ON PHASE TRANSITIONS IN A MODEL SYSTEM

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We consider the properties of a multi-component classical non-linear field which depends on one spatial coordinate. In the particular case of a two-component field this model describes the properties of a string acted upon by an external potential. The thermodynamic characteristics of this system are evaluated and we show that at a certain temperature, determined by the character of the non-linear term, the system undergoes a phase transition. The problem of the temperature behavior of the system turns out to be equivalent to the problem of the dependence of the eigenvalues of the Schrodinger operator on the coupling constant; the role of the latter is played by the inverse square of the temperature. We find the correlation functions of the system and show that at the phase transition point the characteristic correlation length of the fluctuations turns out to be infinite.

1. INTRODUCTION

IT is well known that phase transition in many-body systems are connected with the interactions between the particles. If the interaction has an ordering character, then at certain temperatures, when the disordering influence of the thermal motion is dominated by the interaction, the system goes into an ordered state—it undergoes a phase transition.

Near second-order phase transition points the fluctuations are correlated over very long distances. This makes it impossible to apply here the usual methods for decoupling correlation functions, as these methods are based upon a neglect of contributions from volumes in which the correlations are important.

There is at present no rigorous microscopic theory of phase transitions. It seemed therefore of interest to us to study all kinds of rigorously calculable model systems in which a phase transition takes place when this transition has all assumedly characteristic features of phase transitions in real systems. Onsager's paper [1] was the first publication of this kind in which the phase transition problem of the two-dimensional Ising model was solved exactly.

The aim of the present paper is the study of one of such exactly calculable systems. It is natural to expect that a system of coupled anharmonic oscillators may undergo a phase transition. We shall in the following study this system.

2. THE THERMODYNAMIC PROPERTIES OF THE SYSTEM

We consider a classical field $\mathbf{u} = \{u_1, \ldots, u_n\}$, which in general is a multicomponent field, depends on one coordinate x and the time t, and is described by the Hamiltonian

$$H = \int_0^L \frac{\rho}{2}\left(\frac{\partial \mathbf{u}}{\partial t}\right)^2 dx + \frac{\varkappa}{2}\int_0^L \left(\frac{\partial \mathbf{u}}{\partial x}\right)^2 dx + \int_0^L V(\mathbf{u})\,dx. \quad (2.1)$$

Here, L is the interval along which the field $\mathbf{u}$ is defined; the first term is the kinetic energy, and ρ is the linear density; the second term is the energy of the elastic coupling, and κ is the elastic constant [1)]; $V(\mathbf{u})$ is the potential energy of the field.

If $\mathbf{u}$ is a two-component vector, the Hamiltonian (2.1) describes a string in an external field $V(\mathbf{u})$, or, what comes to the same, a string, the energy of which depends on the displacement from its equilibrium position in a non-linear way.

The Hamiltonian (2.1) can be considered to be a generalized Thirring Hamiltonian. (In Thirring's model, the field $\mathbf{u}$ has one component, and $V(u) = \alpha u^2 + \beta u^4$.)

In the following we shall resort to the visualisable case of a string, so that we impose the natural boundary conditions for that case: $u(x = 0)$

[1)] All results obtained in the following can easily be generalized to apply to the anisotropic case when the elastic constants form a tensor $\varkappa_{ik}$.

$= u(x = L) = 0$, corresponding to a string with fixed end points.[2)]

The partition function of such a system can be expressed in terms of a functional integral

$$Z = \int \delta u \exp\Big\{-\beta \int_0^L \rho \frac{\dot{u}^2}{2} dx\Big\} \times \int \delta u \exp\Big\{-\beta \int_0^L dx\Big[\frac{\varkappa}{2}\Big(\frac{\partial u}{\partial x}\Big)^2 + V(u)\Big]\Big\}. \quad (2.2)$$

Here $\beta = 1/T$ is the inverse temperature in energy units and the integration is performed under the conditions $u(0) = u(L) = 0$.

