# *QUANTUM CORRELATION FUNCTIONS IN A MAXWELLIAN PLASMA*

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We calculate the probabilities of the relative positions of two particles in a fully ionized Boltzmann plasma with allowance for the quantum effects. These probabilities permit, for example, a very simple determination of the thermodynamic potential of such a plasma, which was previously calculated by Vedenov and Larkin[1] with the aid of a diagram technique. Besides, we determine for this case of an equilibrium plasma explicit momentum- and coordinate-dependent quantum two-particle distribution functions which, as is well known, are the direct quantum analogs of the corresponding classical functions.

## INTRODUCTION

IN the present paper we consider quantum effects in an equilibrium plasma, in which two inequalities are assumed satisfied: $n\lambda^3 \ll 1$ (absence of degeneracy) and $e^2/hv \ll 1$ (Born approximation). Under these conditions the plasma can be assumed to be fully ionized, and the momentum distributions of the particles are close to Maxwellian, so that an essentially classical analysis can be employed.

It is known, however, that a purely classical description of the plasma entails difficulties. For example, the classical statistical integral diverges as a result of the non-integrable singularity possessed by factors of the type $w(r_{12}) = \exp(-e_1e_2/\Theta r_{12})$. The quantity $w(r_{12})$ can be regarded as the probability of mutual position of two charges $e_1$ and $e_2$ with distance $\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$ between them (for a more precise definition see Sec. 1), and when $e_1e_2 < 0$ we have $w(r_{12}) \to \infty$ as $r_{12} \to 0$, meaning that it is "thermodynamically convenient" for two unlike charges to join together. Without introducing artificial models with repulsive small-radius forces, the situation can be improved by taking quantum effects into account, and this is the purpose of the present article.

When the quantum corrections are taken into account, the probability of the relative position of two particles in the plasma can be written in the form $w(r) = w_{cl}(r) + \nu_q(r)$; by virtue of the uncertainty relations, the above-mentioned singularities should not arise and should not cause the joining of the two unlike charges. The quantity $u_q(r)$, which can be called a "quantum correlation function," is calculated in Sec. 1 below. We make use of the fact that the quantum nature of the particles can come into play only at distances on the order of the electronic wavelength, $\lambda \sim h/mv$, which under our conditions is much smaller than the Debye radius. At this dis-distance ($r \ll d$) the interaction between these two charges is not distorted by the Debye screening, so that the quantity $\nu_q(r)$ can be determined relatively simply by using the well known solution of the pure two-body Coulomb problem.

Insofar as the authors know, the probabilities of mutual positions of particles in a plasma were never calculated before with account of quantum effects. Yet it is quite simple, for example, to calculate with their aid the thermodynamic potential of the plasma (see Sec. 2), which was obtained for these conditions by Vedenov and Larkin[1] by a diagram technique (it must be noted that some numerical coefficients of Vedenov and Larkin are in error).

## 1. PROBABILITY OF MUTUAL POSITION OF TWO PARTICLES IN A PLASMA

In a homogeneous isotropic plasma, this probability can always be represented in the form

$$dW_{12} = w(r_{12})\frac{d\mathbf{r}_1}{V}\frac{d\mathbf{r}_2}{V}, \quad r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|, \tag{1.1}$$

where V is the volume. As is well known[3,4] in classical theory $w(r)$ is described by

$$w_{cl}(r) = 1 - \frac{e_1e_2}{\Theta r}e^{-r/d} \approx \exp\left(-\frac{e_1e_2}{\Theta r}e^{-r/d}\right), \tag{1.2}$$

where $\Theta = kT$ is the temperature and d the Debye radius. Even if the last exponential form is used, we find for unlike charges ($e_1e_2 < 0$) that $w(r) \to \infty$ as $r \to 0$, corresponding to joining of the charges.

We must expect this joining not to occur if the quantum effects are taken into account, since the uncertainty relation prevents the charges from coming together. In this case the quantity $w(r)$ can obviously be represented in the form

$$w(r) = w_{cl}(r) + \nu_q(r), \tag{1.3}$$

and the correction term $\nu_q(r)$, which takes into account the particle quantum correlation effects, should compensate for the divergence of the classical probability as $r \to 0$.

To determine $\nu_q(r)$ we make use of the fact that the quantum nature of the particles can be manifest only at distances on the order of the de Broglie wavelength $\lambda \sim h/mv$. We shall assume that the latter is much smaller than the Debye radius. For such distances ($r \sim \lambda \ll d$) we can leave out from the classical formula (1.2) the screening factor $\exp(-r/d)$. This means a purely Coulomb interaction between the two particles in question when $r \ll d$, i.e., we can disregard the influence of the remaining charges of the system. Then, in accordance with quantum statistics, $w(r)$ should be determined by the diagonal elements of the density matrix

$$w(r_{12})_{r\ll d} = \text{const}\cdot \sum_n \exp(-E_n/\Theta)\ |\Psi_n(\mathbf{r}_1, \mathbf{r}_2)|^2, \quad (1.4)$$

and, taking the foregoing into account, we must choose for $\Psi_n(\mathbf{r}_1, \mathbf{r}_2)$ the eigenfunctions of the two-body problem with Coulomb interaction.

