ON AN ALLOWED MODEL IN QUANTUM ELECTRODYNAMICS

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The Lee annihilation model in quantum electrodynamics is considered, in which the only allowed transitions are of the type $\gamma \rightleftarrows e^- + e^+$. The model is so constructed that it leads to a causal description of the processes. The photon state is constructed; electron-positron scattering is discussed (and certain limiting expressions for the cross section for this process are obtained). The conditions for the appearance of ghost states are investigated. The expressions for the amplitudes and cross sections of the processes thus derived can apparently be viewed as majorants of the corresponding quantities in the exact theory in the limit of high energies.

1. INTRODUCTION

In this paper we shall study a model of quantum electrodynamics, which to a large extent preserves the basic features of the exact theory such as relativistic invariance and causality and which therefore may be considered as a more or less realistic model for certain electrodynamic processes. We have in mind an annihilation model of the type of the Lee model, in which the only virtual processes are of the form $\gamma \rightleftarrows e^- + e^+$. The quantum electrodynamics Lee model in the form considered previously[1-3] cannot be ascribed real content because of absence of causality in it. For the same reason the Lee model considered by Mashida[4] and Goldstein[5] cannot be looked upon as realistic. Just like the Hamiltonian of the type (2.1) (without the operators ξ, η, ζ), this model does not lead to a causal description of processes. The reason is that in this model the causal Green's function is replaced by the positive frequency part of the retarded Green's function. The situation here is similar to that discussed in[6].

From the point of view of diagrams describing some arbitrary process (for example the one shown in Fig. 2 below) this means that in place of the Feynman diagrams, which give a causal description of the process, Heitler[7] diagrams are used (in Fig. 2 diagrams a and b). An arbitrary process in the model of[4,5] is described by a certain number of Heitler diagrams, the totality of which does not lead to a causal description. This in turn gives rise to meaningless results, such as an infinite cross section for the scattering of electrons by positrons in the c.m.s., etc.

This difficulty is removed in the present paper. To this end we introduce into the Hamiltonian (2.1) the operators ξ, η, ζ, connected respectively with the electron, positron, and photon, and whose role reduces to the reestablishing of causality in the model by taking into account the missing Heitler diagrams (the totality of the Heitler diagrams now produces some of the Feynman diagrams corresponding to an arbitrary process of the exact theory). At that the model continues to be allowed (for more details on the operators ξ, η, ζ see Appendix I).

The expressions for the amplitudes of processes and cross sections in this model, represented in the form of a perturbation theory series, may be obtained from the conventional theory by selective summation of Feynman diagrams. Thus, for example, the expression that the model gives for the photon Green's function coincides with the expression obtained by Landau[8], and the Compton effect is described by the diagrams which in the conventional theory give the largest contribution to the total cross section for this process. This correspondence shows the region of applicability of the present model. Namely it describes to some degree correctly electrodynamic processes in the region of large energies and momentum transfers. It is difficult to establish the region of applicability directly. Apparently the expressions for the amplitudes and cross sections obtained from the model may be viewed as majorants for the corresponding quantities in quantum electrodynamics.

In this paper we solve on the basis of the model the problem of obtaining the photon state vector, we find the renormalization constant Z, and we study the "ghost" states and their dependence on the cut-off constant. Electron-positron scattering is investigated. Limiting expressions for the cross section of this process are found.

2. PHOTON, RENORMALIZATION CONSTANT, "GHOSTS"

A. The model under consideration deals with processes of the type $\gamma \rightleftarrows e^- + e^+$. The Hamiltonian describing only such processes has the form[1)]

$$H = H_0 + H_i,$$

$$H_0 = \sum_{\mathbf{k},\lambda} |\mathbf{k}|\, c_\lambda^+(\mathbf{k})\, c_\lambda(\mathbf{k}) + \sum_{\mathbf{p},r} E_\mathbf{p}\left(a_r^+(\mathbf{p})\, a_r(\mathbf{p}) + b_r^+(\mathbf{p})\, b_r(\mathbf{p})\right),$$

$$H_i = ie_0 \sum_{\substack{\mathbf{p},\mathbf{k}\\ r,r',\lambda}} (2|\mathbf{k}|V)^{-1/2} \bar{\xi}_\mathbf{p} \bar{u}^r(\mathbf{p})\, \zeta_\mathbf{k} \hat{e}^\lambda(\mathbf{k})\, \eta_{\mathbf{p}-\mathbf{k}} v^{r'}(\mathbf{p}-\mathbf{k})\, a_r^+(\mathbf{p}) \times b_{r'}^+(\mathbf{k}-\mathbf{p})\, c_\lambda(\mathbf{k}) + ie_0 \sum_{\substack{\mathbf{p},\mathbf{k}\\ r,r',\lambda}} (2|\mathbf{k}|V)^{-1/2} \bar{\eta}_{\mathbf{p}-\mathbf{k}} \bar{v}^{r'} \times (\mathbf{p}-\mathbf{k})\, \bar{\zeta}_\mathbf{k} \hat{e}^\lambda(\mathbf{k})\, \xi_\mathbf{p} u^r(\mathbf{p})\, b_{r'}(\mathbf{k}-\mathbf{p})\, a_r(\mathbf{p})\, c_\lambda^+(\mathbf{k}) + \delta m^2 \sum_{\mathbf{k},\lambda} (2|\mathbf{k}|)^{-1} \bar{\zeta}_\mathbf{k} \zeta_\mathbf{k} c_\lambda^+(\mathbf{k})\, c_\lambda(\mathbf{k}). \quad (2.1)$$