We can write the partition function in the form

$$\frac{Z}{Z_0} = \frac{\int \delta u \exp\Big\{-\frac{\beta\varkappa}{2}\int_0^L dx\Big(\frac{\partial u}{\partial x}\Big)^2 - \beta\int_0^L V(u)\,dx\Big\}}{\int \delta u \exp\Big\{-\frac{\beta\varkappa}{2}\int_0^L dx\Big(\frac{\partial u}{\partial x}\Big)^2\Big\}}, \quad (2.3)$$

where Z_0 is the partition function for the case $V = 0$.

We shall not discuss here the problem of the divergence of the free energy $F_0 = -\beta^{-1} \ln Z_0$ of a classical harmonic field. We shall consider that problem later. We are interested in the difference $\Delta F = F - F_0$ and Eq. (2.3) determines just this quantity.

It is well known [2] that the Green function of the Schrodinger equation of a particle with the imaginary time (or the single-particle Bloch equation)

$$\frac{\partial}{\partial\tau}\psi(u) = \Big(\frac{p^2}{2m} + \varphi(u)\Big)\psi(u) \quad (2.4)$$

(here $p = -i\partial/\partial u$) has the form

$$G(u\tau, u_0\tau_0) = \frac{1}{N}\int \prod_{\tau_0 u_0}^{\tau u} du' \exp\Big\{-\frac{m}{2}\int_{\tau_0}^{\tau}\Big(\frac{du'}{d\tau'}\Big)^2 d\tau' - \int_{\tau_0}^{\tau}\varphi(u')\,d\tau'\Big\} \quad (2.5)$$

(N is a normalization constant).

One sees easily that the expression for the partition function (2.3) is the same as expression (2.5), if in the latter we understand by the "time" τ the coordinate x, and if we put the mass m equal to $\beta\kappa$ and replace the potential energy $\varphi(u)$ by $\beta V(u)$. We have thus the relation

$$e^{-\beta(F-F_0)} = G(0L, 00)/G_0(0L, 00), \quad (2.6)$$

where $G(uL, u_0L_0)$ is the Green function satisfying the equation

$$\{\partial/\partial L - p^2/2\beta\varkappa - \beta V(u)\}G(uL, u_0L_0) = \delta(u - u_0)\delta(L - L_0),$$
$$G(uL, u_0L_0) = 0 \qquad \text{when } L < L_0, \quad (2.7)$$

and $G_0(uL, u_0L_0)$ is the Green function of the equation with $V \equiv 0$. It is well known [2] that

$$G(uL, u_0L_0) = \begin{cases} \sum_\lambda \psi_\lambda(u)\psi_\lambda^*(u_0)\,e^{-(L-L_0)E_\lambda} & L \geqslant L_0 \\ 0 & L < L_0 \end{cases} \quad (2.8)$$

Here $\psi_\lambda(u)$ and E_λ are the eigenfunctions and eigenvalues of the "Hamiltonian"

$$h(\beta) = p^2/2\beta\varkappa + \beta V(u), \quad (2.9)$$

and the symbol $\int_\lambda$ must be understood to indicate summation over the discrete energies and integration over the region of the continuous spectrum. Since the "mass" of the particle described by the "Hamiltonian" (2.9) and the field in which it moves depend on the temperature, we have for the eigenvalues $E_\lambda = E_\lambda(\beta)$.

Substituting (2.8) into (2.6), we get [3)]

$$e^{-\beta\Delta F} = \frac{\sum_\lambda \psi_\lambda(0)\psi_\lambda^*(0)\,e^{-LE_\lambda(\beta)}}{(2\pi)^{-n}\int dp \exp(-Lp^2/2\beta\varkappa)} \quad (2.10)$$

(n is the number of components of the field u).

