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THE KINETICS OF WAVES IN A WEAKLY TURBULENT PLASMA

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A kinetic equation for waves in a weakly turbulent plasma is derived in a form that yields simple symmetry relations for various terms of the equation. Some consequences of these relations are discussed. It is shown, in particular, that in several cases the "number" of waves (quasiparticles) is conserved during "non-decay" interaction.

1. INTRODUCTION

At present there are many papers devoted to the nonlinear interaction of waves in a plasma. There are two different approaches to this problem. One is purely dynamic, without using an averaging procedure in any of the intermediate stages (for example [1]). Such a method is convenient in investigations of the nonlinear interaction of a finite number of waves. In a weakly turbulent plasma, it is necessary to go over from the dynamic description to the statistical one, i.e., to resort to averaging over some statistical ensemble during one stage or another. The principles of the statistical approach were first used in the quasilinear theory of waves in a plasma [2,3]. The derivation of the kinetic equations for waves in a weakly turbulent plasma and their application to various concrete cases have been the subject of many papers [3-4]. However, these equations are too cumbersome to investigate in sufficiently general cases, and concrete applications have as a rule the character of more or less crude estimates.

In this paper we derive the kinetic equation for the waves in a form which, in our opinion, is quite convenient for a general investigation and for concrete applications, and whose individual terms admit of sufficiently simple interpretation. It makes it therefore possible to obtain several symmetry relations for the kernel of the kinetic equation, from which follow certain conservation laws which in many cases facilitate the investigation of wave kinetics in a weakly turbulent plasma. In particular, it turns out that if "decays" of the waves are impossible (i.e., $\omega_{k'} + \omega_{k''} \neq \omega_{k'+k''}$), then under certain conditions, which are satisfied in most cases of interest, the nonlinear interaction cannot lead to a change in the total number of waves.

For convenience in exposition we first consider in detail the derivation of the kinetic equations for waves and of the symmetry relations in the cases of potential oscillations (Secs. 2 and 3). In Sec. 4 we consider the conservation laws that follow from the symmetry relations. It is shown in Sec. 5 that all the symmetry relations and the corresponding conservation laws obtained for potential oscillations are valid also in the general case of oscillations with arbitrary polarization. By way of illustration we consider several examples: the conservation laws for nonlinear interaction of potential oscillations in a plasma without a magnetic field and in a plasma with a magnetic field in the presence of longitudinal current (Sec. 4), and the interaction between plasmons and photons in a plasma without a magnetic field (Sec. 6).

2. KINETIC EQUATION FOR WAVES (POTENTIAL OSCILLATIONS)

We consider first the case of potential oscillations in a plasma. The fundamental equations for such oscillations are of the form

$$\mathbf{E} = -\mathrm{grad}\,\varphi, \quad \partial\mathbf{E}/\partial t = -4\pi\mathbf{J}. \tag{2.1}$$

The polarization current density vector $\mathbf{J}$, with allowance for the terms nonlinear in $\mathbf{E}$, can be represented in the form

$$\mathbf{J} = \mathbf{J}^{(1)}\{\mathbf{E}\} + \mathbf{J}^{(2)}\{\mathbf{E}\} + \mathbf{J}^{(3)}\{\mathbf{E}\} + \ldots, \tag{2.2}$$

where $\mathbf{J}^{(n)}\{\mathbf{E}\}$ is some functional of the electric field, of order n.

The currents $\mathbf{J}^{(n)}\{\mathbf{E}\}$ can be expressed in terms of corresponding increments to the plasma particle distribution function. To determine these quantities we start from the kinetic equation for the particle distribution function, which we find expedient to write in the form

$$\partial f_j/\partial t + [\mathcal{H}_j, f_j] = -[\mathcal{H}_j^{int}, f_j], \quad (2.3)$$

where $\mathcal{H}_j$—Hamiltonian of the plasma in the absence of oscillations; $\mathcal{H}_j^{int}$—part of the Hamiltonian describing the interaction of the particles with the field of the waves:

$$\mathcal{H}_j = \frac{1}{2m_j}\left(\mathbf{p} - \frac{e_j}{c}\mathbf{A}_0\right)^2, \quad \mathbf{H}_0 = \text{rot}\,\mathbf{A}_0, \quad (2.4)^*$$

$$\mathcal{H}_j^{int} = \int \rho_j(\mathbf{r})\,\varphi(\mathbf{r})\,d\mathbf{r}, \quad \rho_j(\mathbf{r}) = e_j\delta(\mathbf{r} - \mathbf{r}_j(t)), \quad (2.5)$$

where $\mathbf{H}_0$—intensity of the stationary magnetic field; j = e, i (e—electrons, i—ions). This subscript will be left out from now on if there is no summation with respect to j; $\mathbf{r}(t)$—radius vector of the particle at the instant of time t. We neglect collisions between particles, so that there is no collision integral in (2.3).

Going over to the Lagrangian variables $\mathbf{r}_0$ and $\mathbf{p}_0$ corresponding to the motion of the particle and an external stationary magnetic field, we obtain in place of (2.3) $\partial f/\partial t = -[\mathcal{H}^{int}, f]$; from this we get for the n-th order increment to the distribution function

$$f^{(n)}(\mathbf{r}_0, \mathbf{p}_0; t) = (-1)^n \int_{-\infty}^{t} dt_1 \ldots \int_{-\infty}^{t_{n-1}} dt_n$$
$$\times [\mathcal{H}^{int}(t_1) \ldots [\mathcal{H}^{int}(t_n), f^0] \ldots], \quad (2.6)$$

where $f^0 = f^0(\mathbf{r}_0, \mathbf{p}_0)$—unperturbed distribution function. With the aid of formula (2.6) we can determine the polarization currents $\mathbf{J}^{(n)}$:

$$\mathbf{J}^{(n)}(r, t) = (-1)^n \sum_j n_j \int d\mathbf{r}_1 \ldots d\mathbf{r}_n \int_{-\infty}^{t} dt_1 \ldots \int_{-\infty}^{t_{n-1}} dt_n$$
$$\times \langle [\ldots [\mathbf{v}\rho_j(\mathbf{r}, t), \rho_j(\mathbf{r}_1, t_1)] \ldots \rho_j(\mathbf{r}_n, t_n)]\, f_j^0 \rangle, \quad (2.7)$$

where n_j—particle-number density, and the angle brackets denote integration over the Lagrangian variables $\mathbf{r}_0$ and, $\mathbf{p}_0$ of the particle, while the square brackets are Poisson brackets with respect to these variables:

$$[F, \Phi] = \frac{\partial F}{\partial \mathbf{p}_0}\frac{\partial \Phi}{\partial \mathbf{r}_0} - \frac{\partial F}{\partial \mathbf{r}_0}\frac{\partial \Phi}{\partial \mathbf{p}_0}. \quad (2.8)$$

It follows from (2.7) that the connection between the Fourier components of the n-th order currents and the Fourier components of the potential can be represented by

$$4\pi\mathbf{J}^{(n)}(\mathbf{k}, \omega) = \frac{\omega k}{(2\pi)^{n-1}k^2} \sum_{\mathbf{k}_1+\ldots+\mathbf{k}_n=\mathbf{k}} \int d\omega_1 \ldots d\omega_n \delta(\omega - \omega_1 - \ldots - \omega_n)$$
$$\times \mu^{(n)}(\mathbf{k}, \omega;\, \mathbf{k}_1, \omega_1;\, \ldots;\, \mathbf{k}_n, \omega_n)\,\varphi(\mathbf{k}_1, \omega_1) \ldots \varphi(\mathbf{k}_n, \omega_n), \quad (2.9)$$

where[1])

$$\varphi(\mathbf{k}, \omega) = \int d\mathbf{r} \int_{-\infty}^{\infty} dt\,\varphi(\mathbf{r}, t)\, e^{-i(\mathbf{kr}-\omega t)}, \quad (2.10)$$

$$\mu^{(n)}(\mathbf{k}, \omega;\, \mathbf{k}_1, \omega_1;\, \ldots;\, \mathbf{k}_n, \omega_n)$$
$$= \frac{1}{n!}\sum \mathcal{P} \int d\mathbf{r}_1 \ldots d\mathbf{r}_n \int_{-\infty}^{0} dt_1 \ldots \int_{-\infty}^{t_{n-1}} dt_n$$
$$\times \psi^{(n)}(\mathbf{r}_1, t_1;\, \ldots;\, \mathbf{r}_n, t_n)\exp\left[i\sum_{l=1}^{n}(\mathbf{k}_l\mathbf{r}_l - \omega_l t_l)\right],$$
$$\mathbf{k} = \mathbf{k}_1 + \ldots + \mathbf{k}_n, \quad \omega = \omega_1 + \ldots + \omega_n. \quad (2.11)$$

In (2.11) $\Sigma\mathcal{P}$ denotes the sum over all possible permutations of the wave numbers $\mathbf{k}_1, \ldots, \mathbf{k}_n$ (and accordingly the frequencies). The quantity $\psi^{(n)}(\mathbf{r}_1, t_1, \ldots \mathbf{r}_n, t_n)$ is defined by the formula

$$\psi(\mathbf{r}_1 - \mathbf{r},\, t_1 - t;\, \ldots;\, \mathbf{r}_n - \mathbf{r},\, t_n - t) = (-1)^n 4\pi \sum_j n_j$$
$$\times \langle [\ldots[\rho_j(\mathbf{r}, t),\, \rho_j(\mathbf{r}_1, t_1)] \ldots \rho_j(\mathbf{r}_n, t_n)] f_j^0 \rangle. \quad (2.12)$$

Following [15], we shall call the quantities $\mu(\mathbf{k}, \omega, \ldots, \mathbf{k}_n, \omega_n)$ the n-th order responses. For convenience we have included among the arguments of the responses the wave vectors $\mathbf{k}$ and the frequencies ω, and it must be assumed throughout that the responses $\mu(\mathbf{k}, \omega;\, \mathbf{k}_1, \omega_1;\, \ldots;\, \mathbf{k}_n, \omega_n)$ differ from zero only if

$$\mathbf{k} = \sum_{l=1}^{n} \mathbf{k}_l, \quad \omega = \sum_{l=1}^{n} \omega_l.$$

Going over to the Fourier components in (2.1), we obtain the dynamic equation for the waves with account of the nonlinear effects up to third order inclusive:

$$k\omega\varepsilon(\mathbf{k}, \omega)\varphi(\mathbf{k}, \omega) = 4\pi\mathbf{J}^{(2)}(\mathbf{k}, \omega) + 4\pi\mathbf{J}^{(3)}(\mathbf{k}, \omega), \quad (2.13)$$

where $\mathbf{J}^{(2)}$ and $\mathbf{J}^{(3)}$ are determined from (2.9), and $\epsilon(\mathbf{k}, \omega)$ is the dielectric constant of the plasma for longitudinal oscillations:

$$\varepsilon(\mathbf{k}, \omega) = 1 - \mu^{(1)}(\mathbf{k}, \omega)\,/\,k^2. \quad (2.14)$$

We shall solve (2.13) by successive approximations, choosing for the first approximation the solution of the linearized equation

$$\varepsilon(\mathbf{k}, \omega)\varphi(\mathbf{k}, \omega) = 0. \quad (2.15)$$

If the dispersion equation $\epsilon(\mathbf{k}, \omega) = 0$ has real roots $\omega_{\mathbf{k}}$, then the solution of (2.15) takes the form $\varphi^{(1)}(\mathbf{k}, \omega) = 2\pi\varphi_{\mathbf{k}}\delta(\omega - \omega_{\mathbf{k}})$. In the presence of absorption or instability, $\omega_{\mathbf{k}}$ is complex. In this case the solution of (2.15) can be represented in the form

$$\varphi^{(1)}(\mathbf{k}, \omega) = 2\pi\varphi_{\mathbf{k}}\Delta(\omega - \omega_{\mathbf{k}}), \quad (2.16)$$

1)The normalization volume is set equal to unity everywhere.

*rot = curl.

$$\Delta(\omega-\omega_k)=\frac{1}{2\pi i}\left(\frac{1}{\omega-\omega_k-i\delta}-\frac{1}{\omega-\omega_k+i\delta}\right), \quad (2.16a)$$

where δ is a symbol indicating the rule for going around the poles when integrating the expression in (2.16a) with respect to ω: in the first term in the parentheses the contour circles from below, and in the second from above, regardless of the sign[2)] of Im ω_k. The time-dependent solution of (2.15) can be represented in the form

$$\varphi^{(1)}(\mathbf{k},t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi^{(1)}(\mathbf{k},\omega)e^{-i\omega t}d\omega,$$

which, taking into account the rules for going around the poles in $\varphi^{(1)}(\mathbf{k},\omega)$ leads to $\varphi^{(1)}(\mathbf{k},t) = \varphi_k\exp(-i\omega_k t)$ for all values of t.