Here e_0 is the unrenormalized electron charge, $E = (p^2 + m^2)^{1/2}$, $\hat{e} = \gamma_\mu e_\mu$ (γ_μ are the Dirac matrices), V is the normalization volume, and ξ, η, ζ are certain operators whose meaning will be clarified below. We use the system of units in which $\hbar = c = 1$ and a metric such that the scalar product is given by $pq = \mathbf{p}\cdot\mathbf{q} - p_0 q_0$. The Dirac matrices γ_μ are hermitian: $\gamma_\mu^+ = \gamma_\mu$. The creation and annihilation operators for the electrons, positrons and photons, a^+, a, b^+, b, c^+, c, satisfy the usual commutation relations:

$$\{a_r(\mathbf{p}),\ a_{r'}^+(\mathbf{p}')\} = \{b_r(\mathbf{p}),\ b_{r'}^+(\mathbf{p}')\} = \delta_{rr'}\delta_{\mathbf{p}\mathbf{p}'},$$

$$[c_\lambda(\mathbf{k}),\ c_{\lambda'}^+(\mathbf{k}')] = \delta_{\lambda\lambda'}\delta_{\mathbf{k}\mathbf{k}'} \qquad \{a_r(\mathbf{p}),\ b_{r'}^+(\mathbf{p}')\} = 0,$$

$$[a_r(\mathbf{p}),\ c_\lambda^+(\mathbf{k})] = [b_r(\mathbf{p}),\ c_\lambda^+(\mathbf{k})] = 0. \quad (2.2)$$

It is easy to show that the Hamiltonian H commutes with the operators

$$N_1 = \sum_{\mathbf{k},\lambda} c_\lambda^+(\mathbf{k})\, c_\lambda(\mathbf{k}) + \sum_{\mathbf{p},r} a_r^+(\mathbf{p})\, a_r(\mathbf{p}),$$

$$N_2 = \sum_{\mathbf{k},\lambda} c_\lambda^+(\mathbf{k})\, c_\lambda(\mathbf{k}) + \sum_{\mathbf{p},r} b_r^+(\mathbf{p})\, b_r(\mathbf{p}),$$

so that we have separately conservation of the number of electrons plus photons $N_\gamma + N_{e^-}$ and positrons plus photons $N_\gamma + N_{e^+}$. The conservation of these numbers has as a consequence that the Hilbert space of the state vectors of the Hamiltonian H_0 breaks up into separate sectors, as in[3]. Such sectors are, for example, $|\gamma; e^-, e^+\rangle$ and $|\gamma, e^-; 2e^-, e^+\rangle$. Only these sectors will be considered in the following.

B. We first construct in the sector $|\gamma; e^-, e^+\rangle$ the photon state vector of the total Hamiltonian H:

$$|\gamma_\lambda(k)\rangle = Z^{1/2}(k^2)\, c_\lambda^+(\mathbf{k})\,|0\rangle + \sum_{\mathbf{p},r,r'} \Phi_\lambda^{rr'}(\mathbf{p},\mathbf{k})\, a_r^+(\mathbf{p})\, b_{r'}^+(\mathbf{k}-\mathbf{p})\,|0\rangle, \quad (2.3)$$

where $|0\rangle$ is the vacuum state vector of the Hamiltonian H_0. The state vector (2.3) satisfies the Schrödinger equation[2)]

$$H|\gamma_\lambda(k)\rangle = \omega_0|\gamma_\lambda(k)\rangle. \quad (2.4)$$

Equating in Eq. (2.4) the coefficients of the rays $c_\lambda^+(\mathbf{k})|0\rangle$ and $a_r^+(\mathbf{p})b_{r'}^+(\mathbf{p}-\mathbf{k})|0\rangle$ we obtain

$$Z^{1/2}(k^2)|\mathbf{k}|\delta_{\lambda\lambda'} + ie_0 \sum_{\mathbf{p},r,r'} (2|\mathbf{k}|V)^{-1/2} \bar{\eta}\bar{v}^{r'}(\mathbf{p}-\mathbf{k})\, \bar{\zeta}\hat{e}^{\lambda'}(\mathbf{k}) \times \xi u^r(\mathbf{p})\, \Phi_\lambda^{rr'}(\mathbf{p},\mathbf{k}) + \delta m^2 Z^{1/2}(k^2)(2|\mathbf{k}|)^{-1}\bar{\zeta}\zeta\delta_{\lambda\lambda'} + \alpha k_\lambda k_{\lambda'} = \omega_0 Z^{1/2}(k^2)\,\delta_{\lambda\lambda'}, \quad (2.5a)$$

$$\Phi_\lambda^{rr'}(\mathbf{p},\mathbf{k})(E_\mathbf{p} + E_{\mathbf{k}-\mathbf{p}}) + ie_0 Z^{1/2}(k^2)(2|\mathbf{k}|V)^{-1/2}\bar{\xi}\bar{u}^r(\mathbf{p})\,\zeta\hat{e}^\lambda(\mathbf{k}) \times \eta v^{r'}(\mathbf{p}-\mathbf{k}) = \omega_0\Phi_\lambda^{rr'}(\mathbf{p},\mathbf{k}). \quad (2.5b)$$

In the following we set $e_\mu^\lambda(k) = \delta_\mu^\lambda$, hence $\hat{e}^\lambda = \gamma_\lambda$. The term of the type $\alpha k_\lambda k_{\lambda'}$ has been added[3)] in (2.5a) to make that equation valid for all λ and λ'. (The expression for α will be found below.)