We are interested in a "large" system obtained by the limiting transition $L \to \infty$. In that case it makes sense to speak about the specific free energy $\Delta f = \lim(\Delta F/L)$, $L \to \infty$ which according to Eq. (2.10) is

$$\Delta f = -\frac{1}{\beta}\lim_{L\to\infty}\frac{1}{L}\ln\frac{\sum_k \psi_k(0)\psi_k(0)\,e^{-LE_k} + \int_\lambda^c \psi_\lambda(0)\psi_\lambda(0)\,e^{-LE_\lambda}}{(2\pi)^{-n}\int dp \exp\{-Lp^2/2\beta\varkappa\}}. \quad (2.11)$$

We have here separated off explicitly the summation over the discrete levels E_k of the operator $h(\beta)$ and the symbol $\int_\lambda^c$ indicates integration over the continuous spectrum.

It follows from Eq. (2.11) that with asymptotic

[2)] We note that all that is developed in the following remains valid with minor alterations also for the case of other boundary conditions.

[3)] We assume that the lowest value of the limit of $V(u)$ as $|u| \to \infty$ vanishes. If this limit $V_0 \neq 0$, we can always add to the Hamiltonian (2.1) a term $-LV_0$ and this guarantees that the equation is satisfied.

accuracy

$$-\beta\Delta f = |E_0|, \qquad (2.12)$$

where E_0 is the ground state energy.

The Hamiltonian $h(\beta)$ reduces, by a change in the "energy" units of E, to

$$\tilde{h}(\beta) = \mathbf{p}^2/2 + \beta^2\varkappa V(\mathbf{u}). \qquad (2.13)$$

When the temperature T increases (or β decreases) the depth of the well decreases and the energies of the bound states decrease. When a certain critical value β_c is reached, the well is not able to "retain" a particle—the bound states have disappeared.

Using Eq. (2.12), we have thus

$$-\beta\Delta f = \begin{cases} |E_0| & \beta \geqslant \beta_c \\ 0 & \beta < \beta_c \end{cases}. \qquad (2.14)$$

We see that in the point β_c where the properties of the operator $h(\beta)$ change in such a way that the level corresponding to the ground state of a "particle" with "mass" $\beta\kappa$ in the "field" $\beta V(\mathbf{u})$, disappears, a phase transition of the system takes place.

From Eq. (2.14) for the free energy one obtains easily all thermodynamic characteristics of the system. For the average energy $\Delta\epsilon$, which is equal to

$$\Delta\varepsilon = \partial\beta\Delta f/\partial\beta,$$

we get thus

$$\Delta\varepsilon = \begin{cases} dE_0/d\beta & \beta \geqslant \beta_c \\ 0 & \beta < \beta_c \end{cases}. \qquad (2.15)$$

For the specific heat $\Delta C = -\beta^2 d\Delta\epsilon/d\beta$ we have

$$\Delta C = \begin{cases} -\beta^2 d^2E_0/d\beta^2 & \beta \geqslant \beta_c \\ 0 & \beta < \beta_c \end{cases}. \qquad (2.16)$$

The vanishing of the bound state precedes the approach of the "free energy" $E_0(\beta)$ to zero. The way this approach takes place determines the properties of the thermodynamic characteristics of the system in the transition point. We have schematically indicated in Fig. 1 the temperature dependence of the free energy.

It is clear that if $\partial E_0(\beta)/\partial\beta$ tends to zero as β approaches β_c, the β-dependence of the average $\Delta\epsilon$ near β_c has the character shown in Fig. 2.

If now $\partial^2 E_0(\beta)/\partial\beta^2$ is discontinuous in the transition point β_c, we are dealing with a second-order transition. If, however, a higher-order derivative is discontinuous, we have a higher-order transition.

We note that the criterion for the existence of a sharp phase transition is, as can be seen from