Using (2.16), we obtain for the second and third approximations an expression of the form

$$\varphi^{(2)}(\mathbf{k},\omega)=\frac{1}{2\pi k^2\varepsilon(\mathbf{k},\omega)}\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}}\int d\omega' d\omega''\,\delta(\omega-\omega'-\omega'')$$
$$\times\mu(\mathbf{k},\omega;\mathbf{k}'\omega';\mathbf{k}'',\omega'')\varphi^{(1)}(\mathbf{k}',\omega')\varphi^{(1)}(\mathbf{k}'',\omega''), \quad (2.17)$$

$$\varphi^{(3)}(\mathbf{k},\omega)=\frac{1}{(2\pi)^2k^2\varepsilon(\mathbf{k},\omega)}\sum_{\mathbf{k}'+\mathbf{k}''+\mathbf{k}'''=\mathbf{k}}\int d\omega' d\omega'' d\omega'''$$
$$\times\delta(\omega-\omega'-\omega''-\omega''')$$
$$\times\left\{2\sum_{\mathbf{q}}\int\frac{d\Omega\delta(\omega-\omega'-\Omega)}{q^2\varepsilon(\mathbf{q},\Omega)}\mu(\mathbf{k},\omega;\mathbf{k}',\omega';\mathbf{q},\Omega)\right.$$
$$\times\mu(\mathbf{q},\Omega;\mathbf{k}'',\omega'';\mathbf{k}''',\omega''')$$
$$\left.+\mu(\mathbf{k},\omega;\mathbf{k}',\omega';\mathbf{k}'',\omega'';\mathbf{k}''',\omega''')\right\}$$
$$\times\varphi^{(1)}(\mathbf{k}',\omega')\,\varphi^{(1)}(\mathbf{k}'',\omega'')\,\varphi^{(1)}(\mathbf{k}''',\omega'''). \quad (2.18)$$

We now calculate $d\overline{|\varphi(\mathbf{k},t)|^2}/dt$, confining ourselves to terms up to fourth order in φ_k. The bar denotes averaging over the phases of the initial amplitudes ω_k that enter in the first approximation (2.16). These phases, as in [6,7], are assumed to have random distributions for different **k**.

Representing $\varphi(\mathbf{k},t)$ in the form of a Fourier integral, we obtain

$$\frac{d}{dt}\overline{|\varphi(\mathbf{k},t)|^2}=\frac{1}{(2\pi)^2}\mathrm{Im}\int d\omega\,d\omega'(\omega-\omega')$$
$$\times\overline{\varphi(\mathbf{k},\omega)\varphi^*(\mathbf{k},\omega')}e^{-i(\omega-\omega')t}=2\gamma_k|\varphi_k|^2$$
$$+\frac{1}{(2\pi)^2}\mathrm{Im}\int d\omega\,d\omega'(\omega-\omega')\,[2\overline{\varphi^{(3)}(\mathbf{k},\omega^{(1)})\varphi^{(1)*}(\mathbf{k},\omega')}$$
$$+\overline{\varphi^{(2)}(\mathbf{k},\omega)\varphi^{(2)*}(\mathbf{k},\omega')}]\,e^{-i(\omega-\omega')t}, \quad (2.19)$$

where $\gamma_k = \mathrm{Im}\,\omega_k$—linear increment or decrement of the wave. In terms of fourth order in φ we are justified[3)] in neglecting the imaginary part of ω_k if $|\gamma_k| \ll |\omega_k|$. In this case $\Delta(\omega-\omega_k)$ in (2.16) is replaced by a δ-function. Substituting in (2.19) the values of $\varphi^{(2)}(\mathbf{k},\omega)$ and $\varphi^{(3)}(\mathbf{k},\omega)$ from (2.17) and (2.18), we obtain the kinetic equation for the waves in the form

$$\frac{d}{dt}\overline{|\varphi(\mathbf{k},t)|^2}=2\gamma_k|\varphi_k|^2+\frac{1}{k^2\varepsilon_k'}$$
$$\times\left\{\mathrm{Im}\sum_{\mathbf{k}'}\left[8\int\frac{d\omega\delta(\omega_k-\omega_{k'}-\omega)}{|\mathbf{k}-\mathbf{k}'|^2\varepsilon(\mathbf{k}-\mathbf{k}',\omega)}\right.\right.$$
$$\times\mu(\mathbf{k},\omega_k;\mathbf{k}',\omega_{k'};\mathbf{k}-\mathbf{k}',\omega)$$
$$\times\mu(\mathbf{k}-\mathbf{k}',\omega;\mathbf{k},\omega_k;-\mathbf{k}',-\omega_{k'})$$
$$\left.+6\mu(\mathbf{k},\omega_k;\mathbf{k}',\omega_{k'};\mathbf{k},\omega_k;-\mathbf{k}',-\omega_{k'})\right]|\varphi_k|^2|\varphi_{k'}|^2$$
$$+\frac{4\pi}{\varepsilon_k'}\sum_{\mathbf{k}',\mathbf{k}''}|\mu(\mathbf{k},\omega_k;\mathbf{k}',\omega_{k'};\mathbf{k}'',\omega_{k''})|^2|\varphi_{k'}|^2|\varphi_{k''}|^2$$
$$\left.\times\delta(\omega_k-\omega_{k'}-\omega_{k''})\right\}, \quad (2.20)$$

where

$$\varepsilon_k'=\frac{\partial}{\partial\omega}\varepsilon(\mathbf{k},\omega)_{\omega=\omega_k}.$$

It must be noted in connection with (2.20) that the entire deviation presented above was a formal expansion in powers of the oscillation field, corresponding to usual perturbation theory. However, such an expansion contains divergences that are eliminated by a certain renormalization, corresponding to the transition to the nonlinear theory. Thus, in the expression for $\mu(\mathbf{k},\omega_k;\mathbf{k}',\omega_{k'};\mathbf{k}',\omega_{k'};-\mathbf{k}',-\omega_{k'})$ we encounter a diverging term proportional to

$$\int_{-\infty}^{0}dt'\int_{-\infty}^{t'}dt''\int_{-\infty}^{t''}dt'''\,\langle[[[\rho_{-k}(0),\rho_k(t')]\,\rho_{k'}(t'')]\,\rho_{-k'}(t''')]\,f^0\rangle$$
$$\times\exp\{-i\,[\omega_k t'+\omega_{k'}(t''-t''')]\}, \quad (2.21)$$

[$\rho_k(t)$ is the Fourier component of $\rho(\mathbf{r},t)$] and containing a Poisson bracket $[\rho_{-k}(0),\rho_k(t')]$ identical to the bracket in the first-order current. Analogous terms are included also in higher-order currents. It can be shown [16] that summation of an infinite series consisting of such terms is equivalent to quasilinear renormalization of the first-order current, namely to replacing in (2.12) the unperturbed distribution function f^0 by a slowly-varying distribution function $f^0(t)$. This leads to a corresponding renormalization of

2)It is convenient to regard δ as a certain function of ω, different from zero only in an arbitrarily small interval near the point $\omega = \mathrm{Re}\,\omega_k$, at which point $\delta > |\mathrm{Im}\,\omega_k|$. Then the integration in (2.16) can be carried out along the real axis.

3)This means that we neglect terms of order $\gamma_k\tau^{-1}/\omega_k^2$, where τ—characteristic time of variation of the energy of the wave as a result of the nonlinear interaction.

$\epsilon(k, \omega)$ and replacement of the linear increment by a quasilinear one. Thus, all terms of the type (2.21) will henceforth be omitted, and the distribution function f^0, and accordingly $\epsilon(k, \omega)$, will be assumed the same as in the quasilinear theory[4)]. The complete equation for the background distribution function, with account of both the quasilinear renormalization and the wave interaction, can be found, for example, in [14,16] and will not be considered here.

We note now that the integrand with respect to ω in the right side of (2.20) has a pole at $\omega = \omega_{k''}$, $\mathbf{k}'' = \mathbf{k} - \mathbf{k}'$, since $\epsilon(\mathbf{k}'', \omega_{k''}) = 0$. During the integration this pole must be circuited from above. We can therefore write

$$\varepsilon^{-1}(\mathbf{k}'', \omega) = \mathrm{P}\frac{1}{\varepsilon(\mathbf{k}'', \omega)} - \pi i \frac{\delta(\omega - \omega_{\mathbf{k}''})}{\varepsilon'(\mathbf{k}'', \omega_{\mathbf{k}''})},$$

where P—symbol for the principal value.

It is further convenient to introduce in place of $|\varphi_k|^2$ the "number of waves" (quasiparticles) n_k, defined by the relation

$$n_{\mathbf{k}} = \frac{1}{8\pi}\omega_{\mathbf{k}}^{-1}\frac{\partial}{\partial\omega}[\omega\varepsilon(\mathbf{k}, \omega)]_{\omega=\omega_{\mathbf{k}}} k^2 |\varphi_{\mathbf{k}}|^2 = \frac{1}{8\pi}k^2\varepsilon_{\mathbf{k}}'|\varphi_{\mathbf{k}}|^2, \tag{2.22}$$

so that $n_k\omega_k$ is the spectral density of the oscillation energy. We introduce also the quantities

$$D(\mathbf{k}, \omega) = 8\pi\frac{\varepsilon(\mathbf{k}, \omega)}{|\varepsilon_{\mathbf{k}}'|}, \tag{2.23}$$

$$M_{\mathbf{k}',\mathbf{k}''}(\omega', \omega'') = (8\pi)^{3/2}\frac{\mu(\mathbf{k}, \omega; \mathbf{k}', \omega'; \mathbf{k}'', \omega'')}{|\mathbf{k}^2\varepsilon_{\mathbf{k}}'\mathbf{k}'^2\varepsilon'_{\mathbf{k}'}\mathbf{k}''^2\varepsilon_{\mathbf{k}''}'|^{1/2}}, \tag{2.24}$$

$$N_{\mathbf{k}\mathbf{k}'} = (8\pi)^2\frac{\mu(\mathbf{k}, \omega_{\mathbf{k}'}; \mathbf{k}', \omega_{\mathbf{k}'}; \mathbf{k}, \omega_{\mathbf{k}}; -\mathbf{k}', -\omega_{\mathbf{k}'})}{|\mathbf{k}^2\varepsilon_{\mathbf{k}}'\mathbf{k}'^2\varepsilon_{\mathbf{k}'}'|}. \tag{2.25}$$

The kinetic equation for the waves then takes the form

$$\frac{dn_{\mathbf{k}}}{dt} = 2\gamma_{\mathbf{k}}n_{\mathbf{k}} + \frac{1}{8\pi}$$
$$\times\Bigg\{\mathrm{Im}\sum_{\mathbf{k}'}\Bigg[8\mathrm{P}\frac{M_{\mathbf{k}',\mathbf{k}-\mathbf{k}'}(\omega_{\mathbf{k}'}, \omega_{\mathbf{k}}-\omega_{\mathbf{k}'})M_{-\mathbf{k}',\mathbf{k}}(-\omega_{\mathbf{k}'}, \omega_{\mathbf{k}})}{D(\mathbf{k}-\mathbf{k}', \omega_{\mathbf{k}}-\omega_{\mathbf{k}'})}$$
$$+ 6N_{\mathbf{k}\mathbf{k}'}\Bigg]n_{\mathbf{k}}n_{\mathbf{k}'} + \frac{1}{2}\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}}\Bigg[|M_{\mathbf{k}'\mathbf{k}''}(\omega_{\mathbf{k}'}, \omega_{\mathbf{k}''})|^2 n_{\mathbf{k}'}n_{\mathbf{k}''}$$
$$- 2\frac{\omega_{\mathbf{k}''}}{|\omega_{\mathbf{k}''}|}\mathrm{Re}\,M_{\mathbf{k}'\mathbf{k}''}(\omega_{\mathbf{k}'}, \omega_{\mathbf{k}''})M_{-\mathbf{k}'\mathbf{k}}(-\omega_{\mathbf{k}'}, \omega_{\mathbf{k}})n_{\mathbf{k}}n_{\mathbf{k}'}\Bigg]$$
$$\times\delta(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} - \omega_{\mathbf{k}''})\Bigg\}. \tag{2.26}$$

4)For this reason we have left out from formula (3.12) for $\mu(k, \omega_k; k', \omega_{k'}; k, \omega_k; -k', -\omega_{k'})$ the two terms with $\tilde{\mu}(k, \Omega; k, \Omega''; k', \Omega'; -k', \Omega''')$ and $\mu(k, \Omega; k, \Omega''; -k', \Omega'''; k', \Omega')$.