Separating the motion of the center of mass (plane wave) and integrating in (1.4) over the total momentum, we obtain

$$w(r)_{r\ll d} = \text{const}\cdot \sum_{\mathbf{k}} \exp(-\varepsilon_{\mathbf{k}}^{\text{rel}}/\Theta)\,|\psi_{\mathbf{k}}(r)|^2, \quad (1.5)$$

where $\psi_k(r)$ describes the motion of particles with reduced mass in the field of the stationary Coulomb center, and $\psi_k^{\text{rel}}$ —energy of this relative motion. These functions are well known[5], and for the continuous spectrum they take the form

$$\psi_{\mathbf{k}}(r) = (2\pi)^{-3/2} e^{-\pi/2k}\,\Gamma(1 + i/k)\, e^{i\mathbf{k}\rho} F(-i/k, 1, i(k\rho - \mathbf{k}\boldsymbol{\rho})). \quad (1.6)$$

Here

$$\mathbf{k} = h(\mathbf{v}_1 - \mathbf{v}_2)/e_1e_2, \quad \boldsymbol{\rho} = \mu e_1 e_2 (\mathbf{r}_1 - \mathbf{r}_2) h^{-2}$$

and F —confluent hypergeometric function. The functions (1.6) are normalized by the condition

$$\int \psi_{\mathbf{k}}^*(r)\psi_{\mathbf{k}'}(r)\,d\mathbf{r} = \delta(\mathbf{k} - \mathbf{k}').$$

We assume that the Born approximation condition $e^2/hv \ll 1$ is satisfied. We can then neglect in (1.5) the bound states (for like charges there are no such states at all) and, assuming that in (1.6) k is large and $k\rho$ is arbitrary, and confining ourselves to the first term in the expansion in $1/k$, we obtain, as can be readily verified,

$$\psi_{\mathbf{k}}(r) = \frac{e^{i\mathbf{k}\rho}}{(2\pi)^{3/2}}\left\{1 - \frac{1}{k}\left[\frac{\pi}{2} + iC + i\int_0^1 \frac{d\xi}{\xi}(e^{i(k\rho - \mathbf{k}\boldsymbol{\rho})\xi} - 1)\right]\right\} \quad (1.7)$$

(the Euler constant C is a result of the expansion $\Gamma(1 + i/k) = 1 - iC/k + \ldots$). Formula (1.5) then takes the form

$$w(r)_{r\ll d} = \text{const}\cdot \int d\mathbf{k}\, e^{-\varkappa k^2} \times \left\{1 - \frac{\pi}{k} + \frac{2}{k}\int_0^1 \frac{d\xi}{\xi}\sin[(k\rho - \mathbf{k}\boldsymbol{\rho})\xi]\right\} \quad (1.8)$$

where $\kappa = \mu e_1^2 e_2^2/2\Theta h^2$. The integrals contained here can be readily calculated and we ultimately get

$$w(r)_{r\ll d} = 1 - \frac{e_1e_2}{\Theta r} + \frac{e_1e_2}{hv_T}2\int_{r/\lambda}^{\infty} dt\,\frac{\exp(-t^2)}{t^2}, \quad (1.9)$$

where $v_T = (2\Theta/\mu)^{1/2}$, and $\lambda = h\mu v_T$. The constant in (1.8) is chosen such as to make formula (1.9) assume the classical form for $r \gg \lambda$.

It is obvious that the last term in (1.9) is indeed the quantum correction term of (1.3). Since this term is exponentially small for $r \gg \lambda$, the Debye screening does not influence it. Thus, we can ultimately write

$$w(r) = 1 - \frac{e_1e_2}{\Theta r}e^{-r/d} + \nu_q(r), \quad (1.10)$$

where

$$\nu_q(r) = \frac{2e_1e_2}{hv_T}\int_{r/\lambda}^{\infty} dt\,\frac{\exp(-t^2)}{t^2} = \frac{2e_1e_2}{hv_T}\sqrt{\pi}\left[\Phi\left(\frac{r}{\lambda}\right) - 1 + \frac{\lambda}{r\sqrt{\pi}}\exp\left(-\frac{r^2}{\lambda^2}\right)\right] \quad (1.11)$$

($\Phi(x)$ —error function). It is easy to verify that as $r \to 0$ the quantum term in (1.10) actually cancels out the divergence of the classical term, and in particular

$$w(0) = 1 - 2\sqrt{\pi}e_1e_2/hv_T. \quad (1.12)$$

If the particles in question are identical (electrons), then (1.10) must be supplemented by a term that takes the exchange effects into account. In this case the functions $\Psi_n(\mathbf{r}_1, \mathbf{r}_2)$ which are contained in (1.4) should have a suitable symmetry, and in particular they should be antisymmetrical for electrons. When summing over the spin variables, it is necessary to take the antisymmetrical triplet and symmetrical singlet, so that we have in (1.5) in lieu of $|\psi_k(r)|^2$

$$3|\psi_{\mathbf{k}}(\mathbf{r}) - \psi_{\mathbf{k}}(-\mathbf{r})|^2 + |\psi_{\mathbf{k}}(\mathbf{r}) + \psi_{\mathbf{k}}(-\mathbf{r})|^2 = \frac{1}{(2\pi)^3}\left\{1 - \frac{\pi}{k} + \frac{2}{k}\int_0^1 \frac{d\xi}{\xi}\sin[(k\rho - \mathbf{k}\boldsymbol{\rho})\xi] - \frac{e^{2i\mathbf{k}\rho}}{2}\left[1 - \frac{\pi}{k} + \frac{2}{k}\int_0^1 \frac{d\xi}{\xi}e^{-i\mathbf{k}\boldsymbol{\rho}\xi}\sin k\rho\xi\right]\right\}. \quad (1.13)$$

The first three terms coincide here with the integrand in (1.8) and consequently yield the already mentioned quantum correction $\nu_q(r)$, while the terms with $\exp(2i\mathbf{k}\cdot\boldsymbol{\rho})$ yield the exchange corrections.