From Eq. (2.5b) we obtain first

$$\Phi_\lambda^{rr'}(\mathbf{p},\mathbf{k}) = ie_0 Z^{1/2}(k^2)(\omega_0 - E_\mathbf{p} - E_{\mathbf{k}-\mathbf{p}})^{-1}(2|\mathbf{k}|V)^{-1/2}\bar{\xi}\bar{u}^r(\mathbf{p}) \times \zeta\hat{e}^\lambda(\mathbf{k})\,\eta v^{r'}(\mathbf{p}-\mathbf{k}). \quad (2.5b')$$

Substituting this expression for $\Phi_\lambda^{rr'}(p,k)$ into (2.5a) we find

$$Z^{1/2}(k^2)(|\mathbf{k}| - \omega_0)\delta_{\lambda\lambda'} - e_0^2 Z^{1/2}(k^2) \sum_{\mathbf{p},r,r'} (2|\mathbf{k}|V)^{-1/2} \times (\omega_0 - E_\mathbf{p} - E_{\mathbf{k}-\mathbf{p}})^{-1}\, \bar{\eta}\bar{v}^{r'}(\mathbf{p}-\mathbf{k})\, \bar{\zeta}\hat{e}^{\lambda'}(\mathbf{k})\, \xi u^r(\mathbf{p})\, \bar{\xi}\bar{u}^r \times (\mathbf{p})\,\zeta\hat{e}^\lambda(\mathbf{k})\,\eta v^{r'}(\mathbf{p}-\mathbf{k}) + \delta m^2 Z^{1/2}(k^2)(2|\mathbf{k}|)^{-1}\bar{\zeta}\zeta\delta_{\lambda\lambda'} + \alpha k_\lambda k_{\lambda'} = 0. \quad (2.5a')$$

In (2.5a′) we perform a summation over the polarizations making use, as is usual, of[9]

$$\sum_r \xi u^r(\mathbf{p})\,\bar{\xi}\bar{u}^r(\mathbf{p}) = -\frac{i\hat{p} - m}{2E_\mathbf{p}},$$

whereas

$$\sum_r \frac{\eta v^r(\mathbf{p}-\mathbf{k})\,\bar{\eta}\bar{v}^r(\mathbf{p}-\mathbf{k})}{\omega_0 - E_\mathbf{p} - E_{\mathbf{k}-\mathbf{p}}} = \frac{i(\hat{k} - \hat{p}) + m}{(k-p)^2 + m^2}. \quad (2.6)$$

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1)Throughout the remainder of the paper we write for simplicity $\xi u(p)$ instead of $\xi_p u(p)$.

2)We note that the one-particle electron and positron states are simultaneously eigenvectors of both the Hamiltonians H_0 and H; for this reason the mass of these particles is not renormalized.

3)This can be done in view of the Fermi condition usually imposed on the photon state vectors, which is of the form $k_\lambda c_\lambda^+(k)|0\rangle$. Let us note that Eq. (2.4) is satisfied by the solutions of Eqs. (2.5a) and (2.5b) for arbitrary α.

The last equality defines the operation $\eta\bar{\eta}$. (For more details see Appendix I.) Having this in mind we rewrite Eq. (2.5a′) as

$$Z^{1/2}(k^2)(|\mathbf{k}|-\omega_0)\,\delta_{\lambda\lambda'} + Z^{1/2}(k^2)(2|\mathbf{k}|)^{-1}\bar{\zeta}\zeta\bar{\Pi}_{\lambda\lambda'}(k^2) + \delta m^2 Z^{1/2}(k^2)(2|\mathbf{k}|)^{-1}\bar{\zeta}\zeta\delta_{\lambda\lambda'} + \alpha k_\lambda k_{\lambda'} = 0, \qquad (2.7)$$

where

$$\bar{\Pi}_{\lambda\lambda'}(k^2) = \frac{e_0^2}{V}\sum_{\mathbf{p}} \frac{\mathrm{Sp}\,\{\gamma_\lambda(i\hat{p}-m)\,\gamma_{\lambda'}(i(\hat{k}-\hat{p})+m)\}}{2E_{\mathbf{p}}((k-p)^2+m^2)} = \left(\delta_{\lambda\lambda'} - \frac{k_\lambda k_{\lambda'}}{k^2}\right)\bar{\Pi}(k^2) + A\delta_{\lambda\lambda'}, \qquad (2.8)$$

with

$$\bar{\Pi}(k^2) = \frac{1}{3}\left(\bar{\Pi}_{\lambda\lambda}(k^2) - 4\frac{k_\lambda k_{\lambda'}}{k^2}\bar{\Pi}_{\lambda\lambda'}(k^2)\right),$$

$$A = \frac{k_\lambda k_{\lambda'}}{k^2}\bar{\Pi}_{\lambda\lambda'}(k^2). \qquad (2.9)$$