3. SYMMETRY RELATIONS FOR THE KINETIC EQUATION

The quantities $\psi^{(2)}$ and $\psi^{(3)}$ in (2.12) which determine the second- and third-order responses, satisfy definite symmetry relations which follow from the properties of Poisson brackets. These relations turn out to be quite useful in investigations of the kinetic equation (2.26) for the waves. Let us consider first the properties of the responses of the second order. From (2.12) we can easily obtain

$$\psi^{(2)}(\mathbf{r}', t'; \mathbf{r}'', t'') = -\psi^{(2)}(-\mathbf{r}', -t'; \mathbf{r}''-\mathbf{r}', t''-t'), \tag{3.1}$$

$$\psi^{(2)}(\mathbf{r}', t'; \mathbf{r}'', t'') + \psi^{(2)}(\mathbf{r}''-\mathbf{r}'; t''-t'; -\mathbf{r}', -t') + \psi^{(2)}(-\mathbf{r}'', -t''; \mathbf{r}'-\mathbf{r}''; t'-t'') = 0. \tag{3.2}$$

Equations (3.1) and (3.2) still do not lead directly to any relations for the quantity μ defined by (2.11), since the integration with respect to t is carried out in (2.11) along the semi-axis from $-\infty$ to 0. However, if we introduce the complete Fourier component of the function $\psi^{(2)}$:

$$\tilde{\mu}(\mathbf{k}, \Omega; \mathbf{k}', \Omega'; \mathbf{k}'', \Omega'') = \int d\mathbf{r}'d\mathbf{r}''\int_{-\infty}^{\infty}dt'dt''\psi(\mathbf{r}', t'; \mathbf{r}'', t'')$$
$$\times\exp i(\mathbf{k}'\mathbf{r}' + \mathbf{k}''\mathbf{r}'' - \Omega't' - \Omega''t''),$$
$$\mathbf{k} = \mathbf{k}' + \mathbf{k}'', \quad \Omega = \Omega' + \Omega'', \tag{3.3}$$

then we get for this component from (3.1) and (3.2)

$$\tilde{\mu}(\mathbf{k}, \Omega; \mathbf{k}', \Omega'; \mathbf{k}'', \Omega'') = -\mu(-\mathbf{k}', -\Omega'; -\mathbf{k}, -\Omega; \mathbf{k}'', \Omega''), \tag{3.4}$$

$$\tilde{\mu}(\mathbf{k}, \Omega; \mathbf{k}', \Omega'; \mathbf{k}'', \Omega'') + \tilde{\mu}(-\mathbf{k}', -\Omega'; \mathbf{k}'', \Omega''; -\mathbf{k}, -\Omega) + \tilde{\mu}(-\mathbf{k}'', -\Omega''; -\mathbf{k}, -\Omega; \mathbf{k}', \Omega') = 0. \tag{3.5}$$

The response $\mu^{(2)}$ of interest to us is connected with $\tilde{\mu}^{(2)}$ by the relation

$$\mu(\mathbf{k}, \omega; \mathbf{k}', \omega'; \mathbf{k}'', \omega'') = \frac{1}{2(2\pi i)^2}\int\frac{d\Omega'd\Omega''}{\omega - \Omega + i\varepsilon}$$
$$\times\left\{\frac{\tilde{\mu}(\mathbf{k}, \Omega; \mathbf{k}', \Omega'; \mathbf{k}'', \Omega'')}{\omega'' - \Omega'' + i\varepsilon} + \frac{\tilde{\mu}(\mathbf{k}, \Omega; \mathbf{k}'', \Omega''; \mathbf{k}', \Omega')}{\omega' - \Omega' + i\varepsilon}\right\}. \tag{3.6}$$

To clarify the meaning of the formula (3.6) we consider in greater detail the structure of the quantities $\tilde{\mu}(k, \Omega; k', \Omega'; k'', \Omega'')$. Substituting $\psi^{(2)}$ from (2.12) in (3.3) we obtain, after integrating with respect to $\mathbf{r}'$ and $\mathbf{r}''$,

$$\tilde{\mu}(\mathbf{k}, \Omega; \mathbf{k}', \Omega'; \mathbf{k}'', \Omega'') = \sum_j 4\pi e_j^3 n_j\int_{-\infty}^{\infty}dt'dt''$$

$$\times \langle [[\delta(\mathbf{r}_0), \exp i(\mathbf{k}'\mathbf{r}(t') - \Omega' t')] \times \exp i(\mathbf{k}''\mathbf{r}(t'') - \Omega'' t'')] f_j^0 \rangle. \quad (3.7)$$

Differentiation with respect to the Lagrangian variables $\mathbf{r}_0$ and $\mathbf{p}_0$ is implied in the Poisson brackets, and the angle brackets denote integration over these variables. In the presence of a constant magnetic field we must substitute for $\mathbf{r}(t)$

$$\mathbf{r}(t) = \mathbf{r}_0 + \int_0 \mathbf{v}(t')\,dt' = \mathbf{r}_0 + \frac{[\mathbf{h}\mathbf{v}(t)]}{\omega_H} - \frac{[\mathbf{h}\mathbf{v}_0]}{\omega_H} + \mathbf{h}(\mathbf{h}\mathbf{v}_0)t, \quad (3.8)^*$$

where ω_H—Larmor frequency (of the corresponding particles), and $\mathbf{h}$—unit vector directed along $\mathbf{H}_0$. After integrating with respect to t' and t'', we get in place of the exponentials in (3.7) δ-functions of the form $\delta(\Omega' - k_z' v_z - n'\omega_H)$ and $\delta(\Omega'' - k_z'' v_z - n''\omega_H)$, where n' and n'' are integers. It is important here that the δ-functions contain Ω and $\mathbf{k}$ with the same index. These δ-functions are acted upon by certain differential operators in the momenta $\mathbf{p}$. If there is no external magnetic field, the quantities $\tilde{\mu}$ assume a specially simple form. In this case $\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}t$ and it follows from (3.7) that

$$\tilde{\mu}(\mathbf{k}, \Omega; \mathbf{k}', \Omega'; \mathbf{k}'', \Omega'') = \sum_j \frac{\omega_{0j}^2 e_j}{m_j} \int d\mathbf{v}\,\delta(\Omega' - \mathbf{k}'\mathbf{v})\mathbf{k}\frac{\partial}{\partial \mathbf{v}}\left\{\delta(\Omega'' - \mathbf{k}''\mathbf{v})\mathbf{k}''\frac{\partial f}{\partial \mathbf{v}}\right\}, \quad (3.9)$$

where $\omega_{0j} = (4\pi e_j^2 n_j/m_j)^{1/2}$ is the Langmuir frequency. After substituting (3.9) in (3.6) and integrating with respect to Ω' and Ω'', we obtain an expression of the same type as in (3.9) except that in lieu of $\delta(\Omega - \mathbf{k}\mathbf{v})$ we have $\Omega - \mathbf{k}\cdot\mathbf{v} + i\varepsilon$.

It is clear from the foregoing that, for example in the expansion (3.6) for $\mu(\mathbf{k}' + \mathbf{k}''; \omega_{\mathbf{k}'} + \omega_{\mathbf{k}''}; \mathbf{k}', \omega_{\mathbf{k}'}; \mathbf{k}'', \omega_{\mathbf{k}''})$, the half-residues are due to the resonances between the oscillations and the particles of velocity

$$v = \frac{\omega_{\mathbf{k}'}}{k'}, \quad \frac{\omega_{\mathbf{k}''}}{k''}, \quad \frac{\omega_{\mathbf{k}'} + \omega_{\mathbf{k}''}}{|\mathbf{k}' + \mathbf{k}''|} \quad (\mathbf{H}_0 = 0), \quad (3.10)$$

$$v = \frac{\omega_{\mathbf{k}'} - n'\omega_H}{k_z'}, \quad \frac{\omega_{\mathbf{k}''} - n''\omega_H}{k_z''}, \quad \frac{\omega_{\mathbf{k}'} + \omega_{\mathbf{k}''} - m\omega_H}{k_z' + k_z''} \quad (\mathbf{H}_0 \neq 0). \quad (3.11)$$

The first two cases in (3.10) and (3.11) correspond to resonances between the natural oscillations (with frequencies $\omega_{\mathbf{k}'}$ and $\omega_{\mathbf{k}}''$) and the plasma particles, while the last case corresponds to the resonance between the forced oscillations (with frequency $\omega_{\mathbf{k}'} + \omega_{\mathbf{k}''}$) and the particles. In all the nonlinear terms we shall henceforth neglect the half-residues due to resonances with the natural frequencies. This is justified by the fact that these resonances make contributions already to the linear terms, and the corresponding linear contribution cannot compete with them.

For the third-order response we can also write a relation of the type (3.6)

$$\mu(\mathbf{k}, \omega_{\mathbf{k}}; \mathbf{k}', \omega_{\mathbf{k}'}; \mathbf{k}, \omega_{\mathbf{k}}; -\mathbf{k}', -\omega_{\mathbf{k}'}) = -\frac{1}{6(2\pi i)^3}\int \frac{d\Omega'\,d\Omega''\,d\Omega'''}{\omega_{\mathbf{k}} - \Omega + i\varepsilon}$$
$$\times \left\{ \frac{\tilde{\mu}(\mathbf{k}, \Omega; \mathbf{k}', \Omega'; \mathbf{k}, \Omega''; -\mathbf{k}', \Omega''')}{(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} - \Omega'' - \Omega''' + i\varepsilon)(-\omega_{\mathbf{k}'} - \Omega''' + i\varepsilon)} + \frac{\tilde{\mu}(\mathbf{k}, \Omega; \mathbf{k}', \Omega', -\mathbf{k}', \Omega'''; \mathbf{k}, \Omega'')}{(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} - \Omega'' - \Omega''' + i\varepsilon)(\omega_{\mathbf{k}} - \Omega'' + i\varepsilon)} \right.$$
$$\left. + \frac{\tilde{\mu}(\mathbf{k}, \Omega, -\mathbf{k}', \Omega'''; \mathbf{k}, \Omega'', \mathbf{k}', \Omega')}{(\omega_{\mathbf{k}} + \omega_{\mathbf{k}'} - \Omega'' - \Omega' + i\varepsilon)(\omega_{\mathbf{k}'} - \Omega' + i\varepsilon)} + \frac{\tilde{\mu}(\mathbf{k}, \Omega; -\mathbf{k}', \Omega'''; \mathbf{k}', \Omega'; \mathbf{k}, \Omega'')}{(\omega_{\mathbf{k}} + \omega_{\mathbf{k}'} - \Omega'' - \Omega' + i\varepsilon)(\omega_{\mathbf{k}} - \Omega'' + i\varepsilon)} \right\}, \quad (3.12)$$

$$\tilde{\mu}(\mathbf{k}, \Omega;\ \mathbf{k}', \Omega'; \mathbf{k}'', \Omega''; \mathbf{k}''', \Omega''') = \int d\mathbf{r}'\,d\mathbf{r}''\,d\mathbf{r}''' \int_{-\infty}^{\infty} dt'\,dt''\,dt'''$$
$$\times \psi(\mathbf{r}', t'; \mathbf{r}'', t''; \mathbf{r}''', t''')$$
$$\times \exp i(\mathbf{k}'\mathbf{r}' + \mathbf{k}''\mathbf{r}'' + \mathbf{k}'''\mathbf{r}''' - \Omega' t' - \Omega'' t'' - \Omega''' t'''),$$
$$\mathbf{k} = \mathbf{k}' + \mathbf{k}'' + \mathbf{k}''', \quad \Omega = \Omega' + \Omega'' + \Omega'''. \quad (3.13)$$

Expression (3.12) is written for the responses contained in the right side of the kinetic equation (2.26). We note further that in (3.12) we have left out two terms containing

$\tilde{\mu}(\mathbf{k}, \Omega;\ \mathbf{k}, \Omega'';\ \mathbf{k}', \Omega';\ -\mathbf{k}', \Omega''')$

and $\tilde{\mu}(\mathbf{k}, \Omega;\ \mathbf{k}', \Omega'';\ -\mathbf{k}', \Omega''';\ \mathbf{k}', \Omega')$

(see footnote 4). The quantities $\tilde{\mu}^{(3)}$ contained in (3.12) have symmetry properties analogous to (3.4) and (3.5) (see the appendix). Relations (3.6) and (3.12) will henceforth be called the spectral expansions of the second- and third-order responses.

The spectral expansions are useful in the investigation of the symmetry properties of the responses. As shown in the appendix, it follows from the properties of $\tilde{\mu}(\mathbf{k}, \Omega; \mathbf{k}', \Omega'; \mathbf{k}'', \Omega'')$ and (3.6) that we can obtain the following relation for the response $\mu(\mathbf{k}, \omega; \mathbf{k}', \omega'; \mathbf{k}'', \omega'')$

$$\mu(\mathbf{k}, \omega;\ \mathbf{k}', \omega';\ \mathbf{k}'', \omega'') = \mu^*(\mathbf{k}', \omega' - i0;\ \mathbf{k}, \omega - i0;\ -\mathbf{k}'', -\omega''). \quad (3.14)$$

The symbol $-i0$ following the frequencies in the right side of (3.14) denotes that in the corresponding spectral expansion we replace in the denomi-

*$[\mathbf{h}\ \mathbf{v}_0] = \mathbf{h} \times \mathbf{v}_0$.

nators containing these frequencies the imaginary additions, which determine the rule of going around the poles in integrating by their negative values.

We introduce the quantity

$$L_{k'k''}(\omega', \omega'') = (8\pi)^{3/2} \frac{\mu(\mathbf{k}, \omega - i0; \mathbf{k}', \omega'; \mathbf{k}'', \omega'')}{|k^2\varepsilon_k' k'^2 \varepsilon_{k'}' k''^2 \varepsilon_{k''}'|^{1/2}}. \quad (3.15)$$

It differs from $M_{k'k''}(\omega', \omega'')$ [see (2.24)] only in the sign of the imaginary addition to the frequency ω. It follows from (3.14) that

$$M_{\mathbf{k}-\mathbf{k}',\mathbf{k}'}(\omega_k - \omega_{k'}, \omega_{k'}) = L^*_{\mathbf{k},-\mathbf{k}'}(\omega_k, -\omega_{k'}) \quad (3.16)$$