It is easy to verify that after integration with respect to k the final result can be represented in the form

$$w(r)=1-\frac{e_1e_2}{\Theta r}e^{-r/d}+\nu_q(r)+\nu_q^{\,ex}(r). \qquad (1.14)$$

Here

$$\nu_q^{\,ex}(r)=-\frac{1}{2}\left(\frac{\varkappa}{\pi}\right)^{3/2}\int d\mathbf{k}\,e^{-\varkappa k^2+2i\mathbf{k}\boldsymbol{\rho}}$$

$$\times\left[1-\frac{\pi}{2}+\frac{2}{k}\int_0^1\frac{d\xi}{\xi}e^{-ik\rho\xi}\sin k\rho\xi\right]$$

$$=-\frac{1}{2}e^{-r^2/\lambda^2}-\frac{e^2}{hv_T}\int_{r/\lambda}^{\infty}dt\,\frac{e^{-t^2}-e^{-r^2/\lambda^2}}{t^2-r^2/\lambda^2}. \qquad (1.15)$$

The first term describes the exchange correlation in an ideal gas, and the second the exchange corrections that take the interaction into account. For $r\gg\lambda$ these terms are exponentially small, and as $r\to 0$ both remain finite.

## 2. THERMODYNAMIC POTENTIAL OF THE PLASMA

Using the obtained probabilities, we can readily calculate the thermodynamic potential of the plasma. For our conditions ($n\lambda^3\ll 1$, $e^2/hv\ll 1$) the latter was determined earlier by Vedenov and Larkin[1], who used a diagram technique. Their result can be represented in the form

$$\Omega=\Omega_B+\Delta\Omega_D+n\Theta\left[\frac{\pi^{3/2}}{\sqrt{2}}n\lambda^3-4\pi\frac{e^2}{hv_e}n\lambda^3\right.$$

$$+\xi_d\frac{\lambda}{d}\frac{\sqrt{\pi}}{4(z+1)}\left(z+\sqrt{2}\ln 2+\frac{1}{2\sqrt{2}}\right)$$

$$\left.+\frac{\xi_d^2}{12(z+1)}\left(z^4\ln\frac{1}{z^2\xi_d}+\ln\frac{d}{\lambda}-2z^2\ln\frac{d}{\lambda}\right)\right]. \qquad (2.1)$$

Here $\Omega_B$—Boltzmann potential of a classical ideal gas; $\Delta\Omega_D=-\frac{1}{3}n\Theta(z+1)\xi_d$—ordinary Debye correction; $d=[\Theta/4\pi ne^2(z+1)]^{1/2}$—Debye radius (account of quasineutrality gives $zn_i=n_e=n$); $\xi_d=e^2/\Theta d$—small parameter, and $v_e=(2\Theta/m_e)^{1/2}$ and $\lambda=h/m_ev_e$. The calculation that follows, made with the aid of the probabilities obtained in Sec. 1, leads to an expression similar to (2.1) for $\Omega$, but the numerical coefficients in some of the terms are different.

To determine $\Omega$ we made use of the well known formula

$$\Omega=\Omega_{id}+\frac{1}{V}\int_0^{e^2}\langle H_{int}\rangle\frac{de^2}{e^2}. \qquad (2.2)$$

Here $\Omega_{id}=\Omega_B+n\Theta\pi^{3/2}n\lambda^3/\sqrt{2}$—potential of ideal electron-ion gas with account of the correction term, corresponding to the weak electron degeneracy (see [6]). The ions are regarded classically. Averaging the interaction Hamiltonian

$$H_{int}=z^2e^2\sum_{i<i'}\frac{1}{r_{ii'}}+e^2\sum_{e<e'}\frac{1}{r_{ee'}}-ze^2\sum_{i,e}\frac{1}{r_{ie}} \qquad (2.3)$$

with the aid of the probabilities obtained in Sec. 1 we have, with account of the quasineutrality,

$$\frac{1}{V}\langle H_{int}\rangle=\frac{n^2e^2}{2}\int\frac{d\mathbf{r}}{r}[w_{ii}(r)+w_{ee}(r)-2w_{ei}(r)]. \qquad (2.4)$$

For the probability involved here we get

$$w_{ii}+w_{ee}-2w_{ei}=-\frac{(z+1)^2e^2}{\Theta r}e^{-r/d}$$

$$+[\nu_{ee}+\nu_{ee}^{ex}-2\nu_{ei}]. \qquad (2.5)$$

It is easy to verify that the first term leads to the usual Debye correction, so that (2.2) can be represented in the form

$$\Omega=\Omega_B+n\Theta\frac{\pi^{3/2}}{\sqrt{2}}n\lambda^3+\Delta\Omega_D$$

$$+\frac{n^2}{2}\int_0^{e^2}de^2\int\frac{d\mathbf{r}}{r}[\nu_{ee}(r)+\nu_{ee}^{ex}(r)-2\nu_{ei}(r)]. \qquad (2.6)$$

