THE LIE EQUATIONS FOR THE ROTATION GROUP

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The Lie equations for the three-dimensional rotation group are converted by introduction of additional variables into a set of three uncoupled equations for the determination of a single unknown function. The equations are solved by standard methods and the generating function for the representation coefficients is obtained. The same method is then applied to the Lorentz group and it is shown that the theory of finite dimensional representations follows from elementary properties of differential equations.

1. INTRODUCTION

THE infinitesimal transformations method of Lie has been widely used for the determination of representations of continuous groups. If N is the dimension of the representation, α_i $(i = 1, \ldots, n)$ —the parameters of the group, and I_i the infinitesimal operators, then the Lie method yields a system of equations of first order of the form:

$$\frac{\partial u_r}{\partial \alpha_i} = \sum_{j,s} S_{ji}(\alpha_1, \ldots, \alpha_n)(I_j)_{rs} u_s$$

$$(i, j = 1, \ldots, n; \quad r, s = 1, \ldots, N) \tag{1}$$

for the determination of the functions u_r, which are directly related to the representation coefficients.

Although the derivation of Eq. (1) is not complicated, considerable difficulties are encountered in the solution of these coupled equations, because the functions S_{ji} (if the parameters were chosen in an unfortunate manner) are extremely complicated and, moreover, the number of functions to be determined increases with the dimension of the representation. For these reasons the representations are usually obtained by other means and not by solving these complicated equations.

In this article we show that for the case of two continuous groups with which every physicist is familiar—namely the three dimensional rotations group O_3 and the proper Lorentz group L_p—the Lie equations can be put into a very convenient form and they can in fact be solved if the matrices I_i are replaced by certain analytic operators introduced previously by the author.[1] In the case of O_3 these matrices have the form

$$I_k = -iM_k, \qquad M_1 = J\cos\lambda - \sin\lambda\frac{\partial}{\partial\lambda},$$

$$M_2 = J\sin\lambda + \cos\lambda\frac{\partial}{\partial\lambda}, \quad M_3 = -i\frac{\partial}{\partial\lambda}. \tag{2}$$

It is assumed that the new operators act on the function u constructed out of the functions u_r according to the rule

$$u = u_J e^{iJ\lambda} + u_{J-1}e^{i(J-1)\lambda} + \ldots + u_{-J}e^{-iJ\lambda}. \tag{3}$$

Thus, by introducing an additional variable λ, the Lie equations for O_3 are reduced to a sequence of three uncoupled equations for the determination of the single function u. These equations take on a particularly simple form if one chooses for the rotation parameters the Euler angles; the equations may then be easily solved using the standard theory of equations of first order. In the more general case, when the parameters are not explicit, it can be shown by an analysis of the structure of the equations that if $u^{(1/2)}$ is a solution for $J = \frac{1}{2}$ then $[u^{(1/2)}]^{2J}$ is a solution for an arbitrary (integer or half integer) value of J. This makes it possible to construct representations of higher dimensions from the two-dimensional representation. This second method is applied to the Lorentz group.[2]

2. SOLUTION IN THE CASE OF O_3

Let the matrix $T(g)$, corresponding to the element g of the group, act on an arbitrary vector a and transform it into the vector $u(g) = T(g)a$. It follows from the definition of the representation that $u(f) = T(fg^{-1})u(g)$. By differentiation of this equation with respect to $\alpha_i(f)$ we obtain the equation

$$\frac{\partial u(f)}{\partial \alpha_i(f)} = \sum_j \frac{\partial T(fg^{-1})}{\partial \gamma_j(fg^{-1})} \frac{\partial \gamma_j(fg^{-1})}{\partial \alpha_i(f)} u(g),$$

where $\gamma_i(fg^{-1})$ are the parameters of the element fg^{-1}. If $\alpha_1, \alpha_2, \ldots$ and $\beta_1, \beta_2, \ldots$ are respectively the parameters of the elements f and g^{-1} then γ_j may be written in the form $\gamma_j(\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots)$. Setting in the last equation g = f we obtain the Lie equations:[2]

$$\frac{\partial u}{\partial \alpha_i} = \sum_j S_{ji} I_j u, \qquad S_{ji} = \left[\frac{\partial \gamma_j(fg^{-1})}{\partial \alpha_i(f)}\right]_{g=f}. \tag{4}$$

The symbol u in these equations is used for convenience and stands for a vector, and not for the function u of Eq. (3).