(we recall that we have neglected in the nonlinear terms the half-residues due to resonances with the natural frequencies). We now consider the symmetry relations for the imaginary part of the response $\mu(\mathbf{k}, \omega_k; \mathbf{k}', \omega_{k'}; \mathbf{k}, \omega_k; -\mathbf{k}', -\omega_{k'})$. Putting $\omega_k > 0$ and $\omega_{k'} > 0$, we break it up into two parts:

$$\operatorname{Im}\mu(\mathbf{k}, \omega_k; \mathbf{k}', \omega_{k'}; \mathbf{k}, \omega_k; -\mathbf{k}', -\omega_{k'})$$
$$= \operatorname{Im}\mu^-(\mathbf{k}, \omega_k; \mathbf{k}', \omega_{k'}; \mathbf{k}, \omega_k; -\mathbf{k}', -\omega_{k'})$$
$$+ \operatorname{Im}\mu^+(\mathbf{k}, \omega_k; \mathbf{k}', \omega_{k'}; \mathbf{k}, \omega_k; -\mathbf{k}', -\omega_{k'}), \quad (3.17)$$

$$\mu^-(\mathbf{k}, \omega_k; \mathbf{k}', \omega_{k'}; \mathbf{k}, \omega_k; -\mathbf{k}', -\omega_{k'})$$
$$= -\frac{i}{6(2\pi i)^3} \int \frac{d\Omega' d\Omega'' d\Omega'''}{(\omega_k - \Omega + i\varepsilon)(\omega_k - \omega_{k'} - \Omega'' - \Omega''' + i\varepsilon)}$$
$$\times \left\{ \frac{\operatorname{Im}\widetilde{\mu}(\mathbf{k}, \Omega; \mathbf{k}', \Omega'; \mathbf{k}, \Omega''; -\mathbf{k}', \Omega''')}{-\omega_{k'} - \Omega''' + i\varepsilon} + \frac{\operatorname{Im}\widetilde{\mu}(\mathbf{k}, \Omega; \mathbf{k}', \Omega'; -\mathbf{k}', \Omega'''; \mathbf{k}, \Omega'')}{\omega_k - \Omega'' + i\varepsilon} \right\}, \quad (3.18)$$

$$\mu^+(\mathbf{k}, \omega_k; \mathbf{k}', \omega_{k'}; \mathbf{k}, \omega_k; -\mathbf{k}', -\omega_{k'})$$
$$= -\frac{i}{6(2\pi i)^3} \int \frac{d\Omega' d\Omega'' d\Omega'''}{(\omega_k - \Omega + i\varepsilon)(\omega_k + \omega_{k'} - \Omega'' - \Omega' + i\varepsilon)}$$
$$\times \left\{ \frac{\operatorname{Im}\widetilde{\mu}(\mathbf{k}, \Omega; -\mathbf{k}', \Omega'''; \mathbf{k}, \Omega''; \mathbf{k}', \Omega')}{\omega_{k'} - \Omega' + i\varepsilon} + \frac{\operatorname{Im}\widetilde{\mu}(\mathbf{k}, \Omega; -\mathbf{k}', \Omega''', \mathbf{k}', \Omega'; \mathbf{k}, \Omega'')}{\omega_k - \Omega'' + i\varepsilon} \right\}. \quad (3.19)$$

The quantities $\widetilde{\mu}^{(3)}$, in terms of which the response $\mu(\mathbf{k}, \omega_k; \mathbf{k}', \omega_{k'}; \mathbf{k}, \omega_k; -\mathbf{k}', -\omega_{k'})$ is expressed in (3.12), can have, generally speaking, nonvanishing imaginary and real parts. It turns out, however, that their real parts made no contribution[5)] to $\operatorname{Im}\mu(\mathbf{k}, \omega_k; \mathbf{k}', \omega_{k'}; \mathbf{k}, \omega_k; -\mathbf{k}', -\omega_{k'})$ (see the appendix). We have therefore replaced $\widetilde{\mu}^{(3)}$ everywhere in (3.18) and (3.19) by $i \operatorname{Im}\widetilde{\mu}^{(3)}$. We prove in the appendix that the third-order response satisfies the following symmetry relation:

$$\operatorname{Im}\mu^\pm(\mathbf{k}, \omega_k; \mathbf{k}', \omega_{k'}; \mathbf{k}, \omega_k; -\mathbf{k}', -\omega_{k'})$$
$$= \pm \operatorname{Im}\mu^\pm(\mathbf{k}', \omega_{k'}; \mathbf{k}, \omega_k; \mathbf{k}', \omega_{k'}; -\mathbf{k}, -\omega_k). \quad (3.20)$$

5)We are grateful to A. A. Galeev who called our attention to this.

4. CONSERVATION LAWS

We represent the kinetic equation (2.26) in the form

$$\frac{dn_k}{dt} = 2\gamma_k n_k + S\{n\} + R^-\{n\} + R^+\{n\}, \quad (4.1)$$

where

$$S\{n\} = \frac{1}{16\pi} \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} \Big[|M_{k'k''}(\omega_{k'}, \omega_{k''})|^2 n_{k'} n_{k''} - 2\frac{\omega_{k''}}{|\omega_{k''}|} \operatorname{Re} M_{k'k''}(\omega_{k'}, \omega_{k''}) \times M_{-k'k}(-\omega_{k'}, \omega_k) n_k n_{k'} \Big] \delta(\omega_k - \omega_{k'} - \omega_{k''}), \quad (4.2)$$

$$R^-\{n\} = \sum_{\mathbf{k}'} R^-_{kk'} n_k n_{k'}$$
$$= \frac{1}{\pi} \operatorname{Im} \sum_{\substack{\mathbf{k}' \\ \omega_{k'}>0}} \left[\mathrm{P} \frac{L^*_{-k'k}(-\omega_{k'}, \omega_k) M_{-k'k}(-\omega_{k'}, \omega_k)}{D(\mathbf{k}-\mathbf{k}', \omega_k - \omega_{k'})} + \frac{3}{2} N^-_{kk'} \right] n_k n_{k'}, \quad (4.3)$$

$$R^+\{n\} = \sum_{\mathbf{k}'} R^+_{kk'} n_k n_{k'} =$$
$$\frac{1}{\pi} \operatorname{Im} \sum_{\substack{\mathbf{k}' \\ \omega_{k'}>0}} \left[\mathrm{P} \frac{L^*_{k'k}(\omega_{k'}, \omega_k) M_{k'k}(\omega_{k'}, \omega_k)}{D(\mathbf{k}+\mathbf{k}', \omega_k + \omega_{k'})} + \frac{3}{2} N^+_{kk'} \right] n_k n_{k'}, \quad (4.4)$$

$$N^\pm_{kk'} = (8\pi)^2 \frac{\mu^\pm(\mathbf{k}, \omega_k; \mathbf{k}', \omega_{k'}; \mathbf{k}, \omega_k; -\mathbf{k}', -\omega_{k'})}{|k^2\varepsilon_k' k'^2 \varepsilon_{k'}'|}. \quad (4.5)$$

In (4.3) and (4.4) the quantities M are replaced by L, in accordance with (3.16). The summation in the right sides of (4.3) and (4.4) is carried out only over positive frequencies.

The term $S\{n\}$ describes the "decay" interaction of the waves, for which the following conditions are satisfied

$$\omega_k = \omega_{k'} + \omega_{k''}, \quad \mathbf{k} = \mathbf{k}' + \mathbf{k}''. \quad (4.6)$$

The wave interaction described by the terms $R^-\{n\}$ and $R^+\{n\}$ which, unlike that considered above, can be called "non-decaying," is due to the resonance interaction between the waves and the particles. In this case an important role is played by resonances with the forced oscillations at the frequencies $|\omega_k| - |\omega_{k'}|$, in $R^-\{n\}$ and at the frequencies $|\omega_k| + |\omega_{k'}|$ in $R^+\{n\}$. In most concrete cases the term $R^+\{n\}$ is considerably smaller than $R^-\{n\}$. Relation (3.20) can be rewritten in the form

$$\mathrm{Im}\, N^{\pm}_{\mathbf{kk}'} = \pm\, \mathrm{Im}\, N^{\pm}_{\mathbf{k}'\mathbf{k}}. \qquad (4.7)$$

Taking into account the obvious equalities

$$M^{*}_{\mathbf{k}'\mathbf{k}''}(\omega_{\mathbf{k}'}, \omega_{\mathbf{k}''}) = M_{-\mathbf{k}'-\mathbf{k}''}(-\omega_{\mathbf{k}'}, -\omega_{\mathbf{k}''}),$$
$$L^{*}_{\mathbf{k}'\mathbf{k}''}(\omega_{\mathbf{k}'}, \omega_{\mathbf{k}''}) = L_{-\mathbf{k}'-\mathbf{k}''}(-\omega_{\mathbf{k}'}, -\omega_{\mathbf{k}''}), \qquad (4.8)$$

we easily obtain

$$M_{-\mathbf{k}'\mathbf{k}}(-\omega_{\mathbf{k}'}, \omega_{\mathbf{k}}) = M^{*}_{-\mathbf{kk}'}(-\omega_{\mathbf{k}}, \omega_{\mathbf{k}'}),$$
$$L_{-\mathbf{k}',\mathbf{k}}(-\omega_{\mathbf{k}'}, \omega_{\mathbf{k}}) = L^{*}_{-\mathbf{kk}'}(-\omega_{\mathbf{k}}, \omega_{\mathbf{k}'}). \qquad (4.9)$$

It follows from (4.7) and (4.9) that the part $R^{-}_{\mathbf{kk}'}$ of the kernel of the kinetic equation is antisymmetrical, while the part $R^{+}_{\mathbf{kk}'}$ is symmetrical, i.e., $R^{-}_{\mathbf{kk}'} = -R^{-}_{\mathbf{k}'\mathbf{k}}$ and $R^{+}_{\mathbf{kk}'} = R^{+}_{\mathbf{k}'\mathbf{k}}$. If the "decay" conditions (4.6) cannot be satisfied, and the term $R^{+}\{n\}$ is negligibly small compared with $R^{-}\{n\}$, then the kinetic equation (4.1) takes on the form

$$\frac{dn_{\mathbf{k}}}{dt} = 2\gamma_{\mathbf{k}} n_{\mathbf{k}} + \sum_{\mathbf{k}'} R^{-}_{\mathbf{kk}'} n_{\mathbf{k}} n_{\mathbf{k}'}.$$

Owing to antisymmetry of $R^{-}_{\mathbf{kk}'}$ the change in the total number of quasiparticles is determined by the equation

$$\frac{d}{dt} \sum_{\mathbf{k}} n_{\mathbf{k}} = \sum_{\mathbf{k}} \gamma_{\mathbf{k}} n_{\mathbf{k}},$$

from which we see that in the case when the $\gamma_{\mathbf{k}}$ can be regarded as equal to zero (this can occur, for example, as a result of the particle distribution function acquiring a quasilinear "plateau" in the region of resonance with the natural vibration[2,3]), the total number of waves (quasiparticles) is conserved: $\Sigma_{\mathbf{k}} n_{\mathbf{k}} = \mathrm{const}$.

The quasiparticle conservation law leads to important corollaries. Let the spectrum of vibrations be such that their frequencies change little with variation of **k**. This takes place, for example, for electron Langmuir vibrations for which $\omega_{\mathbf{k}} = \omega_{0e}[1 + (3/2)(kr_D)^2]$, where $r_D = v_e/\omega_{0e}$, and $kr_D \ll 1$. As a result of the law of conservation of the number of quasiparticles, the total energy will be conserved in this case in the first nonvanishing approximation [accurate to $(kr_D)^2$], i.e., in this approximation the nonlinear interaction causes energy to be pumped over from one part of the spectrum to another. If the energy transfer is in this case from the shorter to the longer waves, then in the next higher approximation in $(kr_D)^2$ the nonlinear interaction leads to a net attenuation of the wave energy. On the other hand, if the waves are pumped over in the opposite direction, then we have in the next higher approximation an overall increase in the energy of the waves in the packet[6)]. Such a case is realized, for example, in the presence of currents in the plasma (i.e., when the electron motion relative to the ions exceeds some critical velocity).

Perfectly analogous corollaries follow from the quasiparticle conservation law also for the Drummond-Rosenbluth oscillations[17], which are excited when current flows along the magnetic field in a plasma in which $T_e \sim T_i$, and whose frequency is very close to the Larmor ion frequency.

In the case of ion-sound oscillations without a magnetic field, the dispersion equation takes the form

$$\omega_{\mathbf{k}}^2 = \mathbf{k}^2 T_e / m_i (1 + \mathbf{k}^2 r_D^2)$$

(we put for simplicity $T_i = 0$). The energy-pumping effect plays here the major role only when the wave frequencies are close to ω_{0i}, i.e., $kr_D \gg 1$. In the opposite case, generally speaking, the pumping effect and the total change in energy are of the same order of magnitude.

We now consider the case when $\gamma_{\mathbf{k}} > 0$ for all the waves present. From (4.1) it then follows that

$$\frac{d}{dt} \sum_{\mathbf{k}} n_{\mathbf{k}} > 0,$$

i.e., the linear damping alone cannot compensate for the wave growth due to linear instability, and a stationary state cannot be established in this case. The erroneous conclusions concerning the establishment of a stationary state in this case[8,9] are due to the fact that, owing to the cumbersome initial equations and derivations, no notice was taken of the antisymmetry of the kernel of the kinetic equation for the waves. It must be noted, however, that a stationary state can, in principle, be established if $\gamma_{\mathbf{k}} < 0$ for a fraction of the waves in the packet. (An investigation of the stationary state for several concrete examples is reported in [16].)

Let us consider, finally, another case when $2\gamma_{\mathbf{k}} n_{\mathbf{k}}$ and $R\{n\}$ can be neglected in (4.1), so that the principal role is assumed by the "decay" interaction of the waves (case of a "transparent" medium). Taking (3.16) into account, the "decay" term in the kinetic equation (4.1) can be rewritten in the form

$$S\{n\} = \frac{1}{16\pi} \sum \{ |M_{\mathbf{k}'\mathbf{k}''}(\omega_{\mathbf{k}'}, \omega_{\mathbf{k}''})|^2 (n_{\mathbf{k}'} n_{\mathbf{k}''} - n_{\mathbf{k}} n_{\mathbf{k}'} - n_{\mathbf{k}} n_{\mathbf{k}''})$$
$$\times \delta(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} - \omega_{\mathbf{k}''})\, \delta(\omega_{\mathbf{k}''} - \omega_{\mathbf{k}} - \omega_{\mathbf{k}'})\},$$
$$+ 2|M_{\mathbf{kk}'}(\omega_{\mathbf{k}}, \omega_{\mathbf{k}'})|^2 (n_{\mathbf{k}''} n_{\mathbf{k}} + n_{\mathbf{k}''} n_{\mathbf{k}'} - n_{\mathbf{k}} n_{\mathbf{k}'}) \qquad (4.10)$$

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6)This, however, must not be regarded as a nonlinear instability, since $\Sigma_{\mathbf{k}} n_{\mathbf{k}} = \mathrm{const}$.