The second term can be called the pure exchange correction (it does not depend on the charge), while the last term contains the quantum effects considered in the present article. Using formula (1.11) for the quantities $\nu_{ee}$ and $\nu_{ei}$ and formula (1.15) for $\nu_{ee}^{ex}$, and also taking into account the fact that in the case of electron-electron interaction it is necessary to use a reduced mass $\mu=m_e/2$, whereas in the case of an electron ion interaction ($m_i\gg m_e$) we can simply put $\mu=m_e$, we get

$$\int\frac{d\mathbf{r}}{r}\nu_{ee}=8\pi\frac{e^2}{hv_{ee}}\int_0^{\infty}dr\cdot r\int_{r/\lambda_{ee}}^{\infty}dt\,\frac{\exp(-t^2)}{t^2}=\pi^{3/2}\frac{e^2h}{m^{1/2}\Theta^{3/2}},$$

$$\int\frac{d\mathbf{r}}{r}\nu_{ei}=-8\pi\frac{ze^2}{hv_{ei}}\int_0^{\infty}dr\cdot r\int_{r/\lambda_{ei}}^{\infty}dt\,\frac{\exp(-t^2)}{t^2}$$

$$=-\frac{\pi^{3/2}}{\sqrt{2}}\frac{ze^2h}{m^{1/2}\Theta^{3/2}} \qquad (2.7)$$

and finally

$$\int\frac{d\mathbf{r}}{r}\nu_{ee}^{ex}=-4\pi\lambda_{ee}^2\int_0^{\infty}dx\cdot x\left[\frac{e^{-x^2}}{2}+\frac{e^2}{hv_{ee}}\int_x^{\infty}dt\,\frac{e^{-t^2}-e^{-x^2}}{t^2-x^2}\right]$$

$$=-\pi\frac{h^2}{m\Theta}\left[1-\frac{e^2m^{1/2}}{h\Theta^{1/2}}\sqrt{\pi}\ln 2\right]. \qquad (2.8)$$

Integrating in (2.6) over the charge, we get ultimately

$$\Omega = \Omega_{B} + \Delta\Omega_{D} + n\Theta\left[\frac{\pi^{3/2}}{\sqrt{2}}n\lambda^3 - 2\pi\frac{e^2}{hv_e}n\lambda^3 + \xi_d\frac{\lambda}{d}\frac{\sqrt{\pi}}{4(z+1)}\left(\frac{z}{2}+\frac{1+\ln 2}{2\sqrt{2}}\right)\right]. \quad (2.9)$$

The numerical coefficients of three terms in this result differ from those of expression (2.1) derived of Vedenov and Larkin [1], who let some inaccuracies creep into the calculation and who mistakenly used in some cases the density of the electrons with a single spin direction in lieu of the total electron density. The last term in formula (2.1) of [1], which contains logarithms and is proportional to $\xi_d^2$, cannot be derived from our probabilities, which are accurate only to $\xi_d$. However, since the Planck constant is encountered in the last term only under the logarithm sign, this term is essentially classical and can be obtained from the following considerations.

We write down the classical probability (more accurately, the probability density) of the mutual positions of two particles in the form

$$w_{cl} = 1 - \frac{e_1e_2}{\Theta r}e^{-r/d} + \nu_{cl}', \quad (2.10)$$

where $\nu_{cl}'$ are quantities on the order of $\xi_d^2$. Then the indicated last term should, in accordance with (2.6), be equal to

$$\Delta\Omega_{\xi^2} = \frac{n^2}{2}\int^{e^2} de^2 \int\frac{dr}{r}[\nu_{ii}' + \nu_{ee}' - 2\nu_{ei}']. \quad (2.11)$$

The quantities $\nu_{cl}'$ can be determined by using the fact that when $r \ll d$ the interaction of two particles in question should not be distorted by the presence of the remaining charges and must have a pure Boltzmann form

$$w_{cl} = \exp(-e_1e_2/\Theta r). \quad (2.12)$$

If we add to the exponential here the Debye screening factor $\exp(-r/d)$, then (2.12) will coincide with (2.10) accurate to terms in first order in $\xi_d$.

Thus we can state that, accurate to second-order terms in $\xi_d$, the probability can be represented in the form (see [4,7])

$$w_{cl} = \exp\left(-\frac{e_1e_2}{\Theta r}e^{-r/d}\right)[1 + \xi_d^2\varphi(r) + O(\xi_d^3)], \quad (2.13)$$

and as $r \to 0$ the function $\varphi(r)$ should have singularities that are only weaker than singularities of the $r^{-2}$ type. Only then can formula (2.13) go over into the Boltzmann distribution (2.12) as $r \to 0$. Comparing (2.13) with (2.10) we get

$$\nu_{cl}' = \left(\frac{e_1e_2}{\Theta d}\right)^2\left[\frac{e^{-2r/d}}{2r^2/d^2} + \varphi(r)\right]. \quad (2.14)$$

Substituting these values of $\nu_{cl}'$ in (2.11), we see that the first term, which is proportional to $r^{-2}$, leads to integrals that diverge logarithmically as $r \to 0$, whereas the terms with functions $\varphi(r)$ that contain no singularities of the type $r^{-2}$ will be finite. Confining ourselves to logarithmic accuracy, the latter can be disregarded and we obtain

$$\int\frac{dr}{r}(\nu_{ii}' + \nu_{ee}' - 2\nu_{ei}') = 2\pi\frac{e^4}{\Theta^2}\left(z^4\int_{r_{min}^{ii}}^{d}\frac{dr}{r} + \int_{r_{min}^{ee}}^{d}\frac{dr}{r} - 2z^2\int_{r_{min}^{ei}}^{d}\frac{dr}{r}\right). \quad (2.15)$$

Here $r_{min}$ must be chosen differently in different cases. The ion-ion interaction (classical) is cut off at the minimal approach distance $r_{min}^{ii} \sim z^2e^2/\Theta$, whereas the electron-ion interaction [as follows from the quantum probability (1.9)] is cut off at an electron wavelength $\lambda$, and the electron-electron interaction—at a reduced wavelength $\lambda_{ee} = \sqrt{2}\,\lambda$.

Taking these remarks into account, we can write down for the expression (2.15), with logarithmic accuracy,

$$2\pi\frac{e^4}{\Theta^2}\left(z^4\ln\frac{1}{z^2\xi_d} + \ln\frac{d}{\lambda} - 2z^2\ln\frac{d}{\lambda}\right). \quad (2.16)$$

Further, integrating over the charge in accordance with formula (2.11), we obtain the last term of Vedenov and Larkin's formula (2.1).