For α in (2.7) we obtain

$$\alpha = Z^{1/2}(k^2)(2|\mathbf{k}|)^{-1}\bar{\Pi}(k^2). \qquad (2.10)$$

For a free photon $\omega_0 = |\mathbf{k}|$. Under that condition we get for the quantity δm^2

$$\delta m^2 = -\frac{1}{4}\bar{\Pi}_{\lambda\lambda}(0) = -\frac{e_0^2}{4V}\sum_{\mathbf{p}}\frac{\mathrm{Sp}\,\{\gamma_\lambda(i\hat{p}-m)\,\gamma_\lambda(i(\hat{k}-\hat{p})+m)\}}{2E_{\mathbf{p}}((k-p)^2+m^2)}, \quad k^2 = 0. \qquad (2.11)$$

C. Having, further, in mind that[4)]

$$\langle\gamma_\lambda(\mathbf{k})\,|\,\gamma_{\lambda'}(\mathbf{k}')\rangle = \delta_{\lambda\lambda'}\delta_{\mathbf{kk}'} \quad (k^2=0), \qquad (2.12)$$

we find

$$\delta_{\lambda\lambda'} = Z\delta_{\lambda\lambda'} + \sum_{\mathbf{p},r,r'}\Phi_\lambda^{rr'*}(\mathbf{p},\mathbf{k})\,\Phi_{\lambda'}^{rr'}(\mathbf{p},\mathbf{k}) + \beta k_\lambda k_{\lambda'} \qquad (2.13)$$

(the term $\beta k_\lambda k_{\lambda'}$ may be added in view of the Fermi condition), where $Z = Z(k^2 = 0)$. The renormalization constant Z appears in the definition of the renormalized charge in the form

$$Z^{1/2}e_0 = e. \qquad (2.14)$$

Consequently (2.13) gives for the renormalization constant the expression

$$Z = 1 + \frac{e^2}{24|\mathbf{k}|}\frac{1}{V} \times \sum_{\mathbf{p},r,r'}\frac{\bar{\eta}\bar{v}^{r'}(\mathbf{p}-\mathbf{k})\,\gamma_\lambda\xi u^r(\mathbf{p})\,\bar{\xi}\bar{u}^r(\mathbf{p})\,\gamma_\lambda\eta v^{r'}(\mathbf{p}-\mathbf{k})}{(|\mathbf{k}|-E_{\mathbf{p}}-E_{\mathbf{p}-\mathbf{k}})^2}\,\zeta\bar{\zeta}. \qquad (2.15)$$

Performing in (2.15) the summation over the polarizations, keeping in mind the equality (2.6) and the fact that

$$\bar{\zeta}_{\mathbf{k}}\zeta_{\mathbf{k}}/|\mathbf{k}|(|\mathbf{k}|-E_{\mathbf{p}}-E_{\mathbf{k}-\mathbf{p}}) = 2/(k^2-(E_{\mathbf{p}}+E_{\mathbf{k}-\mathbf{p}})^2) \qquad (2.16)$$

(see Appendix I), and going over in (2.15) to integration over the four-dimensional volume, we obtain for Z the manifestly relativistically invariant expression:[5)]

$$Z = 1 - \frac{2e^2}{3i(2\pi)^4} \times \int d^4p\,\frac{2m^2+p^2-kp}{(p^2+m^2)((k-p)^2+m^2)(\mathbf{k}^2-(E_{\mathbf{p}}+E_{\mathbf{k}-\mathbf{p}})^2)}. \qquad (2.15a)$$

D. Taking into account (2.14) and the fact that

$$\bar{\zeta}_{\mathbf{k}}\zeta_{\mathbf{k}}/|\mathbf{k}|(|\mathbf{k}|-\omega_0) = 2/k^2,$$

one can express Eq. (2.7) in the form

$$h(k^2) = k^2Z + \Pi(k^2) + \frac{k_\lambda k_{\lambda'}}{k^2}\Pi_{\lambda\lambda'}(k^2) - \frac{1}{4}\Pi_{\lambda\lambda}(0) = 0,$$

$$\Pi_{\lambda\lambda'} = Z\bar{\Pi}_{\lambda\lambda'}. \qquad (2.17)$$

The zeros of the function $h(k^2)$ introduced in this manner give the mass spectrum of the states in the model under consideration. The explicit form of this function is

$$h(k^2) = k^2\{1-(e^2/4\pi^2)I(k^2)\}, \qquad (2.18)$$

where[11]

$$I(k^2) = -\left(\frac{5}{9}-\frac{4}{3}\frac{m^2}{k^2}\right) + \frac{2}{3}\left(1-\frac{2m^2}{k^2}\right)\left(1+\frac{4m^2}{k^2}\right)^{1/2} \times \coth^{-1}\left(1+\frac{4m^2}{k^2}\right)^{-1/2} - \theta(-k^2-4m^2) \times \frac{i\pi}{3}\left(1-\frac{2m^2}{k^2}\right)\left(1+\frac{4m^2}{k^2}\right)^{1/2}. \qquad (2.19)$$

This expression was obtained in the limit of no regularization, which corresponds to the cut-off parameter $M = \infty$ (we regularize all expressions using the Pauli-Villars method[11]). With regularization we have $h = h(k^2, M^2)$.