It is convenient to discuss these equations after performing the above-mentioned reduction. In what follows we assume that this has been done. A particular solution of the equations is obtained if all components of the initial vector a, except for the m-th component, are set equal to zero. The general solution will be a linear superposition of 2J + 1 particular solutions of this type with arbitrary coefficients.

For one of these particular solutions, say the m-th, the initial condition is of the form $u(0, 0, 0) = e^{im\lambda}$. It is clear that if only one of the equations is considered then this initial condition is not sufficient to determine the solution. To eliminate from the solution all arbitrary functions it is necessary to take into account all three equations, one after another, in the following manner. In the first equation we set $\alpha_i = 0$ for $i \neq 1$. The initial condition then determines $u(\alpha_1, 0, 0)$ as a function of α_1 and λ. We then take the second equation and set $\alpha_i = 0$ for $i \neq 1$, $i \neq 2$. The function $u(\alpha_1, 0, 0)$ serves as the initial condition for this equation and determines $u(\alpha_1, \alpha_2, 0)$ as a function of $\alpha_1, \alpha_2, \lambda$. Continuing in this fashion we obtain the complete solution, free of any arbitrariness, provided that at no stage of the process does the initial data fall into the eigen manifold. Although strictly speaking the above considerations apply to the group O_3, the general features of the initial value problem are the same in the case of an arbitrary continuous group.

Following these general remarks we pass to the solution of Eqs. (4) in the case of O_3 by choosing the Euler angles θ, ψ, φ as the rotation parameters. The main difficulty in carrying out the program consists in the calculation of the functions S_{ji}. Instead of evaluating them directly as functions of θ, ψ, φ we use a somewhat indirect procedure and calculate them first as functions of three symmetric parameters that are often used in the literature. They are the components $\alpha_1, \alpha_2, \alpha_3$ of a vector in the direction of the axis of rotation and whose length is equal to the angle of rotation. From the definition

$$S_{j1} = \lim_{\varepsilon \to 0} \varepsilon^{-1} \gamma_j(\alpha_1 + \varepsilon, \alpha_2, \alpha_3; -\alpha_1, -\alpha_2, -\alpha_3)$$

we obtain after some calculations

$$\begin{aligned} S_{ii} &= e_i^2 + (1 - e_i^2) w^{-1} \sin w, \\ S_{ij} &= e_i e_j (1 - w^{-1} \sin w) - e_k w^{-1} (1 - \cos w), \\ S_{ji} &= e_i e_j (1 - w^{-1} \sin w) + e_k w^{-1} (1 - \cos w) \end{aligned} \tag{5}$$

(i, j, k—cyclic), where e_1, e_2, e_3 are the direction cosines of the axis of rotation, and w is the angle of rotation.

The Lie equations for O_3 can now be written out explicitly and the first important result that follows from them is

$$\sum_i \alpha_i \left(\frac{\partial}{\partial \alpha_i} - I_i\right) u = 0. \tag{6}$$

This relation has a simple geometrical meaning and substantially simplifies the calculations.

To solve the equations it is necessary to return to the variables θ, ψ, φ, which are connected to the homogeneous Euler parameters

$$\zeta_i = e_i \sin \frac{w}{2} \quad (i = 1, 2, 3), \qquad \chi = \cos \frac{w}{2}$$

by the relations

$$i\zeta_1 + \zeta_2 = \sin \frac{\theta}{2} e^{i(\psi - \varphi)/2}, \qquad i\zeta_3 + \chi = \cos \frac{\theta}{2} e^{i(\psi + \varphi)/2}.$$

After the change of variables the equations assume the form

$$\partial u / \partial \varphi = -\partial u / \partial \lambda, \tag{7a}$$

$$\frac{\partial u}{\partial \psi} = -i \sin \theta \left[J u \cos(\lambda - \varphi) - \sin(\lambda - \varphi) \frac{\partial u}{\partial \lambda}\right] - \cos \theta \frac{\partial u}{\partial \lambda}, \tag{7b}$$

$$\left(\frac{\partial u}{\partial \theta}\right)_{\psi = \varphi = 0} = -iJ u \sin \lambda - i \cos \lambda \frac{\partial u}{\partial \lambda}. \tag{7c}$$

We solve these equations by starting with the last one. The differential equations for the characteristics of Eq. (7c) have the form

$$i d\theta = \frac{d\lambda}{\cos \lambda} = -\frac{du}{J u \sin \lambda},$$

for which two independent integrals are

$$u \varkappa^J (\varkappa^2 + 1)^{-J} = A, \qquad \frac{\varkappa + i}{x - i} e^{-i\theta} = B.$$