## 3. QUANTUM TWO-PARTICLE DISTRIBUTION FUNCTIONS

In addition to the probability of the mutual positions of two particles, some interest attaches also to the total quantum two-particle distribution function. In the classical analysis such a function describes the probability of the distribution of two particles over the momenta and coordinates. In the quantum approach, the uncertainty relation does not allow us to introduce this probability, and the system is customarily described by a density matrix $\rho(q, q')$ (in the coordinate representation). However, a more intuitive description, close to the classical one, is obtained by using the density matrices in the "Wigner representation":

$$F_N^q(p, q) = (2\pi)^{-3N}\int e^{-ip\tau}\rho_N\left(q - \frac{h\tau}{2}, q + \frac{h\tau}{2}\right)d^N\tau, \quad (3.1)$$

which are customarily called "quantum N-particle distribution functions" [2]. Although such a function does not have the meaning of a probability (it can be shown that $F_N$ is real but can have an alternating sign), we can define with the aid of $F_N^q$ the momentum probabilities and the coordinate probabilities, when taken separately, and also the mean values, by means of formulas analogous to the classical ones:

$$w(p)=\int F(p,\ q)\,dq, \quad w(q)=\int F(p,\ q)\,dp,$$
$$\overline{A(p,\ q)}=\int A(p,\ q)F(p,\ q)\,dp\,dq. \tag{3.2}$$

Furthermore, the function $F_N^q(p,q)$ goes over into the classical distribution function in the limit as $h \to 0$. Therefore the functions (3.1) can be regarded as direct quantum analogs of classical N-particle distribution functions.

In the case of a plasma, an equation with account of the polarization of the medium was derived for the quantum two-particle distribution function, by Klimontovich and Temko[8,9], and a solution of this equation, suitable for arbitrary single-particle distribution functions f(p), was obtained by Silin[10]. We shall use this general solution to obtain an explicit expression for the two-particle function in our case of a Maxwellian distribution f(p). It must be borne in mind here that neither the Klimontovich-Temko equation nor the Silin solution take into account exchange effects, which we have seen to be significant at distances on the order of $\lambda$. Therefore the expression obtained below is valid only if the two particles in question are not identical.

Leaving out the complicated derivations, we present the result of substituting the Maxwellian distributions f(p) in Silin's general solution. If we represent $F_2^q(p,q)$ in the form

$$F_2{}^q\ (\mathbf{r}_1,\ \mathbf{p}_1;\ \mathbf{r}_2,\ \mathbf{p}_2)$$
$$=f_\alpha(\mathbf{p}_1)f_\beta(\mathbf{p}_2)\left[1+\frac{e_\alpha e_\beta}{hv_T}\zeta_{\alpha\beta}(\mathbf{r}_1-\mathbf{r}_2,\ \mathbf{p}_1,\ \mathbf{p}_2)\right], \tag{3.3}$$

we can obtain for the spatial Fourier component of the function $\zeta_{\alpha\beta}(\mathbf{r},\mathbf{p}_1,\mathbf{p}_2)$ from Silin's solution, in the case of Maxwellian distributions $f_\alpha(p)$,

$$\zeta_{\mathbf{k}}^{\alpha\beta}(\mathbf{p}_1,\ \mathbf{p}_2)=-\frac{\lambda_{\alpha\beta}\exp\left(-k^2\lambda_{\alpha\beta}{}^2/4\right)}{\pi^2k^2(x_\alpha-x_\beta)}\left\{\frac{\operatorname{sh}x_\alpha\operatorname{ch}x_\beta}{\varepsilon_k{}^+(\mathbf{kv}_2)}-\frac{\operatorname{sh}x_\beta\operatorname{ch}x_\alpha}{\varepsilon_k{}^-(\mathbf{kv}_1)}\right.$$
$$\left.+\operatorname{sh}x_\alpha\operatorname{sh}x_\beta\left[S_k{}^-(\mathbf{kv}_1)-S_k{}^-(-\mathbf{kv}_2)\right]\right\}. \tag{3.4}*$$

We put here

$$v_T=(2\Theta/\mu)^{1/2}\quad(\mu=m_\alpha m_\beta/(m_\alpha+m_\beta)$$

(reduced mass) and, in addition,

$$\lambda_{\alpha\beta}=h/\mu v_T, \qquad x_\alpha=h\mathbf{kv}_1/2\Theta, \qquad x_\beta=h\mathbf{kv}_2/2\Theta;$$
$$\mathbf{v}_1=\mathbf{p}_1/m_\alpha, \qquad \mathbf{v}_2=\mathbf{p}_2/m_\beta.$$

The expressions $\epsilon_k^\pm(\omega)$ are respectively the upper or lower limits on the real $\omega$ axis for the function

$$\varepsilon_k(\omega)=1+(2\pi i/h)\left[F_+(k,\ \omega)-F_-(k,\ \omega)\right], \tag{3.5}$$

which can be regarded as the quantum dielectric constant of a multi-species plasma. Here

$$F_\pm(k,\ \omega)=\frac{1}{2\pi i}\sum_\alpha\frac{4\pi n_\alpha e_\alpha{}^2}{k^2}\int\frac{d\mathbf{p}}{\omega-\mathbf{kp}/m_\alpha}f_\alpha\left(\mathbf{p}\pm\frac{h\mathbf{k}}{2}\right). \tag{3.6}$$