(the summation is over the region $\mathbf{k}' + \mathbf{k}'' = \mathbf{k}$; $\omega_{\mathbf{k}'}, \omega_{\mathbf{k}''} > 0$). The decay part of the kinetic equation was obtained earlier in [4,6] for waves of the form (4.10) from the hydrodynamic equations of a magnetized plasma. We note that if $\hbar \to 0$ (4.10) has the same form as the right side of the kinetic equation for phonons in a solid.7) From (4.10) follow directly the energy and momentum conservation laws $\Sigma n_{\mathbf{k}}\omega_{\mathbf{k}} = \text{const}$ and $\Sigma n_{\mathbf{k}}\mathbf{k} = \text{const}$. The number of quasiparticles $\Sigma n_{\mathbf{k}}$, of course, is not conserved in this case.

In concluding this section we note that the form obtained here for the kinetic equation is very convenient for concrete applications, since the responses μ are relatively easy to calculate. (For more details see [16], where kinetic equations are considered for potential oscillations in a plasma without a magnetic field, and also for Drummond-Rosenbluth oscillations [17] arising in the presence of a longitudinal current in the magnetic field).

5. KINETIC EQUATION FOR WAVES WITH ARBITRARY POLARIZATION

Let us generalize the results obtained above to include the case of oscillations with arbitrary polarization. Let $\mathbf{A}_0$ be the vector potential of the external stationary field, and $\mathbf{A}$ the potential of the alternating field of the oscillations. We shall find it convenient to use a gauge in which the scalar potential is $\varphi = 0$, so that $\mathbf{E} = c^{-1}\partial\mathbf{A}/\partial t$ and $\mathbf{H} = \operatorname{curl}(\mathbf{A} + \mathbf{A}_0)$. From Maxwell's equations we obtain

$$\operatorname{rot}\operatorname{rot}\mathbf{A} + \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = \frac{4\pi}{c}\mathbf{J}. \qquad (5.1)^*$$

The interaction Hamiltonian $\mathcal{H}^{int}$, describing the interaction of the particles with the wave field, is of the form

$$\mathcal{H}^{int} = -\frac{e}{mc}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}_0\right)\mathbf{A} + \frac{e^2}{mc^2}\mathbf{A}^2, \qquad (5.2)$$

$\mathcal{H}^{int}$ is nonlinear in $\mathbf{A}$. In addition, the particle velocity expressed in terms of its momentum

$$\mathbf{v} = \frac{1}{m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}_0 - \frac{e}{c}\mathbf{A}\right),$$

depends not only on the vector potential of the external field, but also on the potential of the oscillation field. In this connection, the expression for the current in terms of $\mathbf{A}$ turns out to be quite cumbersome. Proceeding in the same manner as in the case of longitudinal oscillations, we obtain in the required order in $\mathbf{A}$ the following expressions for the contributions to the average current density

$$J_\alpha^{(1)}(\mathbf{r}, t) = \sum_j \frac{n_j}{c}\left\{\int d\mathbf{r}' \int_{-\infty}^{t} dt' \langle [j_\alpha(\mathbf{r}, t), j_\beta(\mathbf{r}', t')] f^0\rangle \times A_\beta(\mathbf{r}', t') - \frac{e_j^2}{m_j} A_\alpha(\mathbf{r}, t)\right\}, \qquad (5.3)$$

$$J_\alpha^{(2)}(\mathbf{r}, t) = \sum_j \frac{n_j}{c^2}\left\{\int d\mathbf{r}'d\mathbf{r}'' \int_{-\infty}^{t'} dt' \int_{-\infty}^{t'} dt'' \times \langle [j_\alpha(\mathbf{r}, t), j_\beta(\mathbf{r}', t)] j_\gamma(\mathbf{r}'', t'') f^0\rangle A_\beta(\mathbf{r}', t') A_\gamma(\mathbf{r}'', t'') - \frac{e_j}{2m_j}\int d\mathbf{r}' \int_{-\infty}^{t} dt' (\langle [j_\alpha(\mathbf{r}, t), \rho(\mathbf{r}', t')] f^0\rangle A^2(\mathbf{r}', t') + 2\langle [\rho(\mathbf{r}, t), j_\beta(\mathbf{r}', t')] f^0\rangle A_\alpha(\mathbf{r}, t) A_\beta(\mathbf{r}', t')\right\}, \qquad (5.4)$$

$$J_\alpha^{(3)}(\mathbf{r}, t) = \sum_j \frac{n_j}{c^3}\left\{\int d\mathbf{r}'d\mathbf{r}''d\mathbf{r}''' \int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt'' \int_{-\infty}^{t''} dt''' \times \langle [[[j_\alpha(\mathbf{r}, t), j_\beta(\mathbf{r}', t')] j_\gamma(\mathbf{r}'', t'')] j_\delta(\mathbf{r}''', t''')] f^0\rangle \times A_\beta(\mathbf{r}', t') A_\gamma(\mathbf{r}'', t'') \times A_\delta(\mathbf{r}''', t''') - \frac{e_j}{2m_j}\int d\mathbf{r}'d\mathbf{r}'' \int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt'' \times (\langle [[j_\alpha(\mathbf{r}, t), j_\beta(\mathbf{r}', t')]\rho(\mathbf{r}'', t'')] f^0\rangle A_\beta(\mathbf{r}', t') A^2(\mathbf{r}'', t'') + \langle [[j_\alpha(\mathbf{r}, t), \rho(\mathbf{r}', t')] j_\beta(\mathbf{r}'', t'')] f^0\rangle A^2(\mathbf{r}', t') A_\beta(\mathbf{r}'', t'') + \langle [[\rho(\mathbf{r}, t), j_\beta(\mathbf{r}', t')] j_\gamma(\mathbf{r}'', t'')] f^0\rangle A_\alpha(\mathbf{r}, t) A_\beta(\mathbf{r}', t') \times A_\gamma(\mathbf{r}'', t'')) + \frac{e_j^2}{2m_j^2}\int d\mathbf{r}' \int_{-\infty}^{t} dt' \langle [\rho(\mathbf{r}, t), \rho(\mathbf{r}', t')] f^0\rangle \times A_\alpha(\mathbf{r}, t) A^2(\mathbf{r}', t')\right\}, \qquad (5.5)$$

where

$$\mathbf{j}(\mathbf{r}, t) = \frac{e}{m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}_0\right)\delta(\mathbf{r} - \mathbf{r}(t)), \quad \rho(\mathbf{r}, t) = e\delta(\mathbf{r} - \mathbf{r}(t)),$$

$\mathbf{r}(t) = \mathbf{r}(\mathbf{r}_0, \mathbf{p}_0; t)$ is the law of motion of the quasiparticles in the external field; the angle brackets, as in (2.7), denote integration over the Lagrangian variables of the particles.

In the general case the connection between the Fourier components of the currents and the Fourier components of the vector potential can be represented in the form

$$\frac{4\pi}{c} J_\alpha^{(n)}(\mathbf{k}, \omega) = \frac{1}{(2\pi)^{n-1}} \sum_{\mathbf{k}_1 + \ldots + \mathbf{k}_n = \mathbf{k}} \int d\omega_1 \ldots d\omega_n \times \delta(\omega - \omega_1 - \ldots - \omega_n)\, \varkappa_{\alpha\alpha_1\ldots\alpha_n}(\mathbf{k}, \omega; \mathbf{k}_1, \omega_1; \ldots; \mathbf{k}_n, \omega_n) \times A_{\alpha_1}(\mathbf{k}_1, \omega_1) \ldots A_{\alpha_n}(\mathbf{k}_n, \omega_n). \qquad (5.6)$$

7)A quantum approach to the derivation of the decay part of the kinetic equation for waves in a plasma was considered by the Vedenov[18].

*rot = curl.

The quantities κ, like μ, will be called responses and we assume that they differ from zero if

$$\mathbf{k} = \sum_{l=1}^{n} \mathbf{k}_l, \quad \omega = \sum_{l=1}^{n} \omega_l.$$

In deriving the kinetic equation for oscillations with arbitrary polarization we encounter, unlike in the case of longitudinal oscillations, a complication, due to the fact that the initial equation (5.1) is a vector equation. Taking the Fourier transform and confining ourselves to currents up to third order in A, we obtain from (5.1)

$$\left[\frac{\omega^2}{c^2}\varepsilon_{\alpha\beta}(\mathbf{k},\omega) + k_\alpha k_\beta - \mathbf{k}^2\delta_{\alpha\beta}\right]A_\beta(\mathbf{k},\omega) = -\frac{4\pi}{c}[J_\alpha^{(2)}(\mathbf{k},\omega) + J_\alpha^{(3)}(\mathbf{k},\omega)], \quad \varepsilon_{\alpha\beta} = \delta_{\alpha\beta} + \frac{c^2}{\omega^2}\varkappa_{\alpha\beta}. \tag{5.7}$$

We shall solve (5.7) by the method of successive approximations. For the first approximation $\mathbf{A}^{(1)}(\mathbf{k}, \omega)$ we choose the solution of the linearized equation

$$\mathbf{A}^{(1)}(\mathbf{k},\omega) = 2\pi c_{\mathbf{k}}{}^r \mathbf{a}^r(\mathbf{k})\,\Delta(\omega - \omega_{\mathbf{k}}{}^r), \tag{5.8}$$

where $\Delta(\omega - \omega_k)$ is defined by (2.16a), and $\mathbf{a}^r(\mathbf{k})$ are polarization vectors satisfying the linearized equation when $\omega = \omega_k^r$; ω_k^r—natural frequencies of the oscillations, c_k^r—scalar amplitudes, with the index r denoting polarization. The natural frequencies ω_k^r can be determined from the dispersion equation which is obtained by setting the determinant of the left side of (5.7) equal to zero.

As before, we confine ourselves to the case $|\gamma_k| \ll |\omega_k|$. We normalize the polarization vectors $\mathbf{a}(\mathbf{k})$ in such a way that $|c_k|^2$ has the meaning of the density of the number of quanta n_k, i.e., $n_k\omega_k$ is the spectral energy density $W(\mathbf{k})$. As is well known [19], the spectral energy density is given by

$$W(\mathbf{k}) = \frac{1}{8\pi}\left\{k^2\delta_{\alpha\beta} - k_\alpha k_\beta + \frac{\omega_k{}^2}{c^2}\frac{\partial}{\partial\omega}[\omega\varepsilon_{\alpha\beta}(\mathbf{k},\omega)]_{\omega=\omega_k}\right\} \times A_\alpha{}^*(\mathbf{k},\omega)A_\beta(\mathbf{k},\omega), \tag{5.9}$$

from which we easily obtain the following normalization condition for $\mathbf{a}(\mathbf{k})$

$$\frac{1}{8\pi}\left[k^2\delta_{\alpha\beta} - k_\alpha k_\beta + \frac{\omega_k{}^2}{c^2}(\omega\varepsilon_{\alpha\beta})'_{\omega=\omega_k}\right] a_\alpha{}^*(\mathbf{k})\,a_\beta(\mathbf{k}) = |\omega_k|. \tag{5.10}$$

We note that (5.9) and (5.10) are meaningful only for an almost "transparent" medium, i.e., when the antihermitian part $\epsilon_{\alpha\beta}$ can be neglected [19]. This is precisely the case we are considering when we assume that $|\gamma_k| \ll |\omega_k|$.

In order to find the next approximation $\mathbf{A}^{(2)}(\mathbf{k}, \omega)$, it is necessary to retain in the right side of (5.7) terms of second order in c_k, and to replace $\mathbf{A}(\mathbf{k}, \omega)$ in the left side by $\mathbf{A}^{(2)}(\mathbf{k}, \omega)$. With the aid of (5.6) and (5.8) we obtain

$$\left[\frac{\omega^2}{c^2}\varepsilon_{\alpha\beta}(k,\omega) + k_\alpha k_\beta - k^2\delta_{\alpha\beta}\right]A_\beta^{(2)}(k,\omega) = -2\pi\sum_{k',k''}\varkappa_{\alpha\beta\gamma}(\mathbf{k},\omega;\mathbf{k}',\omega_{\mathbf{k}'},\mathbf{k}'',\omega_{\mathbf{k}''})\,a_\beta(k')\,a_\gamma(k'') \times c_{k'}c_{k''}\delta(\omega - \omega_{k'} - \omega_{k''}). \tag{5.11}$$

Henceforth k will denote the aggregate comprising the components of the vector $\mathbf{k}$ and the polarization index r, and summation over k will mean summation over $\mathbf{k}$ and r. The solution of (5.11) is

$$\mathbf{A}^{(2)}(k,\omega) = -2\pi\sum_r \frac{\mathbf{a}^r(k,\omega)}{D^r(k,\omega)}\sum_{k',k''}M_{k'k''}(\omega_{k'},\omega_{k''}) \times c_{k'}c_{k''}\delta(\omega - \omega_{k'} - \omega_{k''}), \tag{5.12}$$

where

$$M_{k_1\ldots k_n}(\omega_1,\ldots,\omega_n) = a_\alpha{}^+(k,\omega)\,a_{\alpha_1}(k_1,\omega_1)\ldots a_{\alpha_n}(k_n,\omega_n) \times \varkappa_{\alpha\alpha_1\ldots\alpha_n}(\mathbf{k},\omega;\mathbf{k}_1,\omega_1;\ldots;\mathbf{k}_n,\omega_n), \tag{5.13}$$

$$D(k,\omega) = \lambda(k,\omega)\,\mathbf{a}^+(k,\omega)\,\mathbf{a}(k,\omega), \tag{5.14}$$

$\lambda(k, \omega)$ are the eigenvalues and $\mathbf{a}(k, \omega)$ the eigenvectors of the equation

$$\left[\frac{\omega^2}{c^2}\varepsilon_{\alpha\beta}(k,\omega) + k_\alpha k_\beta - k^2\delta_{\alpha\beta}\right]a_\beta(k,\omega) = \lambda(k,\omega)\,a_\alpha(k,\omega), \tag{5.15}$$

while $\mathbf{a}^+(k, \omega)$ are the eigenvectors of the equation conjugate to (5.15).