Along with the photon state ($k^2 = 0$) and the continuous spectrum of the electron-positron pair ($k^2 \le -4m^2$, for these values $h(k^2)$ is a complex function) Eq. (2.18) has a root for $k^2 \approx m^2 \times \exp(24\pi^2/e^2)$ (in this connection see Appendix II). This is the so called "ghost" state. If the cut-off parameter $M^2 < m^2\exp(24\pi^2/e^2)$ then the equation $h(k^2, M^2) = 0$ does not have this root. The root $k_g^2 \approx m^2\exp(24\pi^2/e^2)$ appears when $M^2 \gtrsim m^2 \times \exp(24\pi^2/e^2)$, and as M is increased further de-

4)In quantizing the electromagnetic field use was made of the indefinite metric.[9,10]

5)In (2.15a) it is understood that the poles are to be handled according to the Feyman prescription (the last factor in the denominator is independent of p_0). Integration of (2.15a) over p_0 yields expression (2.15).

pends on M very weakly. As in[2] it can be shown that the norm of the ghost state is negative. This is also connected with the fact that for $M^2 > m^2 \times \exp(24\pi^2/e^2)$ the renormalization constant $Z < 0$. As $M \to \infty$ the quantity $Z \to -\infty$.

It is interesting to note the dependence of the position k_g^2 of the ghost on the size of the coupling constant e. In the limit e = 0 the ghost state goes off to infinity: $k_g^2 \to \infty$; for $e \to \infty$ the quantity $k_g^2 \to 0$. Figure 1 is a graph of $\mathrm{Re}\, h(k^2)$ for e = 0, e ≠ 0 and e = ∞.

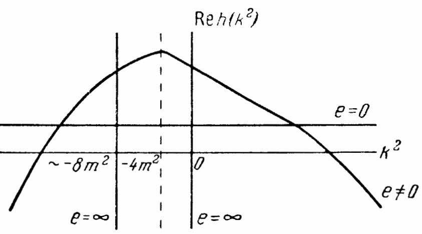

FIG. 1

3. ELECTRON-POSITRON SCATTERING

A. The state vector of an electron-positron pair with momenta p and p′ and spin variables r and r′ is constructed from the vectors of the sector $|\gamma; e^-, e^+\rangle$ in the following manner:

$$|\mathbf{e}_r^-(\mathbf{p}),\ \mathbf{e}_{r'}^+(\mathbf{p}')\rangle = a_r^+(\mathbf{p})\, b_{r'}^+(\mathbf{p}')\,|0\rangle + \sum_{\mathbf{p}, s, s'} N_{rr'}^{ss'}(\mathbf{p}, \mathbf{k}, \mathbf{p}_1)\, a_s^+(\mathbf{p}_1)\, b_{s'}^+(\mathbf{k}-\mathbf{p}_1)\,|0\rangle + Z^{1/2}\sum_\lambda \varphi_{rr'}^\lambda(\mathbf{p}, \mathbf{k})\, c_\lambda^+(\mathbf{k})\,|0\rangle, \qquad (3.1)$$

where $\mathbf{k} = \mathbf{p} + \mathbf{p}'$. The vector (3.1) satisfies the Schrödinger equation

$$H\,|\mathbf{e}_r^-(\mathbf{p})\, \mathbf{e}_{r'}^+(\mathbf{p}')\rangle = (E_\mathbf{p} + E_{\mathbf{p}'})\,|\mathbf{e}_r^-(\mathbf{p})\, \mathbf{e}_{r'}^+(\mathbf{p}')\rangle. \qquad (3.2)$$

Equating in Eq. (3.2) the coefficients of the rays $a_r^+(\mathrm{p})\, b_{r'}^+(\mathrm{p}')|\,0\rangle$ and $c_\lambda^+(\mathrm{k})|\,0\rangle$ we obtain

$$N_{rr'}^{ss'}(\mathbf{p}, \mathbf{k}; \mathbf{p}_1) = ie\,(2\,|\mathbf{k}\,|V)^{-1/2}\sum_\lambda \frac{\bar{\xi}\bar{u}^s(\mathbf{p}_1)\,\zeta\hat{e}^\lambda(\mathbf{k})\,\eta v^{s'}(\mathbf{p}_1-\mathbf{k})}{(E_\mathbf{p} + E_{\mathbf{k}-\mathbf{p}} - E_{\mathbf{p}_1} - E_{\mathbf{k}-\mathbf{p}_1})} \times \varphi_{rr'}^\lambda(\mathbf{p}, \mathbf{k}) \qquad (3.3a)$$

and

$$Z\,(E_\mathbf{p} + E_{\mathbf{k}-\mathbf{p}} - |\,\mathbf{k}\,| - (2\,|\,\mathbf{k}\,|)^{-1}\delta m^2\bar{\zeta}\zeta)\,\varphi_{rr'}^{\lambda'}(\mathbf{p}, \mathbf{k})\,\delta_{\lambda\lambda'} - ie\sum_{\mathbf{p}_1, s, s'}(2\,|\,\mathbf{k}\,|\,V)^{-1/2}\bar{\eta}\bar{v}^{s'}(\mathbf{p}_1-\mathbf{k})\,\bar{\zeta}\hat{e}^{\lambda}\xi u^s(\mathbf{p}_1) \times N_{rr'}^{ss'}(\mathbf{p}, \mathbf{k}; \mathbf{p}_1) = ie\,(2\,|\,\mathbf{k}\,|\,V)^{-1/2}\bar{\eta}\bar{v}^{r'}(\mathbf{p}-\mathbf{k})\,\bar{\zeta}\hat{e}^{\lambda'}(\mathbf{k}) \times \xi u^r(\mathbf{p})\,\delta_{\lambda\lambda'} + \alpha k_\lambda k_{\lambda'}\varphi_{rr'}^{\lambda'}(\mathbf{p}, \mathbf{k}). \qquad (3.3b)$$

The term $\alpha k_\lambda k_{\lambda'}\varphi_{rr'}^{\lambda'}$ may be added in view of the Fermi condition (the α here is the same as in (2.5a)).

