SOVIET PHYSICS JETP VOLUME 20, NUMBER 4 APRIL, 1965

# QUASILINEAR THEORY OF INSTABILITIES CAUSED BY INJECTION OF AN ELECTRON BEAM INTO A SEMI-INFINITE PLASMA

YA. B. FAĬNBERG and V. D. SHAPIRO

Physico-technical Institute, Academy of Sciences, Ukrainian S.S.R.

Submitted to JETP editor March 21, 1964

J. Exptl. Theoret. Phys. (U.S.S.R.) 47, 1389-1404 (October 1964)

We have investigated the spatial structure of the electric field associated with oscillations produced when a beam of fast electrons is injected into a semi-infinite plasma; the directed velocity of the beam electrons is much greater than the thermal velocity of the plasma electrons. It is shown that under these conditions the energy lost by the beam in the excitation of oscillations is accumulated in a narrow surface layer at the boundary of the plasma, in which the relaxation of the beam occurs. The energy density of the oscillations in this layer can be appreciably greater than the energy density in the beam.

1. The theory of two-stream instabilities available at the present time[1-8] explains a number of results that have been observed in experiments on beam-plasma interactions: these include the oscillation excitation conditions, the frequency spectra, the growth rates, and the spreading of the beam distribution function that accompanies the instability (in the case of a low-density beam this spreading results in the formation of a plateau on the distribution function). However certain experimental results such as the spatial structure of electric fields associated with oscillations produced by injection of a beam into a plasma cannot be interpreted within the framework of existing theory. This is the case because in the work cited above primary attention has been given to the development of instabilities in an infinite plasma resulting from "single-shot" injection of a beam of charged particles into the plasma. Inasmuch as the excitation of the oscillations in this case is due to only one group of particles the energy density of the oscillations at the saturation level cannot be greater than the energy density in the beam. Furthermore, the symmetry of the problem implies that the spatial distribution of the oscillations excited by the beam must be uniform. However, in actual investigations of the excitation of plasma oscillations by injection of a beam of fast electrons into a plasma[9,10] there has been observed a highly inhomogeneous distribution of electric fields: the fields are concentrated in a narrow layer (width of several centimeters) at the plasma boundary.

In the present work, in order to explain these experimental results we have analyzed the problem under the assumption of continuous injection of a beam of fast electrons into a semi-infinite plasma; the directed velocity of the electrons u is assumed to be appreciably greater than the thermal velocity of the plasma electrons $v_{T0}$. Under these conditions excitation of oscillations takes place continuously as new fast electrons are injected into the plasma; the total energy of the oscillations can then be much greater than in the "single-shot" case. If the energy transport velocity $v_g$ is much smaller than the beam velocity u, the energy lost by the beam in the excitation of oscillations accumulates in a transition layer at the plasma boundary. The volume density of oscillation energy in this layer $\Sigma|E_k|^2/4\pi$ is then found to be very large and can be much greater than the energy density of the beam $n_0mu^2/2$. In particular, the condition $v_g \ll u$ is satisfied for plasma oscillations because for these oscillations $v_g \sim v_{T0}^2/u$.

As the field strength in the transition layer increases there is a more intense exchange of energy between the beam and the plasma oscillations and the width of the layer diminishes. This process continues up to the point at which the very large field amplitudes establish a stationary distribution for which the energy carried into the layer by the beam is equal to the energy carried away by the plasma oscillations.[6]

In the investigation of the development of the instability we have paid particular attention to the nonlinear stage of the interaction. Since the interaction of the beam electrons with the plasma oscillations is a resonance effect the nonlinear effects appear in the beam much before they appear in the plasma.

In this work we have analyzed the development of the instability in two cases: The injection of a monoenergetic beam into a plasma and the injection of a beam with a smeared-out velocity distribution. In the smeared beam case ($v_\theta \gg \gamma_k/k$, $v_\theta$ is the thermal velocity in the beam and $\gamma_k$ is the growth rate for the most unstable part of the spectrum) with low density ($n_0 \ll N_0$, $N_0$ is the plasma density) the basic nonlinear effect is the distortion of the beam distribution function; this distortion leads to the establishment of a plateau and saturation of the oscillations at an amplitude at which there is still only a small nonlinear interaction between the modes so that the quasilinear approximation holds.[5-7]

The development of the instability in the interaction of an initially monoenergetic beam with a plasma can take place in two stages: first there is a rapid [in a time $\sim \omega_0^{-1}(N_0/n_0)^{1/3}$] smearing of the beam distribution function up to the point at which $v_\theta \gg \gamma_k/k$; this is followed by a slower [in a time $\sim N_0/\omega_0 n_0$] saturation of the oscillations and the establishment of a plateau on the distribution function. The nonlinear interaction between modes can become important when $v_\theta \sim \gamma_k/k$; however, it is important that the largest contribution to the oscillation energy comes from the fields generated at $v_\theta \gg \gamma_k/k$.[8]

It is shown in the present work that the injection of a monoenergetic beam into a plasma gives rise to the appearance of two layers of high field strength; these correspond to the two stages in the development of the instability. The field strength at long times $t > N_0/\omega_0 n_0$ is a maximum in the second layer and the basic part of the energy lost by the beam in the excitation of oscillations is concentrated in this layer.

Since our basic purpose is the investigation of the time behavior of the formation of the transition layers at the plasma boundaries, our work differs from earlier work on the quasilinear theory; in this earlier work one obtained either nonstationary homogeneous solutions[5-8] or stationary solutions;[6] in the present work we consider nonstationary and inhomogeneous solutions of the equations of quasilinear theory.

2. In the quasilinear theory the equations that describe the interaction of a beam and the plasma oscillations excited by the beam are

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = \frac{2\pi e^2}{m^2}\frac{\partial}{\partial v}\left[\sum_{k>0}|E_k|^2\delta(kv-\omega_k)\frac{\partial f}{\partial v}\right], \quad (1)$$

$$\frac{\partial |E_k|^2}{\partial t} + v_g\frac{\partial |E_k|^2}{\partial x} = \frac{4\pi^2 e^2}{mk^2}\omega_k\frac{\partial f}{\partial v}\left(\frac{\omega_k}{k}\right)|E_k|^2. \quad (2)$$

In these equations f is the beam distribution function averaged over distances large compared with the wavelength of the oscillations and over time intervals large compared with the oscillation period; $|E_k|^2$ is the spectral density of the oscillation energy, $\omega_k$ is the frequency of the plasma oscillations:

$$\omega_k \approx \omega_0\left(1+\frac{3k^2T_0}{2m\omega_0^2}\right), \qquad \omega_0 = \left(\frac{4\pi N_0 e^2}{m}\right)^{1/2};$$

$v_g$ is the group velocity:

$$v_g = \frac{d\omega_k}{dk} = \frac{3kT_0}{m\omega_0} \approx \frac{3v_{T0}^2}{v_{\text{ph}}}.$$

Equations (1) and (2) can be derived using the procedure usually employed in the quasilinear theory; however it is simpler to obtain them from the familiar equations for the homogeneous case[5-7] making the obvious substitution $\partial/\partial t \to \partial/\partial t + v\partial/\partial x$ in the equations for f and $\partial/\partial t \to \partial/\partial t + v_g\partial/\partial x$ in the equations for $|E_k|^2$. The oscillation spectrum in Eqs. (1) and (2) is assumed to be one-dimensional[1)] and the waves propagate along the x-axis, which is perpendicular to the plasma boundary, along which the beam moves.