In calculating the nonlinear approximations we neglect throughout the non-hermitian part $\epsilon_{\alpha\beta}(k, \omega_k)$. Continuing the iteration, we easily obtain the third approximation:

$$\mathbf{A}^{(3)}(k,\omega) = 2\pi\sum_r \frac{\mathbf{a}^r(k,\omega)}{D^r(k,\omega)}\sum_{k',k'',k'''}\left\{2\sum_q\int\frac{d\omega'\delta(\omega - \omega_k + \omega')}{D(q,\omega')} \times M_{k'q}(\omega_{k'},\omega')\,M_{k''k'''}(\omega_{k''},\omega_{k'''}) + M_{k'k''k'''}(\omega_{k'},\omega_{k''},\omega_{k'''})\right\} \times c_{k'}c_{k''}c_{k'''}\delta(\omega - \omega_{k'} - \omega_{k''} - \omega_{k'''}); \tag{5.16}$$

$M_{k'k''k'''}(\omega_{k'}, \omega_{k''}, \omega_{k'''})$ and $D(q, \omega')$ are determined by (5.13) and (5.14).

The rest of the procedure for obtaining the kinetic equation for waves with arbitrary polarization coincides fully with that used in Sec. 2 for longitudinal waves. Performing the appropriate calculations, we obtain

$$\frac{dn_k}{dt} = 2\gamma_k n_k$$

$$+\frac{1}{8\pi}\Big\{\operatorname{Im}\sum_{k'}\Big[8P\frac{M_{k',k-k'}(\omega_{k'},\omega_k-\omega_{k'})M_{-k'k}(-\omega_{k'},\omega_k)}{D(k-k',\omega_k-\omega_{k'})}$$

$$+6N_{kk'}\Big]n_kn_{k'}+\frac{1}{2}\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}}\Big[|M_{k'k''}(\omega_{k'},\omega_{k''})|^2n_{k'}n_{k''}$$

$$-2\frac{\omega_{k''}}{|\omega_{k''}|}\operatorname{Re}\{M_{k'k''}(\omega_{k'},\omega_{k''})M_{-k'k}(-\omega_{k'},\omega_k)\}$$

$$\times n_kn_{k'}\Big]\delta(\omega_k-\omega_{k'}-\omega_{k''})\Big\}, \qquad (5.17)$$

$$N_{kk'}=M_{k'\,k-k'}(\omega_{k'},\omega_k-\omega_{k'}). \qquad (5.18)$$

In the derivation of (5.17) we made use of the fact that

$$\frac{\partial}{\partial\omega}D(k,\omega)_{\omega=\omega_k}=8\pi\frac{\omega_k}{|\omega_k|}.$$

This equation can be readily obtained by multiplying both sides of (5.15) by $a_{\beta^+}(k,\omega)$, differentiating with respect to ω, and taking into account the fact that $\lambda(k,\omega_k)=0$ and $\mathbf{a}(k,\omega_k)=\mathbf{a}(k)$. Equation (5.17) differs in form from the kinetic equation for the longitudinal oscillations (2.26) only in the additional summation over the polarizations in the right side. It can be proved that for longitudinal waves the quantities $D(k,\omega)$, $M_{k'k''}(\omega',\omega'')$, and $N_{kk'}$, calculated with the aid of (5.13), (5.14), and (5.18), go over into the corresponding quantities determined in (2.23)—(2.25). This proof is cumbersome and will not be presented here.

We now show how to generalize the symmetry relations, established in Secs. 3 and 4 for longitudinal oscillations, to the case of oscillations with arbitrary polarization. We break up the responses into two parts:

$$\varkappa_{\alpha\ldots\alpha_n}(\mathbf{k},\omega;\ldots;\mathbf{k}_n,\omega_n)=\overset{0}{\varkappa}_{\alpha\ldots\alpha_n}(\mathbf{k},\omega;\ldots;\mathbf{k}_n,\omega_n)$$

$$+\varkappa'_{\alpha\ldots\alpha_n}(\mathbf{k},\omega;\ldots;\mathbf{k}_n,\omega_n). \qquad (5.19)$$

The part $\kappa^0_{\alpha\ldots'\alpha_n}(\mathbf{k},\omega;\ldots;\mathbf{k}_n,\omega_n)$ of the response is defined in terms of Poisson brackets of the microcurrents in the unperturbed plasma $\mathbf{j}(\mathbf{r},t)$:

$$\overset{0}{\varkappa}_{\alpha\alpha_1\ldots\alpha_n}(\mathbf{k},\omega;\mathbf{k}_1,\omega_1;\ldots;\mathbf{k}_n,\omega_n)$$

$$=\frac{1}{n!}\sum\mathscr{P}\int d\mathbf{r}_1\ldots d\mathbf{r}_n\int_{-\infty}^{0}dt_1\ldots\int_{-\infty}^{t_{n-1}}dt_n$$

$$\times\psi_{\alpha\alpha_1\ldots\alpha_n}(\mathbf{r}_1,t_1;\ldots;\mathbf{r}_n,t_n)\exp\Big\{i\sum_{l=1}^{n}(\mathbf{k}_l\mathbf{r}_l-\omega_lt_l)\Big\}, \qquad (5.20)$$

$$\psi_{\alpha\alpha_1\ldots\alpha_n}(\mathbf{r}_1-\mathbf{r},t_1-t;\ldots;\mathbf{r}_n-\mathbf{r},t_n-t)$$

$$=\frac{4\pi}{c^{n+1}}\sum_j n_j\langle[\ldots[j_\alpha(\mathbf{r},t),j_{\alpha_1}(\mathbf{r}_1,t_1)]\ldots j_{\alpha_n}(\mathbf{r}_n,t_n)]f_j^0\rangle, \qquad (5.21)$$

where $\mathscr{P}$—permutation operator.

The tensors $\kappa'_{\alpha\alpha_1\ldots\alpha_n}(\mathbf{k},\omega;\ldots;\mathbf{k}_n,\omega_n)$ can be expressed in terms of the lower-order tensors κ^0. Thus, for example, it follows from (5.4) and (5.5) that

$$\varkappa'_{\alpha\beta\gamma}(\mathbf{k},\omega;\mathbf{k}',\omega';\mathbf{k}'',\omega'')$$

$$=\frac{1}{2}\Big[\frac{\delta_\alpha}{\delta A_{0\beta}}\varkappa_{\alpha\gamma}{}^0(\mathbf{k}-\mathbf{k}',\omega-\omega';\mathbf{k}'',\omega'')$$

$$+\frac{\delta_\alpha}{\delta A_{0\gamma}}\varkappa_{\alpha\beta}{}^0(\mathbf{k}-\mathbf{k}'',\omega-\omega'';\mathbf{k}',\omega')$$

$$+\frac{\delta_\beta}{\delta A_{0\gamma}}\varkappa_{\alpha\beta}{}^0(\mathbf{k},\omega;\mathbf{k}'+\mathbf{k}'',\omega'+\omega'')\Big], \qquad (5.22)$$

$$\varkappa'_{\alpha\beta\gamma\delta}(\mathbf{k},\omega;\mathbf{k}',\omega';\mathbf{k}'',\omega'';\mathbf{k}''',\omega''')$$

$$=\frac{1}{3}\Big[\frac{\delta_\alpha}{\delta A_{0\beta}}\overset{0}{\varkappa}_{\alpha\gamma\delta}(\mathbf{k}-\mathbf{k}',\omega-\omega';\mathbf{k}'',\omega'';\mathbf{k}''',\omega''')$$

$$+\frac{\delta_\alpha}{\delta A_{0\gamma}}\overset{0}{\varkappa}_{\alpha\delta\beta}(\mathbf{k}-\mathbf{k}''\ \ \omega-\omega'';\mathbf{k}''',\omega''';\mathbf{k}',\omega')$$

$$+\frac{\delta_\alpha}{\delta A_{0\delta}}\overset{0}{\varkappa}_{\alpha\beta\gamma}(\mathbf{k}-\mathbf{k}''',\omega-\omega''';\mathbf{k}',\omega';\mathbf{k}'',\omega'')$$

$$+\frac{\delta_\gamma}{\delta A_{0\beta}}\overset{0}{\varkappa}_{\alpha\gamma\delta}(\mathbf{k},\omega;\mathbf{k}'+\mathbf{k}'',\omega'+\omega'';\mathbf{k}''',\omega''')\text{-}$$

$$+\frac{\delta_\delta}{\delta A_{0\gamma}}\overset{0}{\varkappa}_{\alpha\delta\beta}(\mathbf{k},\omega;\mathbf{k}''+\mathbf{k}''',\omega''+\omega''';\mathbf{k}',\omega')$$

$$+\frac{\delta_\beta}{\delta A_{0\delta}}\overset{0}{\varkappa}_{\alpha\beta\gamma}(\mathbf{k},\omega;\mathbf{k}'''+\mathbf{k}',\omega'''+\omega';\mathbf{k}'',\omega'')\Big]$$

$$+\frac{1}{6}\Big[\frac{\delta_\alpha}{\delta A_{0\beta}}\frac{\delta_\gamma}{\delta A_{0\delta}}\varkappa_{\alpha\gamma}{}^0(\mathbf{k}-\mathbf{k}',\omega-\omega';\mathbf{k}''+\mathbf{k}''',\omega''+\omega''')$$

$$+\frac{\delta_\alpha}{\delta A_{0\gamma}}\frac{\delta_\delta}{\delta A_{0\beta}}\varkappa_{\alpha\delta}{}^0(\mathbf{k}-\mathbf{k}'',\omega-\omega'';\mathbf{k}'''+\mathbf{k}',\omega'''+\omega')$$

$$+\frac{\delta_\alpha}{\delta A_{0\delta}}\frac{\delta_\beta}{\delta A_{0\gamma}}\varkappa_{\alpha\beta}{}^0(\mathbf{k}-\mathbf{k}'''\ \omega-\omega''';\mathbf{k}'+\mathbf{k}'',\omega'+\omega'')\Big], \qquad (5.23)$$

where $\delta_\alpha/\delta\mathbf{A}_0$ denotes differentiation of the current with index α with respect to $\mathbf{A}_0$. For example, the term in the right side of (5.22) takes the form

$$\frac{\delta_\alpha}{\delta A_{0\beta}}\varkappa_{\alpha\gamma}{}^0(\mathbf{k}-\mathbf{k}',\omega-\omega';\mathbf{k}'',\omega'')$$

$$=\frac{4\pi}{c^2}\sum_j n_j\int d\mathbf{r}'\int_{-\infty}^{0}dt'\Big\langle\Big[\frac{\partial j_\alpha(\mathbf{r},t)}{\partial A_{0\beta}},j_\gamma(\mathbf{r}',t')\Big]f^0\Big\rangle$$

$$\times\exp\{i(\mathbf{k}''\mathbf{r}'-\omega''t')\}.$$

We introduce the quantities

$$M^0_{k'k''}(\omega',\omega'')=-a_\alpha{}^+(k,\omega)a_\beta(k',\omega')a_\gamma(k'',\omega'')$$

$$\times\overset{0}{\varkappa}_{\alpha\beta\gamma}(\mathbf{k},\omega;\mathbf{k}',\omega;'\,\mathbf{k}'',\omega''), \qquad (5.24)$$

$$\overset{0}{N}_{kk'} = a_\alpha^*(k) a_\beta(k') a_\gamma(k) a_\delta^*(k') \times \overset{0}{\varkappa}_{\alpha\beta\gamma\delta}(\mathbf{k}, \omega_k; \mathbf{k}', \omega_{k'}; \mathbf{k}, \omega_k; -\mathbf{k}', -\omega_{k'});$$

$$\mathbf{a}(\mathbf{k}) = \mathbf{a}(k, \omega_k). \qquad (5.25)$$

From a comparison of (5.20), (5.21) with (2.11), (2.12) we see that these quantities have the same structure as the quantities $M_{k'k''}(\omega', \omega'')$ and $N_{kk'}$ which we considered above. It follows directly that they satisfy symmetry relations of the type (3.16) and (4.7). As regards the quantities $M'_{k'k''}(\omega', \omega'')$ and $N'_{kk'}$, which are determined by the parts $\varkappa'$ of the responses, analogous relations for them can be directly obtained by representing $\varkappa'$ in the form (5.22), (5.23). It is necessary to put

$$N'^{\pm}_{kk'} = a_\alpha^*(k) a_\beta(k') a_\gamma(k) a_\delta^*(k') \times \varkappa'^{\pm}_{\alpha\beta\gamma\delta}(\mathbf{k}, \omega_k; \mathbf{k}', \omega_{k'}; \mathbf{k}, \omega_k; -\mathbf{k}', -\omega_{k'}), \qquad (5.26)$$