Substituting (3.3a) in (3.3b) we obtain the equation for $\varphi_{rr'}^\lambda$:

$$Z\,(E_\mathbf{p} + E_{\mathbf{k}-\mathbf{p}} - |\,\mathbf{k}\,| - \delta m^2\,(2\,|\,\mathbf{k}\,|)^{-1}\bar{\zeta}\zeta)\,\varphi_{rr'}^{\lambda'}(\mathbf{p}, \mathbf{k})\,\delta_{\lambda\lambda'} - (2\,|\,\mathbf{k}\,|)^{-1}\,\Pi_{\lambda\lambda'}(k^2)\,\varphi_{rr'}^{\lambda'}(\mathbf{p}, \mathbf{k}) - \alpha k_\lambda k_{\lambda'}\varphi_{rr'}^{\lambda'}(\mathbf{p}, \mathbf{k}) = ie\,(2\,|\,\mathbf{k}\,|V)^{-1/2}\bar{\eta}\bar{v}^{r'}(\mathbf{p}-\mathbf{k})\,\bar{\zeta}\hat{e}^{\lambda'}(\mathbf{k})\,\xi u^r(\mathbf{p})\,\delta_{\lambda\lambda'} \qquad (3.4)$$

($k = p + p'$), where $\Pi_{\lambda\lambda'}(k^2)$ has the form (2.17). Further, keeping (2.16) in mind we rewrite (3.4) in the form

$$h(k^2)\,\varphi_{rr'}^\lambda(\mathbf{p}, \mathbf{k}) = ie\,(2\,|\,\mathbf{k}\,|\,V)^{-1/2}k^2\,(|\,\mathbf{k}\,| - E_\mathbf{p} - E_{\mathbf{k}-\mathbf{p}})^{-1}\bar{\eta}\bar{v}^{r'} \times (\mathbf{p}-\mathbf{k})\,\bar{\zeta}\hat{e}^\lambda(\mathbf{k})\,\xi u^r(\mathbf{p}), \qquad (3.5)$$

where $h(k^2)$ is given by Eq. (2.17). Substituting the expression for $\varphi_{rr'}^\lambda$ from (3.5) into (3.3a) we find

$$N_{rr'}^{ss'}(\mathbf{p}, \mathbf{k};\ \mathbf{p}_1) = -\frac{e^2}{V}\sum_\lambda \frac{\bar{u}^s(\mathbf{p}_1)\,\hat{e}^\lambda(\mathbf{k})\,v^{s'}(\mathbf{p}_1-\mathbf{k})\,\bar{v}^{r'}(\mathbf{p}-\mathbf{k})\,\hat{e}^\lambda(\mathbf{k})\,u^r(\mathbf{p})}{h(k^2)\,(E_\mathbf{p} + E_{\mathbf{k}-\mathbf{p}} - E_{\mathbf{p}_1} - E_{\mathbf{k}-\mathbf{p}_1})}. \qquad (3.6)$$

B. The quantity $N_{rr'}^{ss'}$ obtained in this manner is related to the matrix element for electron-positron scattering by

$$\langle \mathbf{p}, r;\ \mathbf{p}', r'\,|\,S\,|\,\mathbf{q}, s;\ \mathbf{q}', s'\rangle = \langle 0\,|\,a_r(\mathbf{p})\,b_{r'}(\mathbf{p}')\,|\,\mathbf{e}_s^-(\mathbf{q})\,\mathbf{e}_{s'}^+(\mathbf{q}')\rangle = \delta_{rs}\delta_{r's'}\delta_{\mathbf{p}\mathbf{q}}\delta_{\mathbf{p}'\mathbf{q}'} + N_{rr'}^{ss'}(\mathbf{p}, \mathbf{k};\ \mathbf{q})\,\delta_{\mathbf{p}+\mathbf{p}', \mathbf{q}+\mathbf{q}'} \qquad (3.7)$$

(S—the scattering matrix). We then have for the differential cross section

$$d\sigma = (2\pi)^{-6}|M|^2\delta^4(p + p' - q - q')\,d\mathbf{q}\,d\mathbf{q}', \qquad (3.8)$$

where

$$M = -i(2\pi)^2e^2\sum_\lambda h^{-1}(k^2)\,\bar{u}^s(\mathbf{q})\,\hat{e}^\lambda\bar{v}^{s'}(-\mathbf{q}')\,\bar{v}^{r'}(-\mathbf{p}')\,\hat{e}^\lambda u^r(\mathbf{p}). \qquad (3.9)$$

Averaged over the polarizations of the electron and positron the cross section has in the high energy limit E ≫ m the form (in the c.m.s.)