In the derivation of Eqs. (1) and (2) it is assumed that the energy of the oscillations excited in the beam is small compared with the thermal energy of the plasma electrons:

$$\varepsilon = \frac{1}{N_0T_0}\sum_k |E_k|^2 \ll 1. \quad (3)$$

under these conditions the variation in the macroscopic parameters of the plasma such as the density, temperature etc., which determine the frequency and group velocity of the plasma oscillations,[2)] is small; in Eqs. (1) and (2) the frequency of the plasma oscillations $\omega_k$ and $v_g$ can then be taken as constant during the development of the instability. At the same time the change in the distribution function of the beam due to the interaction of resonance particles ($v \approx v_{\text{ph}}$) with the os-

[1)]The oscillation spectrum can be regarded as one-dimensional when the maximum growth rate occurs for waves that propagate along the beam; this is the case, for example, when there is a strong magnetic field in the direction of motion of the beam. [6,11]

[2)]The contribution of the beam to the real part of the dielectric constant can be neglected since $\omega_k$ and $\nu_g$ are independent of the beam parameters. In Eqs. (1) and (2), the criteria that must be satisfied if one is to neglect terms associated with the real part of the dielectric constant of the beam for the homogeneous case are $\gamma_k/kv_\theta \sim n_0u_0^3/N_0v_\theta^3 \ll 1$. In the inhomogeneous case there is a somewhat more stringent restriction which is obtained from the condition that the change in the $v_g$ due to the beam must be small $\gamma_k/kv_\theta \ll v_g/u_0$.

cillations is extremely important, as are the nonlinear effects associated with this interaction. Hence nonlinear effects appear in the beam much before they appear in the plasma. This difference is due to the fact that the interaction of the plasma with the oscillations is not of a resonance nature. For this reason the plasma can be described by hydrodynamic analysis using the moments of the velocity distribution function averaged over time and space $F_0$.

Using the nonlinear kinetic equation for $F_0$ we obtain the following system of equations for these quantities: [3)]

$$\frac{\partial N}{\partial t}+\frac{\partial}{\partial x}(NU)=0, \qquad \frac{\partial T}{\partial t}=-\frac{e}{N}\sum_k E_k G_1^* + E_k^* G_1,$$

$$m\frac{\partial U}{\partial t}=-\frac{e}{2N}\sum_k E_k G_0^* + E_k^* G_0 - \frac{1}{N}\frac{\partial}{\partial x}(NT) - eE_0, \tag{4}$$

where

$$N = M_0 = \int F_0 dv, \qquad U = \frac{M_1}{N} = \frac{1}{N}\int F_0 v\, dv,$$

$$T = \frac{m}{N} M_2 = \frac{m}{N}\int F_0 v^2\, dv, \qquad G_n = \int F_k v^n\, dv.$$

Here we have introduced the following notation: N is the plasma density, U is the directed velocity, T is the plasma temperature, $G_n$ represents the moments of the k-th mode of the oscillating part of the plasma electron distribution function.

The quantities $G_n$ are determined in terms of the moments $M_n$ from the solutions of the following equations: [4)]

$$\frac{\partial G_n}{\partial t} - i\omega_k G_n + ikG_{n+1} + \frac{\partial G_{n+1}}{\partial x} = -\frac{ne}{m} M_{n-1} E_k \quad (n = 0, 1, 2, 3). \tag{5}$$

This system of equations follows very simply from the linearized kinetic equation for $F_k$. In the derivation of Eq. (4) we have neglected terms of order $\epsilon^2$. In this same approximation, we can use the linearized equations in determining $G_n$; furthermore, on the right side of Eq. (5) the moments $M_n$ are replaced by their initial values: $N = N_0 = \text{const}$, $T = T_0 = \text{const}$ while U is set equal to zero.

The solution of Eq. (5) is obtained by making use of the fact that the $G_n$ (like the oscillation amplitudes $E_k$) vary slowly over times $\sim 1/\omega$ and distances $\sim 1/k$ that is to say, the following conditions are satisfied:

$$\left|\frac{1}{\omega G_n}\frac{\partial G_n}{\partial t}\right| \ll 1, \qquad \left|\frac{1}{kG_n}\frac{\partial G_n}{\partial x}\right| \ll 1;$$

hence we need only retain first-order terms in the small parameter. The solution is

$$G_0 = -\frac{ekN_0}{m\omega_k^2}\left(iE_k + \frac{2}{\omega_k}\frac{\partial E_k}{\partial t} + \frac{1}{k}\frac{\partial E_k}{\partial x}\right),$$

$$G_1 = -\frac{eN_0}{m\omega_k}\left(iE_k + \frac{1}{\omega_k}\frac{\partial E_k}{\partial t} + \frac{6kT_0}{m\omega_k^2}\frac{\partial E_k}{\partial x}\right). \tag{6}$$

The last term in the expression for $G_1$ is important in the equation for the energy of the oscillations excited by the beam [(cf. Eq. (48)] but can be neglected in Eq. (4). Substituting $G_1$ from Eq. (6) in Eq. (4) we obtain an equation for the plasma temperature:

$$\frac{\partial T}{\partial t} = \frac{1}{N_0}\frac{\partial W}{\partial t}, \qquad W = \sum_k \frac{|E_k|^2}{4\pi} \tag{7}$$

(W is the oscillation energy density), whence

$$T = T_0 + W / N_0 \tag{7'}$$

(the contribution of the initial amplitudes has been neglected).