$$\varkappa'_{\alpha\beta\gamma\delta}(\mathbf{k}, \omega_k; \mathbf{k}', \omega_{k'}; \mathbf{k}, \omega_k; -\mathbf{k}' -\omega_{k'}) = \frac{1}{3}\left[\frac{\delta_\alpha}{\delta A_{0\beta}} \overset{0}{\varkappa}_{\alpha\gamma\delta}(\mathbf{k}-\mathbf{k}', \omega_k - \omega_{k'}; \mathbf{k}, \omega_k; -\mathbf{k}', -\omega_{k'}) + \frac{\delta_\delta}{\delta A_{0\gamma}} \overset{0}{\varkappa}_{\alpha\delta\beta}(\mathbf{k}, \omega_k; \mathbf{k}-\mathbf{k}', \omega_k - \omega_{k'}; \mathbf{k}', \omega_{k'})\right] + \frac{1}{6}\frac{\delta_\alpha}{\delta A_{0\beta}}\frac{\delta_\gamma}{\delta A_{0\delta}} \overset{0}{\varkappa}_{\alpha\gamma}(\mathbf{k}-\mathbf{k}', \omega_k - \omega_{k'}; \mathbf{k}-\mathbf{k}', \omega_k - \omega_{k'}), \qquad (5.27)$$

$$\varkappa'^{+}_{\alpha\beta\gamma\delta}(\mathbf{k}, \omega_k; \mathbf{k}', \omega_{k'}; \mathbf{k}, \omega_k; -\mathbf{k}', -\omega_{k'}) = \frac{1}{3}\left[\frac{\delta_\alpha}{\delta A_{0\delta}} \overset{0}{\varkappa}_{\alpha\beta\gamma}(\mathbf{k}+\mathbf{k}', \omega_k + \omega_{k'}; \mathbf{k}', \omega_{k'}; \mathbf{k}, \omega_k) + \frac{\delta_\gamma}{\delta A_{0\beta}} \overset{0}{\varkappa}_{\alpha\gamma\delta}(\mathbf{k}, \omega_k; \mathbf{k}+\mathbf{k}', \omega_k + \omega_{k'}; -\mathbf{k}', -\omega_{k'})\right] + \frac{1}{6}\frac{\delta_\alpha}{\delta A_{0\delta}}\frac{\delta_\beta}{\delta A_{0\gamma}} \varkappa_{\alpha\beta}^0(\mathbf{k}+\mathbf{k}', \omega_k + \omega_{k'}; \mathbf{k}+\mathbf{k}', \omega_k + \omega_{k'}). \qquad (5.28)$$

We have left out from (5.27) and (5.28) the terms connected with the quasilinear renormalization. Thus, the symmetry relations (3.16) and (4.7) and all their corollaries remain in force also in the case of oscillations with arbitrary polarization.

6. INTERACTION BETWEEN LONGITUDINAL AND TRANSVERSE OSCILLATIONS IN A PLASMA WITHOUT A MAGNETIC FIELD

In conclusion we consider, by way of a simple example, the interaction between plasmons and photons in a plasma without a stationary magnetic field. Using the formulas obtained in Sec. 5 for the matrix elements $M_{k'k''}(\omega_{k'}, \omega_{k''})$, $N_{kk'}$, and $D(k, \omega)$, contained in the kinetic equation for the waves, we find that in our case these quantities are expressed as follows:

$$M_{k'k''}(\omega', \omega'') = -a_\alpha^+(k, \omega) a_\beta(k', \omega') a_\gamma(k'', \omega'') \sum_j \frac{\omega_{0j}^2 e_j}{2m_j c^3} \times \int d\mathbf{v}\left\{\left[\frac{k_\delta'' v_\gamma (k_\alpha' v_\beta + k_\beta v_\alpha + \mathbf{k}\mathbf{k}' v_\beta \partial/\partial k_\alpha)}{\omega'' - \mathbf{k}''\mathbf{v} + i\varepsilon} + \frac{k_\delta' v_\beta (k_\alpha'' v_\gamma + k_\gamma v_\alpha + \mathbf{k}\mathbf{k}'' v_\gamma \partial/\partial k_\alpha)}{\omega' - \mathbf{k}'\mathbf{v} + i\varepsilon}\right]\frac{1}{\omega - \mathbf{k}\mathbf{v} + i\varepsilon} + \frac{\delta_{\alpha\beta} v_\gamma k_\delta''}{\omega'' - \mathbf{k}''\mathbf{v} + i\varepsilon} + \frac{\delta_{\alpha\gamma} v_\beta k_\delta'}{\omega' - \mathbf{k}'\mathbf{v} + i\varepsilon} + \frac{\delta_{\beta\gamma} v_\alpha k_\delta}{\omega - \mathbf{k}\mathbf{v} + i\varepsilon}\right\}\frac{\partial f}{\partial v_\delta}, \qquad (6.1)$$

$$N_{kk'}^- = -\sum_j \frac{\omega_{0j}^2 e_j^2}{6m_j^2 c^4}\int d\mathbf{v}\left\{a_\alpha(k) a_\beta(k') \times \left[\left(k_\alpha' v_\beta + k_\beta v_\alpha + \frac{v_\alpha v_\beta \mathbf{k}\mathbf{k}'}{\omega_k - \mathbf{k}\mathbf{v}}\right)\frac{1}{\omega_k - \mathbf{k}\mathbf{v}} + \delta_{\alpha\beta}\right]\right\}^2 \times \frac{\mathbf{k} - \mathbf{k}'}{\omega_k - \omega_{k'} - (\mathbf{k} - \mathbf{k}')\mathbf{v} + i\varepsilon}\frac{\partial f}{\partial \mathbf{v}}, \qquad (6.2)$$

$$D(k, \omega) = \frac{\omega^2}{c^2}|a(k, \omega)|^2 - [\mathbf{k}\mathbf{a}(k, \omega)]^2 + \sum_j \frac{\omega_{0j}^2 \omega}{c^2}\int \frac{\mathbf{v}\mathbf{a}(k, \omega)}{\omega - \mathbf{k}\mathbf{v} + i\varepsilon}\mathbf{a}(k, \omega)\frac{\partial f}{\partial \mathbf{v}} d\mathbf{v}, \qquad (6.3)$$

where $\omega_{0j}^2 = 4\pi e_j/m_j$, and the polarization vectors are normalized in accordance with (5.10). It is important here that the contribution from the term R^+ can be neglected, so that the law of conservation of the total number of quasi-particles of all polarizations holds true here for a nonlinear "non-decay" interaction.

With the aid of (5.17) and (6.1)—(6.3) we can consider a great variety of plasmon and photon interaction processes. We shall not discuss all the possible cases, and confine ourselves to a question of interest in astrophysics (see, for example, [20,21]), that of transformation of plasmons into photons (we shall henceforth denote the plasmon wave vector by **p** and that of the photon by **q**).

We consider first the formation of photons with a frequency close to the plasma frequency ω_{0e}. This process is described by a "non-decay" term in the kinetic equation for the waves. Substituting in (6.1) and (6.2) the Maxwellian distribution function, we readily obtain an equation describing the process in question. It turns out here that the main contribution is made by the scattering of plasmons from ions (compare with the analogous situation in [14]). The equation takes the form

$$\frac{dn_\mathbf{q}}{dt} = -\frac{\sqrt{\pi}}{(2\pi)^3}\frac{\omega_{0e}^2}{n_e T}\int dp \frac{[\mathbf{p}\mathbf{q}]^2}{p^2 q^2}\frac{\Omega}{p v_i}\exp\left[-\left(\frac{\Omega}{p v_i}\right)^2\right] \times \left\{\left[X\left(\frac{\Omega}{p v_i}\right) - 2\right]^2 + \pi\left(\frac{\Omega}{p v_i}\right)^2 \exp\left[-2\left(\frac{\Omega}{p v_i}\right)^2\right]\right\}^{-1} n_\mathbf{p} n_\mathbf{q}, \qquad (6.4)^*$$

*[p q] = p × q.

where

$$X(z) = 2ze^{-z^2}\int_0^z e^{t^2}dt, \quad \Omega = \omega_{\mathbf{q}} - \omega_{\mathbf{p}},$$

$$\omega_{\mathbf{q}}^2 = \omega_{0e}^2 + c^2q^2, \quad \omega_{\mathbf{p}}^2 = \omega_{0e}^2\left(1 + \frac{3}{2}p^2r_D^2\right), \quad (6.5)$$

$v_j = (2T/m_j)^{1/2}$—thermal velocities of the electrons and ions (we assume for simplicity that the plasma is isothermal) and $r_D = v_e/\omega_{0e}$—Debye radius of the electrons.

From (6.4) we see that the process of wave transformation proceeds in a direction from the high-frequency oscillations to the low-frequency ones. The maximum rate of growth of the number of photons corresponds to the wave-number region

$$\frac{v_e}{c}(p_{min} - \delta p) < q < \frac{v_e}{c}p_{min}, \quad \delta p \sim r_D^{-1}\sqrt{\frac{m_e}{m_i}}. \quad (6.6)$$

These photons interact effectively with the plasmons in the interval from p_{min} to $p_{min} + \delta p$, and the effective growth time of the photons in the region (6.6) is of the order of

$$\tau_i \sim \left[\frac{1}{6(4\pi)^3}\frac{\omega_{0e}^2 n_{\mathbf{p}}}{n_e T}p^2 r_D^{-1}\sqrt{\frac{m_e}{m_i}}\right]^{-1}. \quad (6.7)$$

An estimate of the contribution from the scattering by electrons leads to a photon growth time (due to the electrons only) which is

$$\left(\frac{m_e}{m_i}\right)^{1/2}\left(\frac{c}{v_e}\right)^2\frac{1}{p\Delta\Gamma r_D^4}$$

times larger than (6.7), where $\Delta\Gamma$—phase volume of the plasmon wave packet.

Let us consider also the process of merging of two plasmons to produce a photon with frequency of the order of $2\omega_{0e}$ (and wave number $q \sim \sqrt{3}\,\omega_{0e}/c$). This process is described by the "decay" term in the kinetic equation. From (6.1) and (5.17) we get

$$\frac{dn_{\mathbf{q}}}{dt} = \frac{\pi}{8}\frac{\omega_{0e}v_e^2}{n_e T}\sum_{\mathbf{p}'+\mathbf{p}=\mathbf{q}}\frac{(p^2 - p'^2)[\mathbf{pp}']^2}{p^2p'^2q^2}n_{\mathbf{p}}n_{\mathbf{p}'}\delta(\omega_{\mathbf{q}} - \omega_{\mathbf{p}'} - \omega_{\mathbf{p}}). \quad (6.8)$$

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APPENDIX

Let us prove (3.14) and (3.20). According to (3.6) we have

$$\mu(\mathbf{k}, \omega; \mathbf{k}', \omega', \mathbf{k}'', \omega'') = \frac{1}{2(2\pi i)^2}\int\frac{d\Omega' d\Omega''}{\omega - \Omega + i\varepsilon}\left\{\frac{\tilde{\mu}(\mathbf{k}, \Omega; \mathbf{k}', \Omega'; \mathbf{k}'', \Omega'')}{\omega'' - \Omega'' + i\varepsilon} + \frac{\tilde{\mu}(\mathbf{k}, \Omega; \mathbf{k}'', \Omega''; \mathbf{k}', \Omega')}{\omega' - \Omega' + i\varepsilon}\right\}. \quad (A.1)$$

Transforming $\tilde{\mu}(\mathbf{k}, \Omega; \mathbf{k}', \Omega'; \mathbf{k}'', \Omega'')$ in (A.1) with the aid of (3.4), and $\mu(\mathbf{k}, \Omega; \mathbf{k}'', \Omega''; \mathbf{k}', \Omega')$ with the aid of (3.5), and interchanging the integration variables, we obtain

$$\mu(\mathbf{k}, \omega; \mathbf{k}', \omega'; \mathbf{k}'', \omega'') = \frac{1}{2(2\pi i)^2}\int\frac{d\Omega' d\Omega''}{\omega' - \Omega + i\varepsilon}\times\left\{\frac{\tilde{\mu}^*(\mathbf{k}', \Omega; \mathbf{k}, \Omega'; -\mathbf{k}'', \Omega'')}{-\omega'' - \Omega'' - i\varepsilon} + \frac{\tilde{\mu}^*(\mathbf{k}', \Omega; -\mathbf{k}'', \Omega''; \mathbf{k}, \Omega')}{\omega - \Omega' + i\varepsilon}\right\}$$
$$= \mu^*(\mathbf{k}', \omega' - i0; \mathbf{k}, \omega - i0; -\mathbf{k}'', -\omega''). \quad (A.2)$$