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{(2\pi)^2}\left|\sum_\lambda \bar{u}^s(\mathbf{q})\,\hat{e}^\lambda v^{s'}(-\mathbf{q}')\,\bar{v}^{r'}(-\mathbf{p}')\,\hat{e}^\lambda u^r(\mathbf{p})\right|^2 \times E^2\,[(\mathrm{Re}\,h(k^2))^2 + (\mathrm{Im}\,h(k^2))^2]^{-1}. \qquad (3.10)$$

If the cut-off parameter $M^2 < m^2\exp(24\pi^2/e^2)$, then in the limit $E \to \infty$

$$d\sigma\,/\,d\Omega \sim e^4E^{-2}. \qquad (3.11)$$

If $M^2 > m^2\exp(24\pi^2/e^2)$, then for $E^2 \ll m^2 \times \exp(24\pi^2/e^2)$ the cross section has the form (3.11), and for $E^2 \gg m^2\exp(24\pi^2/e^2)$

$$d\sigma\,/\,d\Omega \sim E^{-2}\ln^{-2}(E\,/\,m). \qquad (3.12)$$

In that case the cross section is independent of the coupling constant e. Such a behavior is due to the presence in the theory of the ghost state. In the neighborhood of $E^2 \sim m^2\exp(24\pi^2/e^2)$ the cross section (3.10) has a resonant character:

$$\frac{d\sigma}{d\Omega} \sim e^4 E^{-2}\left[\left(1-\frac{e^2}{24\pi^2}\ln\left(\frac{E}{m}\right)^2\right)^2+\frac{\pi^2}{9}e^4\right]^{-1}. \quad (3.13)$$

In conclusion the authors express their gratitude to Professor A. I. Akhiezer for discussion of the problems here considered.

APPENDIX I

Let us clarify the meaning of the operations ξ, η, ζ. First of all, the spinor amplitudes $u^r(-\mathbf{p}+\mathbf{k})$ and $v^r(\mathbf{p}-\mathbf{k})$ satisfy the Dirac equation for free particles, with

$$\sum_{r=1,2} v^r(\mathbf{p}-\mathbf{k})\bar{v}^r(\mathbf{p}-\mathbf{k}) = -\frac{i\gamma(\mathbf{k}-\mathbf{p})-\gamma_4 E_{\mathbf{k}-\mathbf{p}}+m}{2E_{\mathbf{k}-\mathbf{p}}},$$

$$\sum_{r=1,2} u^r(-\mathbf{p}+\mathbf{k})\bar{u}^r(-\mathbf{p}+\mathbf{k}) = \frac{i\gamma(\mathbf{k}-\mathbf{p})-\gamma_4 E_{\mathbf{k}-\mathbf{p}}+m}{2E_{\mathbf{k}-\mathbf{p}}},$$

where $E_{\mathbf{k}-\mathbf{p}} = [(\mathbf{k}-\mathbf{p})^2+m^2]^{1/2}$, so that

$$\sum_{r=1,2}(u^r\bar{u}^r+v^r\bar{v}^r)=\gamma_4. \quad (I.1)$$

We do not define the operations η and $\bar{\eta}$ separately. But the operation $\eta\bar{\eta}$ is defined in the following manner:

$$\sum_{r=1,2}\eta_{\mathbf{p}-\mathbf{k}}\bar{\eta}_{\mathbf{p}-\mathbf{k}}\frac{v_\alpha^r(\mathbf{p}-\mathbf{k})\bar{v}_\beta^r(\mathbf{p}-\mathbf{k})}{(|\mathbf{k}|-E_\mathbf{p}-E_{\mathbf{k}-\mathbf{p}})}$$

$$=\sum_{r=1,2}\left(\frac{v_\alpha^r(\mathbf{p}-\mathbf{k})\bar{v}_\beta^r(\mathbf{p}-\mathbf{k})}{|\mathbf{k}|-E_\mathbf{p}-E_{\mathbf{k}-\mathbf{p}}}+\frac{u_\alpha^r(-\mathbf{p}+\mathbf{k})\bar{u}_\beta^r(-\mathbf{p}+\mathbf{k})}{|\mathbf{k}|-E_\mathbf{p}+E_{\mathbf{k}-\mathbf{p}}}\right)=\left[\frac{i(\hat{k}-\hat{p})+m}{(k-p)^2+m^2}\right]_{\alpha\beta} \quad (I.2)$$

[where we make use of the relations (I.1)]. This operation, consequently, symmetrizes the Heitler propagator, converting it into the Feynman propagator. In this manner causality is restored. In the case of the Compton effect the operation $\eta\bar{\eta}$ means taking into account along with the Heitler diagram a in Fig. 2 also the diagram b. Together these diagrams give the Feynman diagram c.

The operation $\bar{\xi}\xi$ has a similar meaning:

$$\bar{\zeta}_\mathbf{k}\zeta_\mathbf{k}\frac{1}{|\mathbf{k}|(|\mathbf{k}|-\omega_0)}=\frac{1}{|\mathbf{k}|(|\mathbf{k}|-\omega_0)}+\frac{1}{-|\mathbf{k}|(-|\mathbf{k}|-\omega_0)}=\frac{2}{k^2}, \qquad k^2=\mathbf{k}^2-\omega_0^2 \quad (I.3)$$

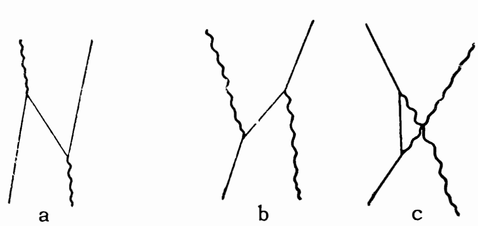

FIG. 2

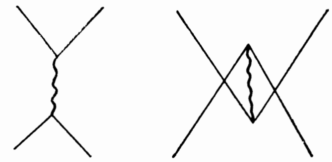

FIG. 3

In the case of electron-positron scattering this operation means taking along with the diagram a of Fig. 3 also the diagram b.