The functions $\epsilon_k(\omega)$ and $F_\pm(k,\omega)$ themselves are analytic on the entire complex plane $\omega$, with the exception of the real axis, on which their discontinuities are equal to, in accordance with (3.5) and (3.6),

$$\Delta\varepsilon_k(\omega)=\varepsilon_k{}^+(\omega)-\varepsilon_k{}^-(\omega)$$
$$=(2\pi i/h)\left[\Delta F_+(k,\ \omega)-\Delta F_-(k,\ \omega)\right],$$
$$\Delta F_\pm(k,\ \omega)=F_\pm{}^+(k,\ \omega)-F_\pm{}^-(k,\ \omega)$$
$$=-\sum\frac{4\pi n_\alpha e_\alpha{}^2}{k^2}\int\delta\left(\omega-\frac{\mathbf{kp}}{m_\alpha}\right)f_\alpha\left(\mathbf{p}\pm\frac{h\mathbf{k}}{2}\right)d\mathbf{p}. \tag{3.7}$$

Finally, the function $S_k^-(\omega)$ in (3.4) is determined by the integral

$$S_k{}^-(\omega)=\frac{1}{h}\int\frac{d\omega'}{\omega'-(\omega-i\nu)}\,\frac{\Delta F_+(k,\ \omega')+\Delta F_-(k,\ \omega')}{\varepsilon_k{}^+(\omega')\,\varepsilon_k{}^-(\omega')}$$
$$(\nu\to+0), \tag{3.8}$$

the evaluation of which for the concrete case of a Maxwellian distribution is precisely our problem.

Using (3.7), we can easily verify that the following relation holds true for a Maxwellian distribution

$$\Delta F_++\Delta F_-=-\frac{h}{2\pi i}\Delta\varepsilon_k\operatorname{cth}\frac{h\omega}{2\Theta}, \tag{3.9}*$$

and therefore the integral (3.8) can be reduced to

$$S_k{}^-(\omega)=\frac{1}{2\pi i}\int\frac{\operatorname{cth}(h\omega'/2\Theta)\,d\omega'}{\varepsilon_k{}^+(\omega')\,(\omega'-\omega)}-\frac{1}{2\pi i}\int\frac{\operatorname{cth}(h\omega'/2\Theta)\,d\omega'}{\varepsilon_k{}^-(\omega')\,(\omega'-\omega)}, \tag{3.10}$$

where the integration contour circles the poles $\omega'=\omega$ and $\omega'=0$ from above. Using further the known expansion of coth x into partial fractions

$$\operatorname{cth}x=\frac{1}{x}+\sum_{n=1}^\infty\left(\frac{1}{x+i\pi n}+\frac{1}{x-i\pi n}\right) \tag{3.11}$$

and closing the integration contours by means of a semicircle of large radius of the regions of analyticity of the functions $\epsilon^+$ and $\epsilon^-$ from above in the first integral and from below in the second, we obtain

*sh = sinh, ch = cosh.

*cth = coth.

$$S_k^-(\omega)=\frac{\operatorname{cth}(h\omega/2\Theta)}{\varepsilon_k^-(\omega)}-\frac{2\Theta/h\omega}{\varepsilon_k^-(0)}$$

$$-\sum_{n=1}^{\infty}\left[\frac{1}{\varepsilon_k^-(-i\pi n\,2\Theta/h)\,(h\omega/2\Theta+i\pi n)}\right.$$

$$\left.+\frac{1}{\varepsilon_k^+(i\pi n\,2\Theta/h)\,(h\omega/2\Theta-i\pi n)}\right]. \quad (3.12)$$

We then get ultimately from (3.4)

$$\zeta_k^{\alpha\beta}(\mathbf{p}_1,\mathbf{p}_2)=-\frac{\lambda_{\alpha\beta}}{\pi^2k^2}\exp\left(-\frac{k^2\lambda_{\alpha\beta}^2}{4}\right)\operatorname{sh}x_\alpha\operatorname{sh}x_\beta\left\{\frac{1}{x_\alpha x_\beta\varepsilon_k(0)}\right.$$

$$+\sum_{n=1}^{\infty}\left[\frac{1}{\varepsilon_k^-(-i\pi n\,2\Theta/h)\,(x_\alpha+i\pi n)\,(x_\beta+i\pi n)}\right.$$

$$\left.\left.+\frac{1}{\varepsilon_k^+(i\pi n\,2\Theta/h)\,(x_\alpha-i\pi n)\,(x_\beta-i\pi n)}\right]\right\}. \quad (3.13)$$

In the case of practical interest when $\lambda \ll d$, this formula can be simplified. In fact, the denominators of the terms of the sum in (3.13) contain the functions $\epsilon_k^{\pm}(\pm i\pi n 2\Theta/h)$. Their arguments $\omega = \pm i\pi n 2\Theta/h$ correspond to very high frequencies. As is well known, at high frequencies the dielectric constants of a plasma [see (3.5)] has an approximate value $\epsilon(\omega) = 1 - (\omega_e/\omega)^2$, where $\omega_e$ —plasma electron frequency, so that these functions can be expressed in the form

$$\varepsilon_k^{\pm}(\pm i\pi n\,2\Theta/h)=1+(\omega_e h/\pi n 2\Theta)^2=1+O(\lambda_e^2/d^2) \quad (3.14)$$

