STATES OF ZERO-MASS FIELDS WITH NON-SCALAR PHASES

R. V. SMIRNOV

Submitted to JETP editor March 14, 1964

J. Exptl. Theoret. Phys. (U.S.S.R.) 47, 1386-1388 (October, 1964)

Three- and four-component phases are introduced which transform respectively like three-dimensional vectors and like four-component spinors. It is shown that solutions of the equations for zero-mass fields can be constructed on the basis of these phases, and that such solutions, like the solutions with scalar phases, are independent of the choice of reference system.

In the attempt to construct a theory of quaternion functions of a quaternion variable in analogy with the theory of complex functions it is natural to take as a foundation a Cauchy theorem, which is easily formulated for this case. On this basis Fueter[1] has constructed a theory of quaternion functions which depend on a quaternion associated with a four-dimensional Euclidean space. If, on the other hand, we use as the variable a quaternion associated with the space of events,

$$X = x_\alpha e_\alpha + ix_4, \qquad \alpha = 1, 2, 3, \tag{1}$$

where e_α are quaternion units with the well known multiplication law,[2] i is the ordinary imaginary unit which commutes with all of the e_α, and x_α are the coordinates of the space, x_4 being time (the speed of light c = 1), then the differential conditions that are equivalent to the quaternion analog of Cauchy's theorem are precisely the equations given in[3] for the fields that make up a quadruplet of zerons. This is the connection between the quadruplet type of particle systematics and the geometrical structure of the space of events of the special theory of relativity.

Quaternions are also a convenient instrument for constructing solutions of the system of equations (1) of[3]. Let $\overline{X}$ be the quaternion obtained from X by replacing i by −i, and let

$$K = k_\alpha e_\alpha + ik_4 \tag{2}$$

be a quaternion with fixed components. With the notation

$$a_4 = -x_1k_1 - x_2k_2 - x_3k_3 + x_4k_4,$$
$$-a_1 = ix_1k_4 - x_2k_3 + x_3k_2 - ix_4k_1 (\text{cyclic in } 1, 2, 3), \tag{3}$$

we can write the product of the quaternions K and $\overline{X}$ in the form

$$X\overline{K} = a_\alpha e_\alpha + a_4. \tag{4}$$

We shall call the set of components of the quaternion (4) the phase.

On the basis of the components of the phase we can construct a solution of the system (1) of[3]. Let C_l be constants, in general complex, and let $f(a_m)$ be a scalar analytic function of the complex argument a_m. That it be analytic is an important requirement, since it assures that for any m the derivative of $f(a_m)$ with respect to its argument exists.

We set

$$y_l = C_l f(a_m), \qquad l, m = 1, 2, 3, 4. \tag{5}$$

Then, as can be easily verified, the set of components (y_1, y_2, y_3, y_4) defined by (5) will for each fixed value of m form a solution of the system in question if the following conditions are satisfied:

$$k_1^2 + k_2^2 + k_3^2 - k_4^2 = 0, \tag{6}$$

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = (-a + \sigma_y b)\begin{pmatrix} C_3 \\ C_4 \end{pmatrix}, \tag{7}$$

where

$$a = \frac{k_1k_3 - ik_2k_4}{k_1^2 + k_2^2}, \qquad b = \frac{-k_1k_4 - ik_2k_3}{k_1^2 + k_2^2},$$

and σ_y is a Pauli matrix. We note that the conditions (6) and (7) do not depend on the choice of the phase component a_m in (5).

A well known solution of this type is the solution with scalar phase, obtained from (5) for m = 4. This solution has the obvious property of invariance: a Lorentz transformation in the space of events does not change the form of the conditions (6) and (7), and C_l and k_l are merely replaced by corresponding quantities with primes. It is understood that k_l and x_l have identical transformation properties.

With a view to the Lorentz invariance and linearity of the system of equations in which we are interested, we can easily perceive that the solution in the form (5) will be invariant in the sense just indicated if the components of the phase transform according to representations of the Lorentz group. Accordingly the problem arises of finding suitable representations of the Lorentz group for the transformation of the components k_l so that the phase components a_m will in turn define a representation. We see at once that the situation that arises in connection with this problem is the same as that on which the quadruplet systematics is based[3]: there exist two ways of securing the invariance of the solutions, and the scalar phase is included in a quadruplet of phases analogous to the quadruplet of zeron fields. The question remains open, however, as to whether all existing possibilities are thus exhausted.