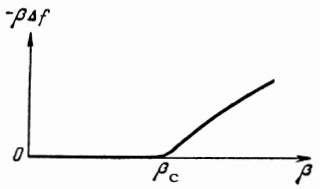

FIG. 1

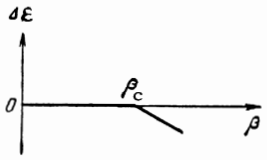

FIG. 2

Eq. (2.11), the inequality $L \gg |E_0|^{-1}$, where E_0 is the ground state "energy." We shall show below that the quantity $|E_0|^{-1}$ determines the largest correlation length of our system near the transition point. When we approach the phase transition point, the quantity E_0 tends to zero, and if the dimensions of the system are finite, the phase transition turns out to be smeared out in temperature, and the interval Δ of the smearing out is determined by the equation

$$|\Delta| \sim \beta_c/\sqrt{L|\Delta C|\beta_c}.$$

If L is large and if there is a second-order phase transition, it follows from this equation that

$$|\Delta| \sim \beta_c/\sqrt{L|\Delta C|\beta_c}.$$

We have thus shown that our system can undergo, from the thermodynamic point of view, a phase transition. Depending on the character of the "potential" $\beta V(\mathbf{u})$ which determines the nonlinearity of the system, the behavior in the transition point can have different forms.

3. CORRELATION FUNCTIONS

We now show that in the phase transition point the correlation functions of our system undergo qualitative changes. We introduce the correlation functions

$$w_m(\mathbf{u}_1x_1, \ldots, \mathbf{u}_mx_m) \equiv \frac{1}{N_m}\int \prod_{x=0}^{x=L} d\mathbf{u} \exp\left\{-\frac{\beta\varkappa}{2}\int_0^L dx\left(\frac{\partial\mathbf{u}}{\partial x}\right)^2 - \beta\int_0^L dx\, V(\mathbf{u})\right\} \prod_{i=1}^{m} \delta(\mathbf{u}(x_i) - \mathbf{u}_i). \qquad (3.1)$$

Here N_m is a normalizing factor,

$$L \geqslant x_1 \geqslant x_2 \geqslant \ldots \geqslant x_m \geqslant 0,$$

and the functional integration is performed under the conditions $\mathbf{u}(x = 0) = \mathbf{u}(x = L) = 0$.

One sees easily that

$$w_m(\mathbf{u}_1x_1, \ldots, \mathbf{u}_mx_m) = N_n^{-1} G(0L, \mathbf{u}_1x_1) G(\mathbf{u}_1x_1, \mathbf{u}_2x_2) \ldots G(\mathbf{u}_mx_m, 00), \qquad (3.2)$$

where G is the Green function of the "Hamiltonian" $h(\beta)$.

From the normalization condition

$$\int w_m(\mathbf{u}_1 x_1, \ldots, \mathbf{u}_m x_m)\, d\mathbf{u}_1 \ldots d\mathbf{u}_m = 1$$

it follows that

$$N_m = \int G(0L, \mathbf{u}_1 x_1) \ldots G(\mathbf{u}_m x_m, 00)\, d\mathbf{u}_1 \ldots d\mathbf{u}_m$$
$$= G(0L, 00). \tag{3.3}$$

We finally get

$$w_m(\mathbf{u}_1 x_1, \ldots, \mathbf{u}_m x_m)$$
$$= G(0L, \mathbf{u}_1 x_1) G(\mathbf{u}_1 x_1, \mathbf{u}_2 x_2) \ldots G(\mathbf{u}_{m-1} x_{m-1}, \mathbf{u}_m x_m)$$
$$\times G(\mathbf{u}_m x_m, 00) \,/\, G(0L, 00). \tag{3.4}$$

Let us consider the first correlation function

$$w_1(\mathbf{u}x) = G(0L, \mathbf{u}x) G(\mathbf{u}x, 00) \,/\, G(0L, 00),$$

or according to Eq. (2.8)

$$w_1(\mathbf{u}x) = \frac{\sum\limits_{\lambda} e^{-(L-x)E_\lambda} \psi_\lambda(0) \psi_\lambda^*(\mathbf{u}) \sum\limits_{\lambda'} e^{-xE_{\lambda'}} \psi_{\lambda'}(\mathbf{u}) \psi_{\lambda'}^*(0)}{\sum\limits_{\lambda} e^{-LE_\lambda} \psi_\lambda(0) \psi_\lambda^*(0)}. \tag{3.5}$$