($\lambda_e = h/p_e$ —quantum thermal wavelength of the electrons and d —Debye radius). Therefore, neglecting small terms of order $\lambda^2/d^2$, we can set the quantities $\epsilon_k^{\pm}(\omega)$ in (3.13) equal to unity. Using (3.11) for the sum in (3.13), we obtain

$$\sum_{n=1}^{\infty}\left[\frac{1}{(x_\alpha+i\pi n)\,(x_\beta+i\pi n)}+\frac{1}{(x_\alpha-i\pi n)\,(x_\beta-i\pi n)}\right]$$

$$=-\frac{1}{x_\alpha x_\beta}-\frac{\operatorname{cth}x_\alpha-\operatorname{cth}x_\beta}{x_\alpha-x_\beta}. \quad (3.15)$$

Then

$$\zeta_k^{\alpha\beta}(\mathbf{p}_1,\mathbf{p}_2)=-\frac{\lambda_{\alpha\beta}}{\pi^2k^2}\exp\left(\frac{-k^2\lambda_{\alpha\beta}^2}{4}\right)\left[\frac{\operatorname{sh}(x_\alpha-x_\beta)}{x_\alpha-x_\beta}\right.$$

$$\left.-\frac{\operatorname{sh}x_\alpha\operatorname{sh}x_\beta}{x_\alpha x_\beta}\left(1-\frac{1}{\varepsilon_k(0)}\right)\right]. \quad (3.16)$$

In the case of a Maxwellian distribution, we have for the quantum dielectric constant at zero frequency, in accordance with (3.5) and (3.6),

$$\varepsilon_k(0)=1+\frac{1}{k^2\Theta}\sum_\alpha 4\pi n_\alpha e_\alpha^2 g\left(\frac{k\lambda_\alpha}{2}\right), \quad (3.17)$$

where we have introduced the function

$$g(z)=e^{-z^2}\int_0^1 e^{z^2x^2}dx. \quad (3.18)$$

If the ions in a two-species plasma are treated in classical manner ($\lambda_i = 0$) then

$$\varepsilon_k(0)=1+\frac{1}{k^2d^2}\,\frac{z+g(k\lambda_e/2)}{z+1}. \quad (3.19)$$

As $h \to 0$ (in this case $x_{\alpha,\beta} \to 0$, $\lambda_\alpha \to 0$), formula (3.16) takes the form

$$\frac{e_\alpha e_\beta}{hv_T}\zeta_k^{\alpha\beta}=-\frac{e_\alpha e_\beta/\Theta}{2\pi^2k^2\varepsilon_k^{\mathrm{cl}}(0)}=-\frac{e_\alpha e_\beta}{2\pi^2\Theta}\,\frac{d^2}{1+k^2d^2}, \quad (3.20)$$

which corresponds to the classical expression (1.2).

When $kd \gg 1$, which corresponds to distances $|\mathbf{r}_1-\mathbf{r}_2|$ much shorter than d, we can put $\epsilon_k(0) = 1$ and then

$$\zeta_k^{\alpha\beta}(\mathbf{p}_1,\mathbf{p}_2)=-\frac{\lambda_{\alpha\beta}}{\pi^2k^2}\exp\left(-\frac{k^2\lambda_{\alpha\beta}}{4}\right)\frac{\operatorname{sh}(x_\alpha-x_\beta)}{x_\alpha-x_\beta}. \quad (3.21)$$

This limiting case, under which the polarization of the medium is not taken into account, can obviously be obtained from an examination of the two-body problem, in a manner similar to that used in Sec. 1 to calculate the probabilities w(r).

To obtain formula (3.21) it is necessary to consider the off-diagonal elements of the density matrix

$$\rho_2(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}_1',\mathbf{r}_2')=\mathrm{const}\cdot\sum_n e^{-E_n/\Theta}\Psi_n^*(\mathbf{r}_1,\mathbf{r}_2)\,\Psi_n(\mathbf{r}_1',\mathbf{r}_2')$$

$$=\mathrm{const}\cdot\exp\left[-\frac{\Theta M}{2h^2}(\mathbf{R}-\mathbf{R}')^2\right]$$

$$\times\sum_{\mathbf{k}}\exp\left(\frac{-\varepsilon_{\mathbf{k}}^{\mathrm{rel}}}{\Theta}\right)\psi_{\mathbf{k}}^*(\mathbf{r})\,\psi_{\mathbf{k}}(\mathbf{r}');$$

$$M=m_1+m_2,\qquad \mathbf{R}=(m_1\mathbf{r}_1+m_2\mathbf{r}_2)/M,\qquad \mathbf{r}=\mathbf{r}_1-\mathbf{r}_2. \quad (3.22)$$

Using again the Born approximation (1.7) for $\psi_k(\mathbf{r})$ and going over then to the Wigner representation in accordance with (3.1) for the spatial Fourier component of the correlation function $\zeta_{\alpha\beta}$ [see (3.3)], we obtain an expression that coincides exactly with (3.21).

In accordance with (3.2), the particle coordinate distribution probability is

$$w(r)=\int d\mathbf{p}_1d\mathbf{p}_2f_\alpha(\mathbf{p}_1)f_\beta(\mathbf{p}_2)\left[1+\frac{e_\alpha e_\beta}{hv_T}\zeta_{\alpha\beta}(\mathbf{r},\mathbf{p}_1,\mathbf{p}_2)\right]$$

$$=1+u(r). \quad (3.23)$$

Consequently, integrating (3.16) over the momenta with weight $f_\alpha(\mathbf{p}_1)f_\beta(\mathbf{p}_2)$, we obtain the Fourier component of the function u(r):

$$u_{\mathbf{k}}=\int d\mathbf{p}_1d\mathbf{p}_2f_\alpha(\mathbf{p}_1)\,f_\beta(\mathbf{p}_2)\,\frac{e_\alpha e_\beta}{hv_T}\zeta_{\mathbf{k}}^{\alpha\beta}(\mathbf{p}_1,\mathbf{p}_2)$$

$$=-\frac{e_\alpha e_\beta}{2\pi^2k^2\Theta}\left[g\left(\frac{k\lambda_{\alpha\beta}}{2}\right)-g\left(\frac{k\lambda_\alpha}{2}\right)g\left(\frac{k\lambda_\beta}{2}\right)\left(1-\frac{1}{\varepsilon_{\mathbf{k}}(0)}\right)\right]. \quad (3.24)$$