Using Eqs. (6) and (7') in Eq. (4) we obtain the following equation for the directed velocity of the plasma particles

$$N_0 m\frac{\partial U}{\partial t} = \frac{1}{4\pi}\frac{\partial}{\partial t}\sum_k \frac{k}{\omega_k}|E_k|^2 - eN_0E_0 - \frac{\partial}{\partial x}(NT_0) - \frac{1}{2}\frac{\partial W}{\partial x}. \tag{8}$$

It follows from Eq. (8) that in the stationary case the mean force exerted on a plasma electron by the wave field is

$$\bar{p} = -\frac{1}{2N_0}\frac{\partial W}{\partial x}, \tag{9}$$

which coincides with the relation obtained in another way by Gurevich and Pitaevskiĭ.[12] Using the results of these authors it is easy to obtain the following expressions for the electron density perturbation and the mean electric field in the stationary case:

$$\delta N = -\frac{1}{4T_0}W, \qquad E_0 = \frac{1}{4N_0 e}\frac{\partial W}{\partial x}. \tag{10}$$

The restrictions on the changes in plasma parameters, which must be satisfied if we are to

---

[3)] In the last equation in (4) we see that there is an average field as a result of the polarization in the plasma. The analogous term in the equation for the distribution function of the beam (1) can be neglected since it is easy to show, using Eq. (10), that (see below):

$$\frac{e}{m}E_0\frac{\partial f}{\partial v}\bigg| v\frac{\partial f}{\partial x} \sim \frac{1}{16\pi N_0 m}\frac{\partial}{\partial x}\sum_k |E_k|^2 \cdot \frac{\partial f}{\partial v}\bigg| v\frac{\partial f}{\partial x} \sim \frac{1}{16\pi N_0 m u v_\theta} \times \sum_k |E_k|^2 \ll 1.$$

[4)] In Eq. (5) we use the small parameter $k\sqrt{T/m}/\omega_k \ll 1$ and form a chain of equations by writing $G_4 = 0$.

neglect the change in frequency and group velocity during the development of the instability, are

$$\delta N \ll N_0, \quad \delta T \ll T_0, \quad U \ll v_g \sim T_0 / mu. \qquad (11)$$

Expressing the quantities $\delta T$, U and $\delta N$ in terms of the oscillation amplitudes using Eqs. (7′), (8) and (10) and substituting the results in Eq. (11) we find that (11) reduces to (3).

3. We now consider the solution of Eqs. (1) and (2). The analysis of these equations is first carried out assuming $v_g = 0$ in which case there is no transport of the energy of the plasma oscillations.[5)] Suppose that during the course of a time interval $t_0$ into a plasma occupying the region $x \geq 0$ we inject particles characterized by the distribution function $f^0(v)$. The boundary conditions on f are then as follows:

$$f(t, 0, v) = f^0(v)\sigma(t_0 - t), \qquad \sigma(t) = \begin{cases} 1, & t \geqslant 0 \\ 0, & t < 0 \end{cases}. \qquad (12)$$

At t = 0 the quantities f and $|E_k|^2$ are determined from the relations

$$f(0, x, v) = f^0(v), \qquad |E_k(0, x)|^2 = |E_k{}^0|^2, \qquad (13)$$

where $|E_k^0|^2$ represents the thermal noise in the plasma.

If $\partial f^0/\partial v > 0$ over some velocity range appreciably greater than the thermal velocity in the plasma the system is unstable and the oscillation energy increases with time. In turn the growth in oscillation energy leads to an enhanced diffusion of beam particles in velocity space which continues as long as a plateau is not formed on the distribution, that is to say, a region in which the distribution function is constant. In an infinite plasma the time during which the beam relaxes and the plateau is established on the distribution function is of order

$$\tau_0 = \frac{N_0}{n_0}\frac{v_\Theta^2}{u_0^2}\frac{1}{\omega_0}.$$

When a beam is injected into a semi-infinite plasma two regions should be distinguished: 1) the region of large x ($x \gg u_0\tau_0$) which the beam particles injected into the plasma reach in a time large compared with the relaxation time; in this region the boundary effects are obviously unimportant, 2) the region of small x ($x \lesssim u_0\tau_0$) in which the distribution of electric field is highly inhomogeneous and in which the relaxation time is a sensitive function of x. The electric field distribution can be investigated by means of the integrals of Eqs. (1) and (2) which determine the spectral energy density of the oscillations excited by the beam in terms of the change in the distribution function. In order to carry out these integrations we first integrate over k in Eq. (1) substituting $|E_k|^2 \partial f(\omega_0/k)/\partial v$ from Eq. (2) in the resulting equation. In this way we obtain

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = \frac{N_0 e^2}{\pi m^2 \omega_0}\frac{\partial}{\partial v}\left(\frac{1}{v^3}\frac{\partial |E_k|^2}{\partial t}\right) \quad \left(k = \frac{\omega_0}{v}\right). \qquad (14)$$

Substituting the function f in the following form in Eq. (14):

$$f(t, x, v) = f^*(t, x, v)\sigma(t_0 - t + x/v), \qquad (15)$$

where $f^*$ is a continuous function of x and t, and integrating Eq. (14) over time we obtain the following relation for $|E_k|^2$:

$$\frac{\partial}{\partial v}\left(\frac{|E_k|^2}{v^3}\right) = \frac{4\pi^2 m}{\omega_0}\Big\{f^*(t, x, v)\sigma\left(t_0 - t + \frac{x}{v}\right) - f^0(v) + v\frac{\partial}{\partial x}\Big[\int_0^{t_0 + x/v} dt' f^*(t', x, v)\Big] + \sigma\left(t_0 - t + \frac{x}{v}\right)v\frac{\partial}{\partial x}\Big[\int_{t_0 + x/v}^{t} dt' f^*(t', x, v)\Big]\Big\}. \qquad (16)$$

In regions of large x ($x \gg u_0\tau_0$) the inhomogeneity in f and $|E_k|^2$ due to the presence of the boundary is unimportant. In this case, when $t > \tau_0$ from Eq. (16) we obtain the integral for the equations of quasilinear theory, which coincides with that obtained earlier for the infinite plasma:[5-7]

$$|E_k|^2 = \frac{4\pi^2 m}{\omega_0} v^3 \int_{v_1} dv' (f^\infty(v') - f^0(v')), \qquad (17)$$

where $f^\infty$ is the distribution function with the plateau: $\partial f^\infty/\partial v = 0$ when $v_1 < v < v_2$ and $\partial f^\infty/\partial v < 0$ outside this range.

Let us now consider the region of small x ($x \lesssim u_0\tau_0$) for $t \gg \tau_0$. The spatial gradients of f and $|E_k|^2$ are rather large in this region: $\Delta x \ll u_0 t$ ($\Delta x$ is the distance over which these quantities change) so that the first two terms on the right side of Eq. (16) can be neglected. Furthermore, we can omit the term x/v compared with t and $t_0$ in the limits of integration and in the argument of the $\sigma$-function. Then, integrating Eq. (16) with respect to x we find

$$\frac{\partial}{\partial v}\left(\frac{1}{v^3}\int_0^{u_0\tau_0} dx |E_k(t, x)|^2\right) = \frac{4\pi^2 m}{\omega_0} v \Big[\int_0^t dt' (f^*(t', u_0\tau_0, v) - f^0(v)) - \sigma(t_0 - t)\int_{t_0}^{t} dt' (f^*(t', u_0\tau_0, v) - f^0(v))\Big]. \qquad (18)$$

---

[5)]Since $v_g \ll u$ the velocity of energy transport by the beam, this approximation applies so long as the spatial gradients of $|E_k|^2$ are not too large.