Let us consider the special case in which the components of the quaternion (1) transform according to a two-parameter Lorentz matrix.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \cos\varphi_3 & -\sin\varphi_3 & 0 & 0 \\ \sin\varphi_3 & \cos\varphi_3 & 0 & 0 \\ 0 & 0 & \operatorname{ch}\varphi_6 & \operatorname{sh}\varphi_6 \\ 0 & 0 & \operatorname{sh}\varphi_6 & \operatorname{ch}\varphi_6 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ x_2' \\ x_4' \end{pmatrix}. \quad (8)^*$$

The representation for the transformation of the components k_l can be chosen to suit our purpose in two ways.

1. The components k_l transform like x_l. In this case a_4 is a scalar, and the remaining components transform with a complex rotation matrix:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} \cos\alpha_3^* & -\sin\alpha_3^* & 0 \\ \sin\alpha_3^* & \cos\alpha_3^* & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1' \\ a_2' \\ a_3' \end{pmatrix}, \quad (9)$$

where $\alpha_3^* = \varphi_3 - i\varphi_6$. In this case the k_l are interpreted as the components of the wave vector or the momentum vector, and the vector phase is the complex momentum, whose real and imaginary parts have well known meanings. Unlike a de Broglie wave (scalar phase), a solution with a component of the vector phase describes the propagation of a particle with constant projections of the velocity and the angular momentum along a fixed direction.

2. Along with the transformation (8) the components k_l are transformed with the following complex Lorentz matrix:

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = \begin{pmatrix} \cos\alpha_3/2 & -\sin\alpha_3/2 & 0 & 0 \\ \sin\alpha_3/2 & \cos\alpha_3/2 & 0 & 0 \\ 0 & 0 & \cos\alpha_3/2 & -i\sin\alpha_3/2 \\ 0 & 0 & -i\sin\alpha_3/2 & \cos\alpha_3/2 \end{pmatrix} \begin{pmatrix} k_1' \\ k_2' \\ k_3' \\ k_4' \end{pmatrix}. \quad (10)$$

Then the components a_l of the phase transform like a four-component spinor:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} \cos\alpha_3^*/2 & -\sin\alpha_3^*/2 & 0 & 0 \\ \sin\alpha_3^*/2 & \sin\alpha_3^*/2 & 0 & 0 \\ 0 & 0 & \cos\alpha_3^*/2 & \sin\alpha_3^*/2 \\ 0 & 0 & -\sin\alpha_3^*/2 & \cos\alpha_3^*/2 \end{pmatrix} \begin{pmatrix} a_1' \\ a_2' \\ a_3' \\ a_4' \end{pmatrix} \quad (11)$$

*sh = sinh, ch = cosh.

and the quantities

$$\frac{1}{\sqrt{2}}\begin{pmatrix} a_3 + ia_4 \\ a_1 + ia_2 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} -a_1 + ia_2 \\ a_3 - ia_4 \end{pmatrix} \quad (12)$$

are two-component spinors. In this case, however, it is not clear how the components k_l and the phase components a_l are to be interpreted physically.

We remark that the consideration of the special case (8) does not restrict the generality of the conclusions.

An example of a noninvariant solution is the solution (5) already given, if we assume that the k_l are not transformed at all. Another example is a solution of the form of (5), but constructed on the basis of the components of the product KX of the quaternions (2) and (1). If we assume that in this case, along with a Lorentz transformation in the space of events, the components k_l transform in accordance with the two hypotheses indicated above, then such solutions will be invariant under space rotations, but not under Lorentz transformations.

The identity which we have noted in the invariance properties of the solutions with scalar, vector, and spinor phases gives reason to suppose that the invariant solutions with nonscalar phases, like those with scalar phases, describe possible states of zero-mass fields.

If in the spirit of this hypothesis we use the solution with the component a_1 of the vector phase in the theory of the two-component neutrino, then the helicity operator is replaced by the operator

$$H_1 = [i(k_2\sigma_3 - k_3\sigma_2) - k_4\sigma_1] / k_1. \quad (13)$$

The components of the vector and spinor phases can also be used for the construction of invariant solutions of the Klein-Gordon equation, but in this case the only suitable form for $f(a_m)$ is $\exp(ia_m)$.

The author is deeply grateful to V. A. Yakubovich for interesting discussions.

[1] R. Fueter, Comment. Math. Helv. 7, 307 (1935).

[2] B. L. van der Waerden, Moderne Algebra, Vol. 2, Berlin, Springer, 1931.

[3] R. V. Smirnov, JETP 46, 1637 (1964), Soviet Phys. JETP 19, 1107 (1964).

Translated by W. H. Furry
198