This expression, however, should coincide with the Fourier component of the function $u^{(1)}(r) = w(r) - 1$, determined by formula (1.10) [compare (1.10) with (3.23)] of Sec. 1:

$$u_{\mathbf{k}}^{(1)} = \frac{1}{8\pi^3}\int d\mathbf{r}e^{-i\mathbf{k}\mathbf{r}}\left\{-\frac{e_\alpha e_\beta}{\Theta r}e^{-r/d} + \frac{e_\alpha e_\beta}{hv_T}2\sqrt{\pi}\left[\Phi\left(\frac{r}{\lambda_{\alpha\beta}}\right)-1+\frac{\lambda_{\alpha\beta}}{r\sqrt{\pi}}\exp\left(-\frac{r^2}{\lambda_{\alpha\beta}^2}\right)\right]\right\}$$

$$= -\frac{e_\alpha e_\beta}{\Theta 2\pi^2 k^2}\left[g\left(\frac{k\lambda_{\alpha\beta}}{2}\right)-\left(1-\frac{1}{\varepsilon_k^{(\mathrm{cl})}(0)}\right)\right]. \qquad (3.25)$$

Under the conditions when $d \gg \lambda$, the difference between (3.24) and (3.25) is insignificant, for when $k\lambda \ll 1$ the functions $g(k\lambda/2)$ are close to unity, and when $k\lambda > 1$, as a result of $kd \gg k\lambda \gtrsim 1$, we can neglect the difference between $\epsilon_k(0)$ and unity.

It is obvious that we can replace with equal accuracy the general formula (3.16) for $\zeta_k^{\alpha\beta}(p_1, p_2)$ by the approximate formula

$$\widetilde{\zeta_k^{\alpha\beta}}(\mathbf{p}_1, \mathbf{p}_2) = -\frac{\lambda_{\alpha\beta}}{\pi^2 k^2}\left[\exp\left(-\frac{k^2\lambda_{\alpha\beta}^2}{4}\right)\frac{\operatorname{sh}(x_\alpha - x_\beta)}{x_\alpha - x_\beta} - \left(1-\frac{1}{\varepsilon_k^{\mathrm{cl}}(0)}\right)\right]. \qquad (3.26)$$

Then, returning to the coordinate representation for the two-particle distribution function (3.3), we get

$$F_2^{\,q}(\mathbf{r}_1-\mathbf{r}_2, \mathbf{p}_1, \mathbf{p}_2) = f_1 f_2\left[1+\frac{e_\alpha e_\beta}{hv_T}\int \widetilde{\zeta_{\mathbf{k}}^{\alpha\beta}}(\mathbf{p}_1, \mathbf{p}_2)e^{i\mathbf{k}\mathbf{r}}d\mathbf{k}\right]$$

$$= f_1 f_2\left[1-\frac{e_\alpha e_\beta}{\Theta r}e^{-r/d}+\frac{e_\alpha e_\beta}{hv_T}\widetilde{\zeta_{\alpha\beta}^{\,q}}(\mathbf{r}, \mathbf{p}_1, \mathbf{p}_2)\right], \qquad (3.27)$$

where

$$\widetilde{\zeta_{\alpha\beta}^{\,q}}(\mathbf{r}, \mathbf{p}_1, \mathbf{p}_2) = -\frac{\lambda_{\alpha\beta}}{\pi^2}\int d\mathbf{k}\frac{e^{i\mathbf{k}\mathbf{r}}}{k^2}\left[\exp\left(-\frac{k^2\lambda_{\alpha\beta}^2}{4}\right)\frac{\operatorname{sh}(x_\alpha - x_\beta)}{x_\alpha - x_\beta}-1\right]$$

$$= \lambda_{\alpha\beta}\left[\frac{2}{r}-\frac{i}{|\mathbf{v}|}\int_{p_+}^{p_-}\Phi\left(\frac{r_\perp}{\lambda_{\alpha\beta}}\operatorname{ch}p\right)dp\right]. \qquad (3.28)$$

We have put here

$$\mathbf{v} = \frac{h}{2\Theta}(\mathbf{v}_1-\mathbf{v}_2), \qquad r_\perp = \left[r^2-\left(\frac{\mathbf{v}}{v}\mathbf{r}\right)^2\right]^{1/2},$$

$$p_\pm = \cosh^{-1}\left[\frac{(\mathbf{r}\pm i\mathbf{v})^2}{r_\perp^2}\right]^{1/2}, \qquad (3.29)$$

where $\Phi(z)$ —error function.

In the limit as $r \to 0$ the integral with the error function can be left out, and the remaining term $\zeta_{\alpha\beta}^{q} = 2\lambda_{\alpha\beta}/r$ cancels the divergence of the classical Debye term in (3.27). In the other limiting case, when $r \gg \lambda$ and $r \gg \nu$, we obtain by using the asymptotic value of the error function

$$\int_{p_+}^{p_-}\Phi\left(\frac{r_\perp}{\lambda}\operatorname{ch}p\right)dp = p_- - p_+ + O\left(e^{-r^2/\lambda^2}\right), \qquad (3.30)$$

where $(p_- + p_+)_{r\gg\nu} = -2i\nu/r$, which compensates for the first term in the square brackets of (3.28).

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188