At the boundary of the region $x = u_0\tau_0$ the relaxation time of the beam remains of order $\tau_0$, that is to say, when $t > \tau_0$ a stationary distribution is established $f^\infty(v)$. Neglecting small terms $\sim \tau_0/t$ in Eq. (18) we can replace the function $f(t', u_0\tau_0, v)$ by the function $f^\infty(v)$ which is independent of t. Then, from (18) we obtain the integral for Eqs. (1) and (2) in the region of small x:

$$\int_0^{u_0\tau_c} dx\,|E_k(t,x)|^2 = \frac{4\pi^2 m}{\omega_0} v^3 J(v) s(t); \tag{19}$$

$$J(v) = \int_{v_1}^{v} dv' \cdot v' (f^\infty(v') - f^0(v')), \tag{19'}$$

$$s(t) = \begin{cases} t & \text{for} \quad t < t_0 \\ t_0 & \text{for} \quad t > t_0 \end{cases} \tag{19''}$$

The height of the plateau in the distribution function $f^\infty$ in Eq. (19) is determined from the conservation of particles in the beam:

$$\int_{v_1}^{v_2} dv \cdot v f^\infty(v) = \int_{v_1}^{v_2} dv \cdot v f^0(v). \tag{20}$$

In the usual way the boundaries of the plateau $v_1$ and $v_2$ are determined from the conditions

$$f^\infty(v_1) = f^0(v_1), \qquad f^\infty(v_2) = f^0(v_2). \tag{21}$$

We note that using Eq. (20) we can obtain an energy conservation relation from Eq. (19):

$$\int_0^{u_0\tau_0} dx W(t,x) = \frac{m}{\omega_0}\int_{v_1}^{v_2} dv \left|\frac{dk}{dv}\right| v^3 J(v) s(t)$$

$$= \int_{v_1}^{v_2} dv \frac{mv^3}{2}(f^0(v) - f^\infty(v)) s(t), \tag{22}$$

that is to say, the change in the total energy of the plasma oscillations in the region $x \lesssim u_0\tau_0$ (the potential energy of the oscillations $W/2$ and the thermal energy of the plasma $N_0\delta T/2 = W/2$) is equal to the energy brought into this region by the beam.

Equations (19) and (22) determine the spectral density and the total energy of the oscillations concentrated in the region $x \lesssim u_0\tau_0$. These quantities continue to increase with time for t large compared with $\tau_0$. Comparing Eq. (19) with Eq. (17) we see that the mean value of $|E_k|^2$ in the region being considered is $t/\tau_0$ times greater than the value of $|E_k|^2$ for $x \gg u_0\tau_0$. However, at large t the field distribution within the region $x \lesssim u_0\tau_0$ is also highly inhomogeneous. As the field strength grows the diffusion in the beam becomes more rapid and the relaxation length is reduced. The layer of peak field strength, whose dimensions are determined by the distance in which the beam relaxation occurs, is then displaced toward the plasma boundary.

The detailed structure of the spatial distribution of $|E_k|^2$ can only be found if the oscillation spectrum is narrow (in $v_{ph}$). In the general case of an arbitrary spectrum we can solve the more limited problem of finding the dependence of relaxation length on time. By definition, if $\xi(\tau)$ is the relaxation length of the beam for $t = \tau$, then for $t > \tau$ the following conditions are satisfied:

$$f(t,\xi,v) = f^\infty(v); \qquad \partial|E_k(t,x)|^2/\partial t = 0 \quad \text{for } x \geqslant \xi. \tag{23}$$

Making use of these conditions and Eq. (14) and making the same assumptions as in obtaining Eq. (19) we find

$$\int_0^{\xi(t)} dx \frac{\partial |E_k(t,x)|^2}{\partial t} = \frac{\pi m^2\omega_0}{N_0 e^2} v^3 J(v) \quad (t < t_0). \tag{24}$$

It follows from this equation that

$$\xi(t)\,\frac{\partial \overline{|E_k|^2}}{\partial t} \approx \frac{m^2\omega_0}{e^2}\frac{n_0}{N_0} v^4, \tag{25}$$

where $\overline{|E_k|^2}$ is the mean value of the spectral density of the oscillation energy in the bounded layer in which beam relaxation occurs. On the other hand, $\xi(\tau)$ can be estimated as the distance in which the beam particles diffuse (in velocity space) into a region of width $\Delta v = v_2 - v_1 \sim v_\theta$

$$\xi(t) \approx (\Delta v)^2 v / \overline{D} = m^2(\Delta v)^2 v^2 / e^2 \overline{|E_k|^2}. \tag{26}$$

Here

$$\overline{D} = \frac{2\pi e^2}{m^2}\sum_k \overline{|E_k|^2}\,\delta(kv - \omega_0) = \frac{e^2}{m^2 v}\overline{|E_k|^2}$$

is the mean value of the diffusion coefficient.

Substituting $\overline{|E_k|^2}$ from Eq. (26) in Eq. (25) we obtain the following approximate equation for $\xi(t)$

$$\frac{d\xi}{dt} + \frac{1}{\tau_0}\xi = 0. \tag{27}$$

Thus,

$$\xi \approx u_0\tau_0 \exp(-t/\tau_0), \qquad \tau_0 = N_0(\Delta v)^2 / n_0 v^2 \omega_0$$

($\tau_0$ is the beam relaxation time in an infinite plasma). The constant of integration is determined from the condition $\xi \approx u_0\tau_0$ for $t \approx \tau_0$. The quantity $\overline{|E_k|^2}$ is obtained from Eq. (26) where we substitute $\xi(t)$ from Eq. (27):

$$\overline{|E_k|^2} \approx n_0 m \frac{v^3}{\omega_0}\exp\left(\frac{t}{\tau_0}\right). \tag{28}$$

Thus, the field strength in the narrow bounded layer $x \lesssim \xi(t)$ increases with time much more rapidly than the mean value in the region $x \lesssim u_0\tau_0$.

We note that since the characteristic time $\tau_0$ for the formation of the layer is relatively large ($\tau_0 \sim 10^{-7}$ sec for $N_0 \approx 10^{11}$ cm$^{-3}$, $n_0 \approx 10^8$ cm$^{-3}$, $\Delta v \sim v$) the time variation of the process can be investigated experimentally.

4. Because of the complexity of the original system (1)—(2) in the general case, we have only been able to obtain approximate expressions for the dimensions of the layer in which there is an appreciable electric field strength $\xi(t)$ and for the value of $\overline{|E_k|^2}$ in this layer. We can also obtain one particular solution of Eqs. (1)—(2) which makes it possible to investigate in greater detail the spatial distribution of the electric fields associated with the oscillations.