APPENDIX II

We give the expression for the function $\Pi_{\lambda\lambda'}(k^2, M^2)$ regularized according to the Pauli-Villars method[11]:

$$\Pi_{\lambda\lambda'}(k^2,M^2)=\frac{ie^2}{8\pi^2}\int_0^1 dx\,F_{\lambda\lambda'}(k^2,M^2;x),$$

where

$$F_{\lambda\lambda'}(k^2,M^2;x)=[2ix(1-x)(k_\lambda k_{\lambda'}-\delta_{\lambda\lambda'}k^2)+im^2\delta_{\lambda\lambda'}]\left\{\ln\left|\frac{(1-x)M^2+xm^2-x(1-x)k^2}{m^2-x(1-x)k^2}\right|+\ln\left|\frac{xM^2+(1-x)m^2-x(1-x)k^2}{M^2-x(1-x)k^2}\right|\right\}$$

$$-2i\delta_{\lambda\lambda'}x\left\{m^2\ln\left|\frac{(1-x)M^2+xm^2-x(1-x)k^2}{m^2-x(1-x)k^2}\right|-M^2\ln\left|\frac{M^2-x(1-x)k^2}{xM^2+(1-x)m^2-x(1-x)k^2}\right|\right\}. \quad (II.1)$$

In order to obtain an explicit expression for the regularized renormalization constant $Z(M^2)$ we make use of the known relation between it and $\Pi(k^2, M^2)$[9]:

$$Z(M^2)=1-\partial\Pi(k^2,M^2)/\partial k^2|_{k^2=0}.$$

So that

$$Z(M^2)=1-\frac{e^2}{24\pi^2}\ln\frac{M^2}{m^2}+\frac{5e^2}{72\pi^2}, \quad Z=\lim_{M\to\infty}Z(M^2). \quad (II.2)$$

Then

$$\frac{h(k^2,M^2)}{k^2}=1-\frac{e^2}{24\pi^2}\ln\frac{M^2}{m^2}+\frac{5e^2}{72\pi^2}$$

$$-\frac{e^2}{4\pi^2}\int_0^1 x(1-x)\,dx\left\{\ln\left|\frac{m^2-(1-x)k^2}{(1-x)M^2+xm^2-x(1-x)k^2}\right|+\ln\left|\frac{M^2-x(1-x)k^2}{xM^2+(1-x)m^2-x(1-x)k^2}\right|\right\}$$

$$+\frac{e^2}{4\pi^2}\frac{m^2}{k^2}\int_0^1 dx\sum_i c_i(x)\times\ln\left|\frac{\alpha_i(x)k^2+\beta_i(x)m^2+\gamma_i(x)M^2}{\delta_i(x)k^2+\rho_i(x)m^2+\sigma_i(x)M^2}\right|, \quad (II.3)$$

where $c_i(x)$, $\alpha_i(x)$, ..., $\sigma_i(x)$ are functions of the type x, $1-x$, $x(1-x)$. The function (2.17) is given by

$$h(k^2) = \lim_{M\to\infty} h(k^2, M^2).$$

We study now the dependence of the position k_g^2 of the ghost on the cut-off constant M. For $k^2 > -4m^2$ and $M^2 \gg k_g^2$ the equation

$$h(k^2, M^2)\ /\ k^2 = 0 \qquad \text{(II.4)}$$

takes the form

$$1 - \frac{e^2}{24\pi^2}\ln\frac{k^2}{m^2} + O_1\left(\frac{m^2}{k^2}\right) = 0 \quad (k^2 \gg m^2), \qquad \text{(II.5)}$$

so that a root of this equation is $k_g^2 \approx m^2 \times \exp(24\pi^2/e^2) \gg m^2$. For $M^2 \sim k^2$ the equation (II.4) assumes the form

$$1 - \frac{e^2}{24\pi^2}\ln\frac{k^2}{m^2} + \frac{e^2}{3\pi^2}\left(\frac{29}{24} - \ln 2\right) + O_2\left(\frac{m^2}{k^2}\right) = 0 \qquad \text{(II.6)}$$

and the root of this equation is $k_g^2 \approx m^2 \times \exp(24\pi^2(1+\epsilon)/e^2)$, $\epsilon > 0$. If $M^2 \ll k^2$ then Eq. (II.4) goes into

$$1 - \frac{e^2}{24\pi^2}\ln\frac{M^2}{m^2} + \frac{5e^2}{72\pi^2} + O_3\left(\frac{m^2}{k^2}\right) = 0, \qquad \text{(II.7)}$$

which for $M^2 < m^2 \exp(24\pi^2/e^2)$ has no root in the region $k^2 > 0$. For $M^2 > m^2 \exp(24\pi^2/e^2)$ Eq. (II.7) has no root satisfying the condition $k_g^2 \gg M^2$. Thus the "ghost" appears only for a cut-off constant $M^2 \gtrsim m^2 \exp(24\pi^2/e^2)$.

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