We first of all evaluate w_1 for the case of a "free" field when $V = 0$. In that case

$$\psi = (2\pi)^{-n/2} e^{i\mathbf{k}}, \quad E_{\mathbf{k}} = \mathbf{k}^2 / 2\beta\varkappa,$$
$$w_1(\mathbf{u}x) = \left\{\frac{2\pi}{\beta\varkappa}\left(1 - \frac{x}{L}\right)x\right\}^{-n/2} \exp\left\{-\frac{\beta\varkappa \mathbf{u}^2}{2} \frac{1}{(1 - x/L)x}\right\}. \tag{3.6}$$

We are led to a Gaussian distribution with a dispersion

$$\delta = \left[\frac{T}{\varkappa} x\left(1 - \frac{x}{L}\right)\right]^{1/2}. \tag{3.7}$$

As one should expect, the dispersion increases with increasing temperature, and if we treat the field $\mathbf{u}$ as the displacement field of a string, δ increases when we get away from the ends of the string. It is clear that

$$\langle \mathbf{u}^2(x) \rangle = n \frac{T}{\varkappa} x\left(1 - \frac{x}{L}\right)$$

(n: number of components of the field $\mathbf{u}$). The maximum of $\langle \mathbf{u}^2(x) \rangle$ is equal to $nTL/4\varkappa$, and as $L \to \infty$, $\langle \mathbf{u}^2 \rangle_{\max} \to \infty$.

Let us now consider that range of temperatures where the "Hamiltonian" $h(\beta)$ has bound states. In that case

$$w_1(\mathbf{u}x) = \left(\sum_k e^{-(L-x)E_k} \rho_k(0, \mathbf{u}) + \int_\lambda^c e^{-(L-x)E_\lambda} \rho_\lambda(0, \mathbf{u})\right)$$
$$\times \left(\sum_k e^{-xE_k} \rho_k(\mathbf{u}, 0) + \int_\lambda^c e^{-xE_\lambda} \rho_\lambda(\mathbf{u}, 0)\right) \left\{\sum_k e^{-LE_k} \rho_k(0, 0)\right.$$
$$\left. + \int_\lambda^c e^{-LE_\lambda} \rho_\lambda(0, 0)\right\}^{-1}, \tag{3.8}$$

where

$$\rho_\lambda(\mathbf{u}_1, \mathbf{u}_2) = \psi_\lambda(\mathbf{u}_1) \psi_\lambda^*(\mathbf{u}_2).$$

One sees easily that the asymptotic value of this expression as $L \to \infty$ and $L - x \to \infty$ is

$$w_1(\mathbf{u}, x) = e^{xE_0} \frac{\rho_0(0, \mathbf{u})}{\rho_0(0, 0)} G(\mathbf{u}x, 00) \qquad (E_0 < 0).$$

As $x \to \infty$

$$G(\mathbf{u}x, 00) = e^{-xE_0} \rho_0(\mathbf{u}, 0),$$

and hence

$$w_1(\mathbf{u}x) = \psi_0(\mathbf{u}) \psi_0(\mathbf{u}), \tag{3.9}$$

where $\psi_0(\mathbf{u})$ is the wave function of the ground state.

In contradistinction of the free field case ($V = 0$), the mean square deviation

$$\langle \mathbf{u}^2(x) \rangle = \int w_1(\mathbf{u}x) \mathbf{u}^2 d\mathbf{u}$$

turns thus out to be finite as $x \to \infty$ and equal to the mean square of the radius of the bound state,

$$r_0^2 = \int \psi_0(\mathbf{u}) \mathbf{u}^2 \psi_0(\mathbf{u})\, d\mathbf{u}.$$

When we approach the phase transition point, the energy of the ground state tends to zero and the radius to infinity and in the transition point itself when the bound state disappears, $\langle \mathbf{u}^2(x) \rangle \to \infty$ as $x \to \infty$.

Above the transition point, the function $w_1(\mathbf{u}x)$ is, of course, not the same as (3.6), but it retains its basic property that its dispersion increases with increasing x.