Consequently the general solution is

$$u(\theta,0,0) = \varkappa^{-J}(\varkappa^2+1)^J f\left(\frac{\varkappa+i}{\varkappa-i}e^{-i\theta}\right)$$

where f is an arbitrary function, $\kappa = e^{i\lambda}$, and $\psi = \varphi = 0$. The form of f is determined by the initial condition $u(0, 0, 0) = \kappa^m$, so that

$$u(\theta, 0, 0) = \varkappa^{-J}\left(-\varkappa\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^{J-m}$$
$$\times\left(\varkappa\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)^{J+m}. \qquad (8)$$

This will serve as the initial condition for the Eq. (7b) if φ is set there equal to zero. By standard techniques[1)] we find

$$u(\theta,\psi,0) = e^{-im\psi}u(\theta,0,0). \qquad (9)$$

Equation (7a) presents no problems. It simply states that u is a function of $\lambda - \varphi$. Thus the solution of the initial value problem for Eqs. (7a)—(7c) is

$$u(\theta,\psi,\varphi) = e^{-im\psi}(\varkappa e^{-i\varphi})^{-J}\left(-\varkappa e^{-i\varphi}\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^{J-m}$$
$$\times\left(\varkappa e^{-i\varphi}\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)^{J+m}. \qquad (10)$$

The function $u(\theta, \psi, \varphi)$ may be looked upon as the generating function for the coefficients of the representation. Indeed, if it is expanded in powers of κ then the coefficient of $\kappa^{m'}$, multiplied by the quantity

$$[(J-m')!(J+m')!]^{1/2}[(J-m)!(J+m)!]^{-1/2}, \qquad (11)$$

gives the matrix element $D_{m'm}{}^J$ of the representation. The necessity of multiplying by the above factor is due to the change in basis, connected with the use of the operators (2).

3. ANOTHER METHOD OF SOLUTION FOR O_3 AND ITS GENERALIZATION TO THE LORENTZ GROUP

In order to successfully apply the above method to the group L_p it is necessary to find a set of parameters in terms of which the Lie equations can be easily solved. Since the author does not know whether such a set of parameters exists in the case of L_p, it is useful to formulate the theory in such a manner that knowledge of specific parameters is not needed. This is not hard to accomplish by dividing Eqs. (4) by u. The function ln u then satisfies the inhomogeneous equations (k = 1, 2, 3):

$$i\frac{\partial \ln u}{\partial \alpha_k} + (S_{1k}\sin\lambda - S_{2k}\cos\lambda + iS_{3k})\frac{\partial \ln u}{\partial\lambda}$$
$$= J(S_{1k}\cos\lambda + S_{2k}\sin\lambda). \qquad (12)$$

From here it is seen that if $u^{(1/2)}$ is a solution of the equation for $J = 1/2$, then $u^{(J)} = [u^{(1/2)}]^{2J}$. This conclusion, clearly, does not depend on the choice of the parameters. This then makes it possible to construct higher dimensionality representations from the two-dimensional representation $D^{1/2}$ in the following manner.

Let the matrices $D^{1/2}$ be of the form $\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}$ and let them operate on an arbitrary vector $\begin{pmatrix}\xi\\ \eta\end{pmatrix}$. According to our rule the function $u^{(1/2)}$ should be taken in the form

$$u^{(1/2)} = e^{i\lambda/2}(\alpha\xi+\beta\eta) + e^{-i\lambda/2}(\gamma\xi+\delta\eta),$$

and, according to the remark just made, we have

$$u^{(J)} = [e^{i\lambda/2}(\alpha\xi+\beta\eta) + e^{-i\lambda/2}(\gamma\xi+\delta\eta)]^{2J}. \qquad (13)$$

From here it is easy to obtain the matrices D^J if the initial vector is known. Setting in Eq. (13) $\alpha = \delta = 1$, $\beta = \gamma = 0$, we see that the components of the initial vector a are

$$a_m = \xi^{J+m}\eta^{J-m}(2j)!\,/\,[(J+m)!(J-m)!],$$

which up to factors coincides with the quantities obtained from the standard theory.

It is interesting to see how the two approaches are related. If we expand $u^{(J)}$ in a power series of the form

$$\sum_{m,m'}^{j} e^{i\lambda m'}a_m D_{m'm},$$

we obtain

$$D^J_{m'm} = \sum_s \frac{(J+m)!(J-m)!}{s!(J+m-s)!(J-m'-s)!(m'-m+s)!}$$
$$\times\, \alpha^{J+m-s}\delta^{J-m'-s}\beta^{m'-m+s}\gamma^s.$$

After multiplication by the factor (11) this quantity coincides with the conventional expression for it. It is easy to see that the procedure may be inverted and the representation $D^{1/2}$ may be obtained from D^1 by extraction of the square root of the homogeneous quadratic form in ξ and η.

