-2\xi) = \sum_{k=0}^{l}\alpha_{lk}\xi^k, \quad \alpha_{lk} = (-)^k\frac{(l+k)!}{(k!)^2(l-k)!},$$

$$V_l(z) - (-)^l V_l(-z) = 2W_{l-1}(z),$$

$$W_{l-1}(z) = \frac{1}{2}\int_{-1}^{1}\frac{P_l(z) - P_l(z')}{z - z'}dz'$$

and formula (7.127) from [6]. The polynomial $W_{l-1}(z)$ is defined by formula (8.831) of [6]. Comparatively awkward calculations lead to the result:

$$R_l^c(z) = P_l(z)\Big[\frac{1-\xi^{-i\eta}}{i\eta} - 2i\eta\sum_{k=1}^{l}\frac{1}{k^2+\eta^2} - \int_0^{\xi}\frac{1-t^{-i\eta}}{1-t}dt - i\eta e^{2i(\sigma_0-\sigma_l)}\xi^{-i\eta}f_l(\xi,\eta)\Big]$$
$$+ (-)^l V_l(-z) - 2i\eta e^{2i(\sigma_0-\sigma_l)}W_{l-1}(z)\sum_{k=0}^{l}\frac{\alpha_{lk}}{k-i\eta}\xi^{k-i\eta} \qquad \text{(A4)}$$

Here $z = \cos\theta$, $\xi = \sin^2(\theta/2)$,

$$f_l(\xi,\eta) = \begin{cases} 0 & \text{for } l = 0,\\ \sum_{n=0}^{l-1}\xi^n\sum_{k=n+1}^{l}\frac{\alpha_{lk}}{(k-n)(k-i\eta)} & \text{for } l \geqslant 1 \end{cases} \qquad \text{(A5)}$$

We note some properties of $R_l^c(z)$.

1. For $\eta \to 0$ $R_l^c(z) \to R_l(z)$, and, moreover, for $\eta \ll 1$ $R_l^c(z)$ significantly differs from $R_l(z)$ only for angles $0 \le \theta \le \theta_c$, $\theta_c \sim \exp(-1/\eta)$.

2. For $\theta \to 0$ $R_l(z)$ has a singularity of a power type $\sim[\sin(\theta/2)]^{-2i\eta}$ in contrast to the logarithmic singularity $R_l(\cos\theta) \sim \ln\sin(\theta/2)$.

3. In the experimentally interesting range of angles θ close to 180° CAL is small in the absence of Coulomb interaction since $R_l(-1) = 0$.

In accordance with (A4) we have

$$R_l^c(-1) = (-)^l\Big[\psi(1) - \psi(1-i\eta) - 2i\eta\sum_{k=1}^{l}\frac{1}{k^2+\eta^2} - i\eta e^{2i(\sigma_0-\sigma_l)}f_l(1,\eta)\Big], \qquad \text{(A6)}$$

$$f_l(1,\eta) = \sum_{k=1}^{l}\frac{\alpha_{lk}}{k-i\eta}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)$$

and for $\eta \gtrsim 1$ CAL can have an appreciable value in the case of backward scattering.

4. The general character of the behavior of $R_l^c(\cos\theta)$ can be seen from the following asymptotic formula:

$$R_l^c(\cos\theta) \approx (-)^l\exp(2i\sigma_l)\times\Big[\frac{2i\eta}{l+1/2}J_0\Big(\Big(l+\frac{1}{2}\Big)\varphi\Big) - \varphi J_1\Big(\Big(l+\frac{1}{2}\Big)\varphi\Big)\Big]$$

$$\varphi = \pi - \theta, \quad 0 \leqslant \varphi \leqslant \pi l^{-1},$$

which is valid for $l \gg 1$ (J_0 and J_1 are Bessel functions). Comparison with the results of numerical calculations in accordance with formula (A4) shows that already for $l = 4$ the accuracy of the asymptotic formula is $\sim 10\%$ in the range of angles $90° < \theta < 180°$ which is sufficient for qualitative conclusions.

[1] L. Rebolia and G. Viano, Nuovo cimento **26**, 1426 (1962).

[2] V. S. Popov and É. I. Dolinskiĭ, JETP **46**, 1829 (1964), Soviet Phys. JETP **19**, 1232 (1964).

[3] Jones, Phillips, Harris, and Beckner, Nuclear Phys. **37**, 1 (1962).

[4] L. Hulthen and M. Sugawara, Collection of articles "Structure of the Atomic Nucleus," IIL, 1959, Sec. 21.

[5] T. Regge, Nuovo cimento **14**, 951 (1959); **18**, 947 (1960).

[6] I. S. Gradshteĭn and I. M. Ryzhik, Tablitsy integralov, summ, ryadov i proizvedeniĭ (Tables of Integrals, Sums, Series and Products), Fizmatgiz, 1962.

[7] A. Lane and R. Thomas, Theory of Nuclear Reactions at Low Energies (Russ. Transl., IIL, 1960) Ch. 4.

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