Let us now turn to the evaluation of the second - the "two-particle" - distribution function $w_2(\mathbf{u}_1 x_1, \mathbf{u}_2 x_2)$. It is well known that the phase transition is connected with a rearrangement of correlations. We shall now show that in our model such a rearrangement takes place.

From Eq. (3.5) we have

$$w_2(\mathbf{u}_1 x_1, \mathbf{u}_2 x_2) = G(0L, \mathbf{u}_1 x_1) G(\mathbf{u}_1 x_1, \mathbf{u}_2 x_2)$$
$$\times G(\mathbf{u}_2 x_2, 00) / G(0L, 00) =$$
$$\frac{\int e^{-(L-x_1)E_\lambda} \rho_\lambda(0, \mathbf{u}_1) \int e^{-(x_1 - x_2)E_{\lambda'}} \rho_{\lambda'}(\mathbf{u}_1, \mathbf{u}_2) \sum\limits_{\lambda''} e^{-x_2 E_{\lambda''}} \rho_{\lambda''}(\mathbf{u}_2, 0)}{\sum\limits_{\lambda} e^{-LE_\lambda} \rho_\lambda(0, 0)} \tag{3.10}$$

($L \geq x_1 \geq x_2 \geq 0$).[4]

[4] If $x_2 > x_1$, one sees easily that

$$w_2(\mathbf{u}_1 x_1, \mathbf{u}_2 x_2) = \frac{G(0L, \mathbf{u}_2 x_2) G(\mathbf{u}_2 x_2, \mathbf{u}_1 x_1) G(\mathbf{u}_1 x_1, 00)}{G(0L, 00)} = w_2(\mathbf{u}_2 x_2, \mathbf{u}_1 x_1).$$

The same rule also holds for the general expression (3.4) for w_n: if $x_i > x_{i-1}, x_{i-2}, \ldots, x_{i-k}$, we have

$$\tilde{w}_m(\mathbf{u}_1 x_1, \ldots, \mathbf{u}_{i-k} x_{i-k}, \ldots, \mathbf{u}_{i-1} x_{i-1}, \mathbf{u}_i x_i, \mathbf{u}_{i+1} x_{i+1}, \ldots, \mathbf{u}_m x_m)$$
$$= w_m(\mathbf{u}_1 x_1, \ldots, \mathbf{u}_i x_i, \mathbf{u}_{i-k} x_{i-k}, \ldots, \mathbf{u}_{i-1} x_{i-1}, \mathbf{u}_{i+1} x_{i+1}, \ldots, \mathbf{u}_m x_m).$$

We shall be interested, apart from the second correlation function, also in the conditional probability $w(\mathbf{u}_1 x_1 | \mathbf{u}_2 x_2)$ that there is a displacement $\mathbf{u}_1$ in the point x_1 when there is a displacement $\mathbf{u}_2$ in the point x_2.

It is clear that

$$w(\mathbf{u}_1 x_1 | \mathbf{u}_2 x_2) = w_2(\mathbf{u}_1 x_1, \mathbf{u}_2 x_2) / w_1(\mathbf{u}_2 x_2). \quad (3.11)$$

Comparing Eqs. (3.11), (3.10), and (3.5), we get

$$w(\mathbf{u}_1 x_1 | \mathbf{u}_2 x_2) = \frac{G(0L, \mathbf{u}_1 x_1)}{G(0L, \mathbf{u}_2 x_2)} G(\mathbf{u}_1 x_1, \mathbf{u}_2 x_2). \quad (3.12)$$

For the "free"-field case ($V = 0$) we get by using (3.12) in the limit as $L \to \infty$

$$w(\mathbf{u}_1 x_1 | \mathbf{u}_2 x_2) = \left\{ \frac{2\pi}{\beta\varkappa} |x_1 - x_2| \right\}^{-n/2} \times \exp\left\{ -\frac{\beta\varkappa}{2} \frac{(\mathbf{u}_1 - \mathbf{u}_2)^2}{|x_1 - x_2|} \right\} \quad (3.13)$$

It follows from this expression that as $|x_1 - x_2| \to \infty$,

$$w(\mathbf{u}_1 x_1 | \mathbf{u}_2 x_2) \sim |x_1 - x_2|^{-n/2} \to 0. \quad (3.14)$$

We see that in the free field case the correlation between displacements $\mathbf{u}$ in different points decreases with distance.