The above described method can be directly generalized to the case of the group L_p. Although none of the results to follow depend on the choice of the parameters it is inconvenient to keep them

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[1)] The general solution of (7b) for $\varphi = 0$ is

$$u(\theta,\psi,0) = \varkappa^{-J}\left(\varkappa-\cot\frac{\theta}{2}\right)^J\left(\varkappa+\tan\frac{\theta}{2}\right)^J$$
$$\times\, g\left(\frac{\varkappa+\tan(\theta/2)}{\varkappa-\cot(\theta/2)}e^{-i\psi}\right),$$

where the arbitrary function g may depend on θ as a parameter.

arbitrary from the very beginning. We therefore choose six parameters α_i, the first three of which are the projections of the "rotation vector" of the Lorentz frame O′ relative to the frame O, and the last three of which coincide up to factors of 1/c with the components of the velocity of the frame O′ with respect to the frame O. If the I_i ($i = 1, \ldots, 6$) stand for the corresponding infinitesimal operators then, as is known, we may form the six linear combinations

$$A_k^{(1)} = -{}^1/_2(I_k + iI_{3+k}),$$

$$A_k^{(2)} = -{}^1/_2(I_k - iI_{3+k}) \quad (k = 1,\ 2,\ 3),$$

which satisfy the commutation relations

$$[A_j^{(1)}, A_k^{(1)}] = A_l^{(1)}, \qquad [A_j^{(2)}, A_k^{(2)}] = A_l^{(2)}$$

(j, k, l—cyclic), with any of the $A^{(1)}$ commuting with any of the $A^{(2)}$. Therefore the matrices $A^{(1)}$, $A^{(2)}$ may be replaced by the operators (2); it is only necessary to use two different variables λ_1, λ_2 and two values of J in the construction of $M_k^{(1)}$ and $M_k^{(2)}$. The Lie equations for L_p then take the form

$$\partial u / \partial \alpha_i = [\sigma_{ki} M_k^{(1)} - \sigma_{ki}^* (-1)^k M_k^{(2)}] u,$$

$$\sigma_{ki} = iS_{ki} + S_{3+k,\, i}. \tag{14}$$

The solution of these equations will be denoted by $u^{(J_1 J_2)}(\lambda_1, \lambda_2)$. If one replaces λ_1, λ_2 by $-\lambda_1$, $-\lambda_2$ and goes over to the complex conjugate of Eq. (14) one sees easily that the quantity $u^{(J_1J_2)*}(-\lambda_1, -\lambda_2)$ satisfies the same equation with the operators $M_k^{(1)}$ and $M_k^{(2)}$ interchanged. Since the original function is unchanged by this operation it follows that the representation $D^{J_2J_1}$ is equivalent to the complex conjugate of $D^{J_1J_2}$. This result is well known.

Further, dividing Eq. (14) by u and thus converting it into a system of inhomogeneous equations

$$\frac{\partial \ln u}{\partial \alpha_i} + (\sigma_{1i} \sin \lambda_1 - \sigma_{2i} \cos \lambda_1 + i\sigma_{3i}) \frac{\partial \ln u}{\partial \lambda_1}$$
$$+ (\sigma_{1i}^* \sin \lambda_2 + \sigma_{2i}^* \cos \lambda_2 + i\sigma_{3i}^*) \frac{\partial \ln u}{\partial \lambda_2}$$
$$= J_1(\sigma_{1i} \cos \lambda_1 + \sigma_{2i} \sin \lambda_1) + J_2(\sigma_{1i}^* \cos \lambda_2 - \sigma_{2i}^* \sin \lambda_2),$$

we see that

$$u^{(J_1J_2)} = [u^{(^1/_2 0)}]^{2J_1} [u^{(0\, ^1/_2)}]^{2J_2}.$$

Consequently representations of higher dimensionality may be obtained from $D^{1/2\ 0}$ and $D^{0\ 1/2}$ by the method used in the case of O_3.

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[1] S. D. Majumdar, Proc. Phys. Soc. 72, 635 (1958).
[2] B. L. van der Waerden, Gruppentheoretische Methode in der Quantenmechanik, Julius Springer, Berline, 1932, pp. 62-63, 78-87.

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