We take Eqs. (1) and (14) as the original equations. Since we are considering the region with large spatial gradients ($x \ll u_0\tau_0$) we can neglect $\partial f/\partial t$ compared with $v\,\partial f/\partial x$ in these equations. The following system results

$$v\frac{\partial f}{\partial x} = \frac{e^2}{m^2}\frac{\partial}{\partial v}\left(\frac{|E_k|^2}{v}\frac{\partial f}{\partial v}\right),$$

$$\frac{\partial}{\partial v}\left(\frac{1}{v^3}\frac{\partial |E_k|^2}{\partial t}\right) = \frac{\pi\omega_0 m^2}{N_0 e^2} v\frac{\partial f}{\partial x}. \tag{29}$$

In the development of an instability the limits of the spectrum in $v_{ph}$ are usually displaced in the course of time. However, there exists a class of problems with fixed boundaries: for example, when $\partial f/\partial v \to -\infty$ at two points $v = u \pm v_0$. The oscillations in these problems are generated only in the phase velocity range $u - v_0 < v < u + v_0$ and the distribution function remains unchanged outside this range.[6)] If the range of velocities corresponding to this width is small $v_0 \ll u$, following[6] we can write f and $|E_k|^2$ in the form

$$f(t, x, v) = f^\infty + A(t, x)(v - u)$$
$$(A(t = 0) \equiv \partial f^0/\partial u),$$

$$|E_k(t, x)|^2 = {}^1\!/_2 B(t, x)\left[v_0^2 - (v - u)^2\right]$$
$$(k = \omega_0/v). \tag{30}$$

Using Eq. (29) we can write the following system of equations for A(t, x) and B(t, x):

$$\frac{\partial A}{\partial x} = -\frac{e^2}{m^2u^2}AB,$$

$$\frac{\partial B}{\partial t} = -\frac{\pi\omega_0 m^2}{N_0 e^2}u^4\frac{\partial A}{\partial x}. \tag{31}$$

It is easily shown that the solution of this system is

$$A(t, x) = (\partial f^0/\partial u)\left(\xi^2(t)/(x^2 + \xi^2(t))\right],$$

$$B(t, x) = (\pi\omega_0 m^2/N_0 e^2)u^4(\partial f^0/\partial u)\,\tau_0\{x/[x^2 + \xi^2(t)]\}; \tag{32}$$

Here

$$\xi(t) = u\tau_0\exp(-t/\tau_0), \quad \tau_0 = \frac{2}{\pi}\frac{N_0}{u^2}\left(\frac{\partial f^0}{\partial u}\right)^{-1}\frac{1}{\omega_0} \approx \frac{N_0}{n_0}\frac{v_0^2}{u^2}\frac{1}{\omega_0}.$$

Thus, when $x > u\tau_0\exp(-t/\tau_0)$ a plateau is established on the distribution function; for a fixed value of x $|E_k|^2$ first increases with t but approaches the following constant value when $t > \tau_0 \ln(u\tau_0/x)$

$$|E_k^\infty|^2 = {}^1\!/_2(\pi\omega_0 m^2/N_0 e^2)u^4(\partial f^0/\partial u)(\tau_0/x)$$
$$\times\left[v_0^2 - (v - u)^2\right],$$

which increases as x is reduced.

The change in the spatial distribution of the oscillation energy as a function of time is shown in Fig. 1. The maximum value of $|E_k|^2$ in the plasma increases with time and is displaced toward the boundary. Using Eq. (32) we can obtain the total oscillation energy for a given k in the region[7)]

$$\int_0^{u\tau_0} dx\,|E_k(t, x)|^2 = \frac{\pi\omega_0 m^2}{N_0 e^2}u^4\frac{\partial f^0}{\partial u}\frac{v_0^2 - (v - u)^2}{2}\tau_0\int_0^{u\tau_0}\frac{x\,dx}{x^2 + \xi^2(t)}$$

$$= \frac{4\pi^2 m}{\omega_0}u^4\frac{\partial f^0}{\partial u}\frac{v_0^2 - (v - u)^2}{2}t, \tag{33}$$

which coincides with Eq. (19) obtained in the general case.

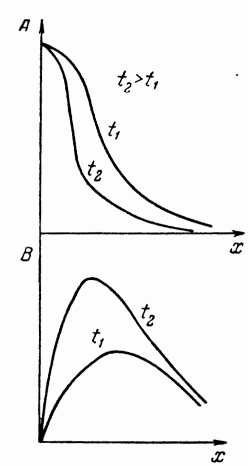


FIG. 1. A and B as functions of x for various values of t.

5. In all of the preceding analysis we have assumed $v_g = 0$. However, although $v_g \sim v_{T0}^2/u_0 \ll u_0$, if the spatial gradients of $|E_k|^2$ are large there may be important terms $v_g\,\partial|E_k|^2/\partial x$ that

6)The existence of a class of problems with fixed boundaries has been indicated earlier by Vedenov.[6].

7)In this case the approximation expression (32) can be extrapolated to $x \sim u\tau_0$ because when $t \gg \tau_0$ these values of x give a small contribution in the integral (33).

imply significant transport of energy of the plasma oscillations. These terms impose a limitation on the growth of $|E_k|^2$. If the $v_g \partial|E_k|^2/\partial x$ term is retained in the equation for the spectral density of the oscillation energy Eq. (24) is replaced by

$$\int_0^{\xi(t)} dx \frac{\partial |E_k(t, x)|^2}{\partial t} + v_g |E_k(t, \xi)|^2 = \frac{4\pi^2 m}{\omega_0} v^3 J(v). \quad (34)$$

In this case, the expression for $\xi(t)$ is given by $d\xi/dt + \xi/\tau_0 = v_g$, i.e.,

$$\xi \approx u_0\tau_0 \exp(-t/\tau_0) + v_g\tau_0[1 - \exp(-t/\tau_0)]. \quad (35)$$

When $t \to \infty$

$$\xi \to \xi^{min} = v_g\tau_0 \approx \frac{u_0}{\omega_0}\frac{N_0T_0}{n_0mu_0^2}, \qquad \xi^{min} \gg \lambda \sim \frac{u_0}{\omega_0}$$

if the condition $N_0T_0 \gg n_0mu_0^2$ is satisfied. This condition is less stringent than that given in (3) because the energy density of the oscillations is appreciably greater than the energy density in the beam. As before, the time dependence of $|E_k|^2$ in the bounded layer is determined by Eq. (26) in which we must use $\xi(t)$ from Eq. (35).

The maximum amplitude of the oscillations in the bounded layer is reached when $\xi \to \xi^{min}$. This amplitude is

$$|E_k^{max}|^2 = (4\pi^2 m/\omega_0)(v^3/v_g)J(v). \quad (36)$$

The relation that has been derived has a simple physical meaning: at the maximum amplitude the energy removed from the layer $x < \xi^{min}$ by the waves is equal to the energy carried into this layer by the particles in the beam. The maximum amplitude of the oscillations given by Eq. (36) was found by Vedenov[6] who considered the stationary problem $\partial/\partial t = 0$, $\partial/\partial x \neq 0$ in the quasilinear approximation. In this case a bounded layer with high field intensity does not arise since the field given by Eq. (36) is displaced with velocity $v_g$ from the boundary into the depth of the plasma and when $t > L/v_g$ (L is the dimension of the region occupied by the plasma) the stationary distribution found in[6] is obtained.