One shows easily that a similar situation occurs also in the case when $V \neq 0$ but when the operator $h(\beta) = \mathbf{p}^2/2\beta\kappa + \beta V(\mathbf{u})$ has only a continuous spectrum.

A completely different situation arises when the operator $h(\beta)$ has bound states. We get from Eq. (3.12) in the limit as $L \to \infty$

$$w(\mathbf{u}_1 x_1 | \mathbf{u}_2 x_2) = \frac{\psi_0(\mathbf{u}_1)}{\psi_0(\mathbf{u}_2)} \Big[\psi_0(\mathbf{u}_1)\psi_0(\mathbf{u}_2) + \sum_{k \neq 0} e^{-|x_1 - x_2|(E_k - E_0)} \psi_k(\mathbf{u}_1)\psi_k(\mathbf{u}_2) + e^{|x_1 - x_2|E_0} \int_\lambda^c e^{-|x_1 - x_2|E_\lambda} \psi_\lambda(\mathbf{u}_1)\psi_\lambda^*(\mathbf{u}_2) \Big]. \quad (3.15)$$

Here $E_k < 0$ are the energies of the bound states and the $\psi_k(\mathbf{u})$ the eigenfunctions of the operator $h(\beta)$ corresponding to them.

Comparing (3.15), (3.11), and (3.9) we see that

$$f_2(\mathbf{u}_1 x_1, \mathbf{u}_2 x_2) \equiv w_2(\mathbf{u}_1 x_1, \mathbf{u}_2 x_2) - w_1(\mathbf{u}_1 x_1) w_2(\mathbf{u}_2 x_2) = \psi_0(\mathbf{u}_1)\psi_0(\mathbf{u}_2) \int_\lambda^1 e^{-|x_1 - x_2|(E_\lambda - E_0)} \psi_\lambda(\mathbf{u}_2)\psi_\lambda^*(\mathbf{u}_1), \quad (3.16)$$

where the symbol $\int_\lambda^1$ indicates a summation over all bound states, bar the ground state, and integration over the continuous spectrum.

The function $f_2(\mathbf{u}_1 x_1, \mathbf{u}_2 x_2)$ characterizes the correlation of the fluctuations of the field $\mathbf{u}$. One sees easily from Eqs. (3.15) and (3.16) that the largest characteristic correlation length of the fluctuations is $|E_0 - E_1|^{-1}$ where E_1 is the energy of the first excited state. If the Hamiltonian has only one bound state, one uses $|E_0|^{-1}$ for the characteristic length.

As $|x_1 - x_2| \to \infty$ the second and third terms in the square brackets in (3.15) decrease exponentially, and

$$w(\mathbf{u}_1 x_1 | \mathbf{u}_2 x_2) = \psi_0(\mathbf{u}_1)\psi_0(\mathbf{u}_1). \quad (3.17)$$

We have also

$$w_2(\mathbf{u}_1 x_1, \mathbf{u}_2 x_2) \xrightarrow[|x_1 - x_2| \to \infty]{} (\psi_0(\mathbf{u}_1))^2 (\psi_0(\mathbf{u}_2))^2 = w_1(\mathbf{u}_1 x_1) w_1(\mathbf{u}_2 x_2). \quad (3.17')$$

We see that at temperatures below the critical T_c the conditional probability $w(\mathbf{u}_1 x_1 | \mathbf{u}_2 x_2)$ and the second distribution function $w_2(\mathbf{u}_1 x_1, \mathbf{u}_2 x_2)$ do not vanish as $|x_1 - x_2| \to \infty$ (as would be the case for $T > T_c$) but tend to a finite limit, determined by Eqs. (3.17) and (3.17'): long-range order occurs in the system.