To prove (3.20) we need the following formulas

$$\tilde{\mu}(\mathbf{k}, \Omega; \mathbf{k}', \Omega'; \mathbf{k}'', \Omega''; \mathbf{k}''', \Omega''') = -\tilde{\mu}(-\mathbf{k}', -\Omega'; -\mathbf{k}, -\Omega; \mathbf{k}'', \Omega''; \mathbf{k}''', \Omega'''), \quad (A.3)$$

$$\tilde{\mu}(\mathbf{k}, \Omega; \mathbf{k}', \Omega'; \mathbf{k}'', \Omega''; \mathbf{k}''', \Omega''') - \tilde{\mu}(\mathbf{k}, \Omega; \mathbf{k}', \Omega'; \mathbf{k}''', \Omega'''; \mathbf{k}'', \Omega'')$$
$$= -\tilde{\mu}(-\mathbf{k}'', -\Omega''; \mathbf{k}''', \Omega'''; -\mathbf{k}, -\Omega; \mathbf{k}', \Omega') + \tilde{\mu}(-\mathbf{k}'', -\Omega''; \mathbf{k}''', \Omega''', \mathbf{k}', \Omega'; -\mathbf{k}, -\Omega), \quad (A.4)$$

$$\tilde{\mu}(\mathbf{k}, \Omega; \mathbf{k}', \Omega'; \mathbf{k}'', \Omega''; \mathbf{k}''', \Omega''') = \tilde{\mu}(\mathbf{k}, \Omega; \mathbf{k}'', \Omega''; \mathbf{k}', \Omega'; \mathbf{k}''', \Omega''')$$
$$-\tilde{\mu}(-\mathbf{k}', -\Omega'; \mathbf{k}'', \Omega''; -\mathbf{k}, -\Omega; \mathbf{k}''', \Omega'''), \quad (A.5)$$

which follow from the properties of the Poisson brackets and are analogous to (3.4) and (3.5). We show first that the real parts of the quantities in the numerators of the spectral expansion (3.12) make no contribution to Im $\mu(\mathbf{k}, \omega_{\mathbf{k}}; \mathbf{k}', \omega_{\mathbf{k}'}; \mathbf{k}, \omega_{\mathbf{k}}; -\mathbf{k}', -\omega_{\mathbf{k}})$. To this end it is sufficient to prove that the quantity $\mu_{\mathrm{P}}(\mathbf{k}, \omega_{\mathbf{k}}: \mathbf{k}', \omega_{\mathbf{k}'}; \mathbf{k}, \omega_{\mathbf{k}}; -\mathbf{k}', \omega_{\mathbf{k}'})$, which is obtained if all the poles in (3.12) are integrated in the sense of the principal value, is real. According to (3.12) we have

$$\mu_{\mathrm{P}}(\mathbf{k}, \omega_{\mathbf{k}}; \mathbf{k}', \omega_{\mathbf{k}'}; \mathbf{k}, \omega_{\mathbf{k}}; -\mathbf{k}', -\omega_{\mathbf{k}'})$$
$$= -\frac{1}{6(2\pi i)^3}\mathrm{P}\int\frac{d\Omega' d\Omega'' d\Omega'''}{\omega_{\mathbf{k}} - \Omega}\times\left\{\frac{\tilde{\mu}(\mathbf{k}, \Omega; \mathbf{k}'\Omega'; \mathbf{k}, \Omega''; -\mathbf{k}', \Omega''')}{(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} - \Omega'' - \Omega''')(-\omega_{\mathbf{k}'} - \Omega''')}\right.$$
$$+ \frac{\tilde{\mu}(\mathbf{k}, \Omega; \mathbf{k}', \Omega'; -\mathbf{k}', \Omega'''; \mathbf{k}, \Omega'')}{(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} - \Omega'' - \Omega''')(\omega_{\mathbf{k}} - \Omega'')}$$
$$+ \frac{\tilde{\mu}(\mathbf{k}, \Omega; -\mathbf{k}', \Omega'''; \mathbf{k}, \Omega''; \mathbf{k}', \Omega')}{(\omega_{\mathbf{k}} + \omega_{\mathbf{k}'} - \Omega'' - \Omega')(\omega_{\mathbf{k}'} - \Omega')}$$
$$+ \frac{\tilde{\mu}(\mathbf{k}, \Omega; -\mathbf{k}', \Omega'''; \mathbf{k}', \Omega'; \mathbf{k}, \Omega'')}{(\omega_{\mathbf{k}} + \omega_{\mathbf{k}'} - \Omega'' - \Omega')(\omega_{\mathbf{k}} - \Omega'')}$$
$$+ \frac{\tilde{\mu}(\mathbf{k}, \Omega; \mathbf{k}, \Omega''; \mathbf{k}', \Omega'; -\mathbf{k}', \Omega''')}{(-\Omega' - \Omega''')(-\omega_{\mathbf{k}'} - \Omega''')}$$
$$\left. + \frac{\tilde{\mu}(\mathbf{k}, \Omega; \mathbf{k}; \Omega''; -\mathbf{k}', \Omega'''; \mathbf{k}', \Omega')}{(-\Omega' - \Omega''')(\omega_{\mathbf{k}'} - \Omega')}\right\}. \quad (A.6)$$

We have retained for the time being the renormalization terms in (A.6). The right side of (A.6) can be rewritten in the form

$$-\frac{1}{6(2\pi i)^3}\mathrm{P}\int d\Omega' d\Omega'' d\Omega'''$$
$$\times\left\{\frac{\tilde{\mu}(\mathbf{k},\Omega;\mathbf{k}',\Omega';-\mathbf{k}',\Omega''';\mathbf{k},\Omega'')-\tilde{\mu}(\mathbf{k},\Omega;\mathbf{k}',\Omega';\mathbf{k},\Omega'';-\mathbf{k}',\Omega''')}{(\omega_{\mathbf{k}}-\Omega)(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}-\Omega''-\Omega''')(\omega_{\mathbf{k}}-\Omega'')}\right.$$
$$+\frac{\tilde{\mu}(\mathbf{k},\Omega;-\mathbf{k}',\Omega''';\mathbf{k}',\Omega';\mathbf{k},\Omega'')-\tilde{\mu}(\mathbf{k},\Omega;-\mathbf{k}',\Omega''';\mathbf{k},\Omega'';\mathbf{k}',\Omega')}{(\omega_{\mathbf{k}}-\Omega)(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'}-\Omega''-\Omega')(\omega_{\mathbf{k}}-\Omega'')}$$
$$+\frac{\tilde{\mu}(\mathbf{k},\Omega;\mathbf{k}',\Omega';\mathbf{k},\Omega'';-\mathbf{k}',\Omega''')}{(\omega_{\mathbf{k}}-\Omega)(\omega_{\mathbf{k}}-\Omega'')(-\omega_{\mathbf{k}'}-\Omega''')}+\frac{\tilde{\mu}(\mathbf{k},\Omega;-\mathbf{k}',\Omega''';\mathbf{k},\Omega'';\mathbf{k}',\Omega')}{(\omega_{\mathbf{k}}-\Omega)(\omega_{\mathbf{k}}-\Omega'')(\omega_{\mathbf{k}'}-\Omega')}$$
$$\left.+\frac{\tilde{\mu}(\mathbf{k},\Omega;\mathbf{k},\Omega'';\mathbf{k}',\Omega';-\mathbf{k}',\Omega''')}{(\omega_{\mathbf{k}}-\Omega)(-\Omega'-\Omega''')(-\omega_{\mathbf{k}'}-\Omega''')}+\frac{\tilde{\mu}(\mathbf{k},\Omega;\mathbf{k},\Omega'';-\mathbf{k}',\Omega''';\mathbf{k}',\Omega')}{(\omega_{\mathbf{k}}-\Omega)(-\Omega'-\Omega''')(\omega_{\mathbf{k}'}-\Omega')}\right\}. \tag{A.7}$$

If the numerator in the first term of the curly brackets is transformed in accordance with (A.4), and then and the integration variables are interchanged, we obtain

$$-\frac{\tilde{\mu}^*(\mathbf{k},\Omega;\mathbf{k}',\Omega';-\mathbf{k}',\Omega''';\mathbf{k},\Omega'')-\tilde{\mu}^*(\mathbf{k},\Omega;\mathbf{k}',\Omega';\mathbf{k},\Omega'';-\mathbf{k}',\Omega''')}{(\omega_{\mathbf{k}}-\Omega)(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}-\Omega''-\Omega''')(\omega_{\mathbf{k}}-\Omega'')}. \tag{A.8}$$

From this we see that this term makes only a real contribution to $\mu_{\mathrm{P}}(\mathbf{k}, \omega_{\mathbf{k}}; \mathbf{k}', \omega_{\mathbf{k}'}; \mathbf{k}, \omega_{\mathbf{k}}; -\mathbf{k}', -\omega_{\mathbf{k}'})$. We prove the same thing in exactly the same way for the second term. By transforming the numerator of the third term in the curly brackets in (A.7) with the aid of (A.5), and the fourth with the aid of (A.3), the remaining part of $\mu_{\mathrm{P}}(\mathbf{k}, \omega_{\mathbf{k}}; \mathbf{k}', \omega_{\mathbf{k}'}; \mathbf{k}, \omega_{\mathbf{k}}; -\mathbf{k}', -\omega_{\mathbf{k}'})$ can be represented in the form

$$-\frac{1}{6(2\pi i)^3}\mathrm{P}\int d\Omega' d\Omega'' d\Omega''' \left\{\frac{\tilde{\mu}(\mathbf{k},\Omega;\mathbf{k},\Omega'';\mathbf{k}',\Omega';-\mathbf{k}',\Omega''')-\tilde{\mu}(-\mathbf{k}',-\Omega';\mathbf{k},\Omega'';-\mathbf{k},-\Omega;-\mathbf{k},\Omega''')}{(\omega_{\mathbf{k}}-\Omega)(\omega_{\mathbf{k}}-\Omega'')(-\omega_{\mathbf{k}'}-\Omega''')}\right.$$
$$\left.-\frac{\tilde{\mu}(\mathbf{k}',-\Omega''';-\mathbf{k},-\Omega;\mathbf{k},\Omega'';\mathbf{k}',\Omega')}{(\omega_{\mathbf{k}}-\Omega)(\omega_{\mathbf{k}}-\Omega'')(\omega_{\mathbf{k}'}-\Omega')}+\frac{\tilde{\mu}(\mathbf{k},\Omega;\mathbf{k},\Omega'';\mathbf{k}',\Omega';-\mathbf{k}',\Omega''')}{(\omega_{\mathbf{k}}-\Omega)(-\Omega'-\Omega''')(-\omega_{\mathbf{k}'}-\Omega''')}+\frac{\tilde{\mu}(\mathbf{k},\Omega;\mathbf{k},\Omega'',-\mathbf{k}',\Omega''';\mathbf{k}',\Omega')}{(\omega_{\mathbf{k}}-\Omega)(-\Omega'-\Omega''')(\omega_{\mathbf{k}'}-\Omega')}\right\}$$
$$=-\frac{i}{3(2\pi i)^3}\mathrm{P}\int d\Omega' d\Omega'' d\Omega'''\left\{\frac{\mathrm{Im}\,\tilde{\mu}(\mathbf{k}',\Omega;-\mathbf{k},\Omega';\mathbf{k},\Omega'';\mathbf{k}',\Omega''')}{(-\omega_{\mathbf{k}}-\Omega')(\omega_{\mathbf{k}}-\Omega'')(\omega_{\mathbf{k}'}-\Omega''')}+\frac{\mathrm{Im}\,\tilde{\mu}(\mathbf{k},\Omega;\mathbf{k},\Omega'';-\mathbf{k}',\Omega''';\mathbf{k}',\Omega')}{(\omega_{\mathbf{k}}-\Omega)(-\Omega'-\Omega''')(\omega_{\mathbf{k}'}-\Omega')}\right\} \tag{A.9}$$

in such a way that this part is also real. We now proceed directly to prove (3.20). We write formula (3.18) in the form

$$\mu^-(\mathbf{k},\omega_{\mathbf{k}};\mathbf{k}',\omega_{\mathbf{k}'};\mathbf{k},\omega_{\mathbf{k}};-\mathbf{k}',-\omega_{\mathbf{k}'})=\frac{i}{6(2\pi i)^3}\int\frac{d\Omega' d\Omega'' d\Omega'''}{(\omega_{\mathbf{k}}-\Omega)(-\omega_{\mathbf{k}'}-\Omega''')}$$
$$\times\left\{\frac{\mathrm{Im}\,\tilde{\mu}(\mathbf{k},\Omega;\mathbf{k}',\Omega';-\mathbf{k}',\Omega''';\mathbf{k},\Omega'')-\mathrm{Im}\,\tilde{\mu}(\mathbf{k},\Omega;\mathbf{k}',\Omega';\mathbf{k},\Omega'';-\mathbf{k}',\Omega''')}{\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}-\Omega''-\Omega'''+i\varepsilon}-\frac{\mathrm{Im}\,\tilde{\mu}(\mathbf{k},\Omega;\mathbf{k}',\Omega';-\mathbf{k}',\Omega''';\mathbf{k},\Omega'')}{\omega_{\mathbf{k}}-\Omega''}\right\}. \tag{A.10}$$

The last term in the curly brackets of (A.10) can be neglected, since all its denominators contain only the natural frequencies. Then it follows immediately from (A.4) that

$$\mathrm{Im}\,\mu^-(\mathbf{k},\omega_{\mathbf{k}};\mathbf{k}',\omega_{\mathbf{k}'};\mathbf{k},\omega_{\mathbf{k}};-\mathbf{k}',-\omega_{\mathbf{k}'})=\frac{i}{6(2\pi i)^3}\int\frac{d\Omega' d\Omega'' d\Omega'''}{(\omega_{\mathbf{k}'}-\Omega)(\omega_{\mathbf{k}'}-\omega_{\mathbf{k}}-\Omega''-\Omega'''-i\varepsilon)(-\omega_{\mathbf{k}}-\Omega''')}$$
$$\times\{\mathrm{Im}\,\tilde{\mu}(\mathbf{k}',\Omega;\mathbf{k},\Omega';-\mathbf{k},\Omega''';\mathbf{k}',\Omega'')-\mathrm{Im}\,\tilde{\mu}(\mathbf{k}',\Omega;\mathbf{k},\Omega';\mathbf{k}',\Omega'';-\mathbf{k},\Omega'')\}$$
$$=-\mathrm{Im}\,\mu^-(\mathbf{k}',\omega_{\mathbf{k}'};\mathbf{k},\omega_{\mathbf{k}};\mathbf{k}',\omega_{\mathbf{k}'};-\mathbf{k},-\omega_{\mathbf{k}}).$$

The correctness of (3.20) for Im $\mu^+(\mathbf{k}, \omega_\mathbf{k}; \mathbf{k}', \omega_{\mathbf{k}'}; \mathbf{k}, \omega_\mathbf{k}; -\mathbf{k}', -\omega_{\mathbf{k}'})$ is proved in the same manner.

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