A stationary distribution has evidently not been produced in a number of experimental investigations of beam-plasma interactions.[9,10] For example, in[10] the plasma oscillations were excited by a pulsed beam with pulse length $t_0 \ll L/v_g$. In the experiments of Kharchenko[9] a magnetic field was used so that the modes exhibited anomalous dispersion $d\omega/dk < 0$ in which case the oscillation energy is not transported in the direction of motion of the beam.

6. We now consider the development of an instability arising in the interaction of an initially monoenergetic beam $kv_\theta^0/\gamma_k \ll 1$ and a plasma. The basic difference between this case and the one considered earlier is the fact that in the monoenergetic beam the width of the velocity range in which there is a resonance between the beam particles and the k-th mode is large compared with $v_\theta$ the thermal velocity of the beam $|v - v_{ph}| \lesssim \gamma_k/k$; hence the entire beam is in resonance with the wave.

In this case the beam is treated by a hydrodynamic description based on the moments of the distribution function f: $n = f\int dv$, f is the density, $u = n^{-1}\int dv \cdot vf$ is the directed velocity, $\Theta = n^{-1}\int dv\ m(v-u)^2 f$ is the beam temperature. These quantities are described by the following system of equations obtained from the kinetic equation for f:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu) = 0,$$

$$m\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = -\frac{e}{2n}\sum_k (E_k g_{0k}^* + E_k^* g_{0k}) - \frac{1}{n}\frac{\partial}{\partial x}(n\Theta),$$

$$\frac{\partial\Theta}{\partial t} + u\frac{\partial\Theta}{\partial x} + 2\Theta\frac{\partial u}{\partial x} = -\frac{e}{n}\sum_k (E_k g_{1k}^* + E_k^* g_{1k}) - \frac{1}{n}\frac{\partial Q}{\partial x}. \quad (37)$$

Here we have used the notation

$$Q = \int dv\, m(v-u)^3 f, \qquad g_s = \int dv (v-u)^s f_k.$$

The $g_s$ are the moments of the oscillating parts of the distribution functions $f_k$. The system of equations for the $g_s$ is obtained from the linearized kinetic equation for $f_k$. We have

$$\frac{\partial g_s}{\partial t} + i(ku_0 - \omega_k)g_s + u_0\frac{\partial g_s}{\partial x} + ikg_{s+1} + \frac{\partial g_{s+1}}{\partial x} = -s\frac{eE_k}{m}\mu_{s-1},$$

$$\mu_s = \int dv (v-u)^s f, \qquad s = 0, 1, 2, 3. \quad (38)$$

We first investigate the initial (hydrodynamic) stage in the development of the instability in which the change in the directed velocity and the thermal energy of the beam due to the effect of the oscillations is small so that the following conditions are satisfied for all x:

$$\eta = |k\delta u/(ku_0 - \omega_k)| \ll 1,$$

$$\tilde{\eta} = k^2\Theta/m(ku_0 - \omega_k)^2 \ll 1. \quad (39)$$

In this stage the change in beam parameters does not cause a change in growth rates: the $\gamma_k$ are then determined by the initial values of these parameters, which are assumed to be independent of x. Initially, the oscillation energy increases in time in accordance with the relation $|E_k|^2 = |E_k^0|^2 \exp(2\gamma_k t)$ and is also independent of x.

The solution of Eq. (38) in the case in which $E_k$ is uniform has been obtained earlier.[8] Neglecting terms of order $\eta$ and $\tilde{\eta}$ in the solution we have

$$g_{0k} = -\frac{kg_{1k}}{ku_0 - \omega_k - i\gamma_k}, \quad g_{1k} = -\frac{en_0}{m}\frac{E_k}{i(ku_0 - \omega_k - i\gamma_k)}. \tag{40}$$

Substituting $g_{1k}$ in the last equation of (37) and omitting the terms $\sim \partial Q/\partial x$, $\Theta\partial u/\partial x$, we obtain the following equation for $\Theta$ [8)]

$$\frac{\partial\Theta}{\partial t} + u_0\frac{\partial\Theta}{\partial x} = \frac{2e^2}{m}\sum_k \frac{\gamma_k}{\Delta_k}|E_k(t)|^2,$$
$$\Delta_k = (ku_0 - \omega_k)^2 + \gamma_k^2. \tag{41}$$

The solution of this equation that satisfies the condition $\Theta(0, x) = \Theta(t, 0) = \Theta_0$ is

$$\Theta \approx \Theta_0 + \frac{e^2}{m}\sum_k \frac{|E_k(t)|^2}{\Delta_k}S_k(x)$$
$$\approx \Theta_0 + \frac{2^{2/3}}{8\pi}\alpha\sum_k |E_k(t)|^2 S_k(x);$$
$$S_k(x) = 1 - \exp[-2\gamma_k x/u_0], \quad \alpha = (n_0/N_0)^{1/3} \ll 1. \tag{42}$$

To obtain the last relation we have made use of the fact that $|E_k|^2 \sim \exp(2\gamma_k t)$; thus under the summation sign we can replace the factor in front of the exponential by its value for the most unstable mode $k = \omega_0/u_0$ so that

$$\gamma^0 = 2^{-4/3}\sqrt{3}\alpha\omega_0, \qquad \omega_k - ku_0 = -2^{-4/3}\alpha\omega_0.$$

Substituting the function $g_{0k}$ from Eq. (40) in the equation for $u(x, t)$, neglecting $n^{-1}\partial(n\Theta)/\partial x$ (this requires $k\xi_g^0 \gg 1$) and solving the resulting equation, we have

$$u(x, t) \approx u_0 - \frac{e^2}{m^2}\sum_k |E_k(t)|^2 k\frac{ku_0 - \omega_k}{\Delta_k^2}S_k(x)$$
$$\approx u_0 - \frac{1}{4\pi n_0 m u_0}\sum_k |E_k(t)|^2 S_k(x). \tag{43}$$

The first equation of (37) is then solved for $n(x, t)$

$$n(x, t) = n_0\Big\{1 + \frac{2e^2}{m^2}\sum_k |E_k(t)|^2 k\frac{ku_0 - \omega_k}{\Delta_k^2}$$
$$\times(\gamma_k x/u_0^2)\exp[-2\gamma_k x/u_0]\Big\}. \tag{44}$$

The variables n, u and $\Theta$ as functions of x and t are shown in Fig. 2. It is evident that these quantities are sensitive to x only when $x \lesssim \xi_g^0 \approx u_0/\alpha\omega_0$ in which case

$$|\delta u|, \quad \Theta \sim 1 - \exp(-2x/\xi_g^0);$$
$$n \sim \text{const} + x\exp(-2x/\xi_g^0).$$

Outside of this range n, u and $\Theta$ are essentially independent of x and $\delta n \to 0$ while u and $\Theta$ are given by expressions reported earlier[8] for the case of a beam interacting with an infinite plasma.