Equation (3.17) is also noteworthy because it shows the possibility for an asymptotic decoupling (as $|x_1 - x_2| \to \infty$) of the two-particle distribution function as a product of single-particle ones. This is a particular example of a display of the principle of the weakening of correlations. [3]

It is clear from Eq. (3.15) that the condition that Eqs. (3.17) and (3.17') are satisfied can be written as

$$|x_1 - x_2| \gg |E_0 - E_1|^{-1}, \quad (3.18)$$

where $E_0 - E_1$ is the difference in energy of the ground state and the first excited state of the "Hamiltonian" $h(\beta)$. When β approaches β_c the quantities E_0 and E_1 decrease, and in some region near β_c, where the Hamiltonian $h(\beta)$ has only one bound state with energy E_0, $|E_0|^{-1}$ can be used as the characteristic correlation length. In the transition point itself, where $E_0 = 0$, quadratic fluctuations determined by the function w_2 extend over the whole volume of the system. The higher-order fluctuations behave similarly, since the quantity $|E_0|^{-1}$ plays also for those the role of a characteristic length.

4. CONCLUSION

If we turn to the example of the string, the character of the phase transition studied here becomes clear. Let us consider a short section of the string. This section is acted upon by the po-

tential $V(u)$. At the same time it experiences the action of the portions of the string which adjoin it on the right and on the left. These portions "pull" the section considered, tending to "draw it away" from the potential well. At low temperatures when this action is comparatively weak, the potential $V(u)$ plays the main role, and the oscillations of the separate portion of the string are restricted by this potential: $\langle u^2 \rangle_{max} < \infty$. However, when the temperature is increased, the role of the thermal action increases and at a certain critical temperature the section is "pulled out" of the potential: $\langle u^2 \rangle_{max}$ becomes infinite. This corresponds to a phase transition. From this point of view one can also easily understand the character of the higher-order correlation functions.

The problem may arise of how valid is the subtraction of one infinite quantity (F_0) from another one (F). The divergence of the thermodynamic characteristics of a free string is connected with the fact that there are infinitely many degrees of freedom per unit length. The same situation occurs also in the case of a string in a field $V(u)$, but here the correlation properties of the system below the transition point differ fundamentally from those for the case of a free string or those above the transition point. Since the phase transition considered is determined by the rearrangement of the correlation properties of the system, while the quantities ΔF, $\Delta\epsilon$, and so on, which are obtained as the differences of diverging quantities are just connected with this rearrangement, the procedure adopted here for removing the infinities seems meaningful to us.

Moreover, one can get rid of the above-mentioned divergence by introducing a cut-off in momentum. The free energy of the system then has the form

$$F = F_0 + \Delta F,$$

where the free energy of the string without a field, F_0, is already finite, since the number of degrees of freedom per unit length is proportional to p_0, where p_0 is the cut-off momentum. As far as ΔF is concerned, in the immediate vicinity of the transition point (and that is just the region of interest) the wave function corresponding to the ground state is such that the average radius of the bound state r_0 is much larger than the reciprocal of the cut-off momentum, $1/p_0$, and the parameter $p_0 r_0$ is the larger, the smaller the quantity $\beta - \beta_c$. All results given in the foregoing, obtained in fact under the assumption $p_0 r_0 \to \infty$, remain thus valid.

One must note that the analogy between the partition function of a string and the Green function corresponding to the Hamiltonian $h(\beta)$ has a simple meaning. Writing the Green function as a continuous integral means integrating the exponent over all paths connecting the points u_1 and u_2. The evaluation of the partition function of a string is simply reduced to an integration over all shapes of the latter under the conditions that its end points have displacements u_1 and u_2 which is equivalent to an integration over paths, if we understand "time" as the coordinate of a point of the string and the coordinate of the "particle" as the displacement of the string.

We have shown thus that the system considered by us can undergo a phase transition and this transition is characterized by singularities in the thermodynamic functions and a qualitative change in the correlation functions, consisting in the appearance of long-range order.

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