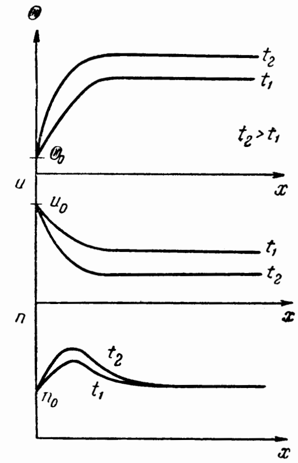


FIG. 2. The density, directed velocity, and temperature of the beam as functions of x and t in the initial stage of development of the instability.

Equations (42), (43), and (44) apply at low oscillation amplitudes in which case the following condition is satisfied:

$$\beta(t) = \frac{1}{\alpha n_0 m u_0^2}\sum_k |E_k(t)|^2 \ll 1.$$

When $\beta \sim 1$ in the region $x > \xi_g^0$

$$|\delta u| \sim \alpha u_0 \sim \frac{ku_0 - \omega_k}{k}; \quad \Theta \sim \alpha^2 m u_0^2 \sim \frac{m(ku_0 - \omega_k)^2}{k^2},$$

i.e., the condition in (39) is violated. The beam is then spread out so much that $kv_\Theta/\gamma_k \sim 1$ and its subsequent relaxation can be treated in the quasi-linear approximation. The beam remains monoenergetic in the region $x \lesssim \xi_g^0$ and when $\beta \sim 1$ (the hydrodynamic layer) the characteristic time for the increase of amplitude in this layer remains of order $1/\gamma^0$.

Thus, when an initially monoenergetic beam is injected into a plasma, at high field amplitudes, in which case $\beta \gtrsim 1$ there arises a boundary $x = \xi_g$ through which a beam with a smeared out velocity distribution enters the plasma $kv_\Theta/\gamma_k \sim 1$. The relaxation of this beam is the same as that described earlier; the oscillation energy in this case is given by Eq. (19) in which the lower limit of integration on x is $\xi_g$ and $f^0$ is the distribution function with $v_\Theta \sim \alpha u_0$ and $u \approx u_0$. The oscillations are concentrated primarily in the region $\xi_g < x < \xi$ where $\xi$ is given by Eq. (35). The beam remains

8) The criteria that must be satisfied when $|E_k|^2$ is independent of x are of the form: $|\delta u| \ll \gamma_k\xi_g^0$, $\Theta \ll mu_0\xi_g^0\gamma_k^2/\omega_0$ where $\xi_g^0$ is the distance over which the beam parameters u and $\Theta$ change in the stage being considered, $\xi_g^0 \sim u_0/\gamma_k$.

monoenergetic in the region $x < \xi_g$ and the characteristic time for the development of the instability in this region is $\sim 1\ \alpha\omega_0$ i.e., much smaller than for large x. The width of this region $\xi_g = \xi_g^0 \sim u_0/\alpha\omega_0$ for $\beta \sim 1$; as $\Sigma|E_k|^2$ increases, $\xi_g$ is reduced.

7. We now obtain an equation that determines the time variation of the energy in the hydrodynamic layer and the thickness of the layer $\xi_g$ for $\beta \gg 1$ when $\xi_g \ll \xi_g^0$. In this case we have from Eq. (38)

$$g_{1k} \approx -\frac{e\bar{E}_k\xi_g}{mu_0}n_0, \qquad g_{0k} \approx -\frac{ke\bar{E}_k\xi_g^2}{mu_0^2}n_0, \tag{45}$$

where $\bar{E}_k$ is the mean field amplitude for $x \lesssim \xi_g$. Substituting Eq. (45) in the equations for u and $\Theta$ (37) we have[9)]

$$u_0\frac{\partial u}{\partial x} \approx -\frac{e^2}{m^2}\sum_k k\,\overline{|E_k|^2}\,\frac{\xi_g^2}{u_0^2},$$

$$u_0\frac{\partial \Theta}{\partial x} \approx \frac{e^2}{m}\sum_k \overline{|E_k|^2}\,\frac{\xi_g}{u_0}. \tag{46}$$

Integrating these equations with respect to x we obtain an approximate expression which gives the variation of $\Theta$ and u within the hydrodynamic layer. Making use of the fact that $\Theta \sim \alpha^2 mu_0^2$ at the boundary of the layer, we obtain from Eq. (46) a relation that gives the layer thickness:

$$\xi_g \approx \frac{u_0}{\alpha\omega_0}\left[\frac{4\pi}{\bar{\beta}(t)}\right]^{1/2} \tag{47}$$

and the retardation of the beam in the layer

$$\delta u = -\alpha^2\xi_g\omega_0. \tag{47'}$$

Equation (47) determines the reduction in $\xi_g$ with increasing $\beta$.

In order to find the time dependence of the oscillation energy in the bounded layer we use the energy conservation relation:

$$\frac{\partial \mathcal{E}}{\partial t} + u_0\frac{\partial}{\partial x}(\mathcal{E} + P) + \frac{\partial W}{\partial t} + \frac{\partial}{\partial x}\sum_k v_g\frac{|E_k|^2}{4\pi} = 0. \tag{48}$$

Here we have used the following notation: $\mathcal{E} = n(mu^2/2 + \Theta/2)$ is the energy density in the beam, $u_0(\mathcal{E} + P)$ is the density of the energy flux, $P = n\Theta$ is the pressure. Equation (48) can be obtained easily by making use of the fact that the density of energy dissipated per unit time in the beam is

$$\frac{1}{2}\sum_k (E_k j_k^{B*} + E_k^* j_k^B) = -\frac{1}{2}\frac{\partial W}{\partial t} - \frac{1}{2}\sum_k (E_k j_k^{p*} + E_k^* j_k^p).$$

In this expression $j_k^p$ is the oscillating part of the plasma particle current

$$j_k^p = -e\int dv\cdot vF_k = -eG_1,$$

where $G_1$ is determined from Eq. (6).

Let us first investigate the case $v_g = 0$. Omitting in Eq. (48) the term $\partial\mathcal{E}/\partial t$ which is unimportant when $\xi_g \ll u_0/\gamma^0$, integrating Eq. (48) from 0 to $\xi_g$ with respect to x and substituting at the boundary of the layer $\delta u$ from (47′) with $\Theta \sim \alpha^2 mu_0^2$, we have

$$\int_0^{\xi_g} dx\frac{\partial W(t,x)}{\partial t} \approx \alpha n_0 mu_0^2\xi_g\gamma^0. \tag{49}$$

From Eqs. (47) and (49) we have approximate equations giving the time dependence of $\bar{W} = \Sigma|\bar{E}_k|^2/4\pi$ and $\xi_g$

$$\bar{W} \approx \alpha n_0 mu_0^2\gamma^0 t, \qquad \xi_g \approx n_0/\alpha\omega_0(\gamma^0 t)^{1/2}. \tag{50}$$

According to Eq. (50) the energy density of the oscillations as well as the total energy in the layer both increase as t increases:

$$\int_0^{\xi_g} dx\,W(t,x) \approx \alpha n_0 mu_0^3(t/\gamma^0)^{1/2}. \tag{51}$$

The increase in oscillation energy continues until the transport of oscillation energy at high amplitudes becomes important. Retaining terms $\sim v_g$ in Eq. (48), in place of Eq. (49) we obtain the following relation:

$$\int_0^{\xi_g} dx\frac{\partial W(t,x)}{\partial t} + \frac{1}{4\pi}\sum_k v_g|E_k(t,\xi_g)|^2 \approx \alpha n_0 mu_0^2\xi_g\gamma^0. \tag{52}$$

The maximum oscillation amplitudes are determined from the condition

$$W^{max}(\xi_g) \approx \alpha\frac{n_0 mu_0^3}{3v_{T0}^2}\xi_g\gamma^0. \tag{53}$$

Substituting $\xi_g$ from Eq. (47) we have finally

$$W^{max}(\xi_g) = \alpha\frac{u_0^{1/3}}{v_{T0}^{4/3}}n_0 mu_0^2, \quad \xi_g^{min} \approx \frac{u_0}{\omega_0}\left(\frac{N_0T_0}{n_0 mu_0^2}\right)^{1/3}. \tag{54}$$

If the condition $n_0 mu_0^2 \ll N_0T_0$ is satisfied the quantity $\xi_g^{min}$ is large compared with the wavelength $u_0/\omega_0$ but much smaller than the minimum width of the second layer given by Eq. (35).

The time growth of the oscillation energy in the second layer, in which there is a quasilinear relaxation of the beam, is given by Eq. (19) when $t > N_0/\omega_0 n_0$. The height of the plateau is obtained from Eq. (20):

$$f^\infty = 2n_0u_0/(v_2^2 - v_1^2). \tag{55}$$

9)In the equation for $\Theta$ the term with $n^{-1}\,\partial Q/\partial x$ can be neglected if $\xi_g \ll \xi_g^0$; this is easily shown if one estimates Q by means of an equation analogous to Eq. (37).

The width of the plateau is approximately

$$v_2 \approx u_0, \qquad v_1 \approx \sqrt{\frac{T_0}{2m}} \ln \left[\frac{N_0}{n_0} \frac{u_0}{(2\pi T_0/m)^{1/2}}\right],$$

i.e., in the case of an initially monoenergetic beam the particles diffuse to a velocity of the order of the thermal velocity in the plasma.

In Eq. (19) we can neglect $f^0$ compared with $f^\infty$ for all $v_{ph}$ with the exception of a narrow range $|v_{ph} - u_0| \sim \alpha u_0$ which gives a small contribution in the oscillation energy ($\sim \alpha$). Thus, from Eq. (19) we have

$$\int_{\xi_g}^{u_0\tau_0} dx\, |E_k(t, x)|^2 = 2\pi^2 n_0 m v^3 \frac{v_2}{\omega_0} \frac{v^2 - v_1^2}{v_2^2 - v_1^2} t \quad (t < t_0). \quad (56)$$

Summing over all modes from $k_{min} = \omega_0/v_2$ to $k_{max} = \omega_0/v_1$ we obtain the total energy of the oscillations

$$\int_{\xi_g}^{u_0\tau_0} dx\, W(t, x) = \frac{1}{4} n_0 m u_0 (u_0^2 - v_1^2) t \approx \frac{1}{4} n_0 m u_0^3 t. \quad (57)$$

$|E_k^{max}(\xi)|^2$ in the second layer is given by Eq. (36):

$$|E_k^{max}(\xi)|^2 = \frac{8}{3} \pi^2 n_0 m v^4 \frac{u_0}{\omega_0} \frac{1}{v_{T0}^2} \frac{v^2 - v_1^2}{v_2^2 - v_1^2}. \quad (58)$$

The maximum value of the oscillation energy density in the second layer can be appreciably greater than the energy density in the beam:

$$W^{max}(\xi) \approx 2n_0 m u_0^4 / 15 v_{T0}^2. \quad (59)$$

Thus, the relaxation of an initially monoenergetic beam at large $t > N_0/\omega_0 n_0$ is characterized by two layers with high field intensities. The dimensions of these layers are determined by Eqs. (54) and (35); the largest field strength obtains in the second layer. The expressions for the energy density (54) and (59) can be compared with the corresponding expressions for the case of single-shot injection: we see that the oscillation energy is considerably greater in the continuous injection case: by a factor of $(u_0/v_{T0})^{4/3}$ in the first layer and a factor of $u_0^2/v_{T0}^2$ in the second.

The authors are indebted to G. Ya. Lyubarskiĭ and A. A. Vedenov for discussion of these results.

[1] A. I. Akhiezer and Ya. B. Faĭnberg, DAN SSSR 69, 555 (1949); JETP 21, 1262 (1951).

[2] D. Bohm and E. P. Gross, Phys. Rev. 75, 1851 (1949).

[3] V. D. Shapiro, 4-th Conference of Ministry of Second. & Higher Education, U.S.S.R. on Radio Engineering, Kharkov, 1960; Izv. VUZ'ov, Radiofizika News of the Universities, Radiophysics 4, 867 (1961); V. D. Shapiro and V. I. Shevchenko, JETP 42, 1515 (1962), Soviet Phys. JETP 15, 1053 (1962).

[4] Yu. A. Romanov and G. F. Filippov, JETP 40, 123 (1961), Soviet Phys. JETP 13, 87 (1961).

[5] Vedenov, Velikhov and Sagdeev, UFN 73, 701 (1961), Soviet Phys. Uspekhi 4, 332 (1961); Conference on Plasma Physics, Salzburg, 1961, Report 199; Nuclear Fusion, Supp. Pt. II, 465 (1962).

[6] A. A. Vedenov, Atomnaya énergiya (Atomic Energy) 13, 1 (1962).

[7] W. E. Drummond and D. Pines, Conference on Plasma Physics, Salzburg, 1961, Report 134; Nuclear Fusion, Supp. Pt. III, 1049 (1962).

[8] V. D. Shapiro, JETP 44, 613 (1963), Soviet Phys. JETP 17, 416 (1963).

[9] Kharchenko, Faĭnberg, Kornilov and Pedenko, ZhTF 34, 1031 (1964), Soviet Phys. Tech. Phys. 9, 798 (1965).

[10] A. K. Berezin, Dissertation, Kharkov State University, 1964.

[11] V. D. Shapiro, Dissertation, Joint Institute for Nuclear Research, 1963.

[12] A. V. Gurevich and L. P. Pitaevskiĭ, JETP 45, 1281 (1963), Soviet Phys. JETP 19, 870 (1964).

Translated by H. Lashinsky
199