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NONLINEAR EFFECTS OF WAVES PROPAGATING IN STATISTICAL MEDIA

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Propagation of stationary waves of small but finite amplitude in unbounded media is studied on the basis of the Vlasov equation. The hydrodynamic flow, second harmonic, and correction to the wave frequency computed from the linear approximation are found. Special attention is paid to the problem of making more precise the coordinate system in which the investigation is carried out. The results are applied to the case in which the medium considered is an electron plasma.

1. INTRODUCTION

THE propagation of stationary waves of small but finite amplitude in homogeneous media is accompanied by various nonlinear effects. These include the appearance of higher harmonics, the effect of the wave amplitude on the relation between the frequency and the propagation vector of the wave, and hydrodynamic flow, i.e., the steady flow produced by the wave.

In the present work we study the propagation of similar waves in unbounded homogeneous media, the behavior of which is described by means of a distribution function $f(\mathbf{r}, \mathbf{v}, t)$ that satisfies the Vlasov equation:[1]

$$\frac{\partial f}{\partial t} + \mathbf{v}\nabla_{\mathbf{r}} f - \frac{1}{m}\nabla_{\mathbf{v}} f \nabla_{\mathbf{r}} \int K(|\mathbf{r}-\mathbf{r}'|) f(\mathbf{r}', \mathbf{v}', t)\, d\mathbf{v}'\, d\mathbf{r}' = 0, \tag{1.1}$$

where $K(|\mathbf{r}-\mathbf{r}'|)$ is the interaction potential energy between the particles which make up the medium; m is the mass of the particle. We shall carry out all the calculations without specifying the interaction potential energy between the particles, but the last section will be devoted to a particular case in which the medium under consideration is an electron plasma.

The nonlinear theory of oscillations of an electron plasma has been considered by many authors (see, for example, [2-4] where one can find references to other works). But in all these researches the authors limit themselves basically to the study of a plasma at zero temperature. Moreover, these works suffer from the general deficiency that the effect of the wave on the "noise" background is not considered in them, i.e., the effect on that part of the distribution function which does not depend either on the time or on the coordinate (we do not have in mind here the so-called "entrained" particles which are considered by some authors). From the macroscopic point of view, the possibility is not considered in these researches of the existence of a hydrodynamic flow, which cannot be ignored if one takes into account a number of hydrodynamic studies,[5,6] and also the works of Vlasov [7] and Andreev.[8]

The effect of the wave on the background noise leads to the result that the noise does not coincide with the equilibrium distribution function of a homogeneous medium. In this connection, special attention should be given separately to the problem of which system of coordinates is being used in the analysis. Since stationary waves in an unbounded medium are being considered here, we can connect that system of coordinates only with the noise (one can connect the set of coordinates with the wave but in essence this contributes nothing new). If we do not ascertain the effect of the wave on the noise, then the latter will be known to us with accuracy to quantities of second order of smallness (if one assumes the amplitude of the wave to be a quantity of first order of smallness). This uncertainty in the choice of a system of coordinates has led different authors to different conclusions on the relation of the wave frequency and its amplitude. It must be noted that, thanks to the Doppler effect, it is always possible to choose such a set of coordinates in which the correction to the frequency obtained from the linear approximation is equal to zero.

2. THE BASIC EQUATIONS. FIRST APPROXIMATION

One of the solutions of Eq. (1.1), under the condition of boundedness of the integral $\int K(\mathbf{r})\, d\mathbf{r}$, is a function that depends only on the velocity, $f = f_0(\mathbf{v})$. This corresponds to the equilibrium distribution in coordinate space with density $\rho_0 = \int f_0(\mathbf{v})\,d\mathbf{v}$. We shall seek a solution of Eq. (1.1) which differs slightly from $f_0(\mathbf{v})$:

$$f(\mathbf{r}, \mathbf{v}, t) = f_0(\mathbf{v}) + f_I(\mathbf{r}, \mathbf{v}, t), \quad f_I \ll f_0. \tag{2.1}$$

Only plane waves are considered in the present work. The direction of the x axis is chosen along the direction of propagation of the waves. In this case, f will depend only on x, v, t. We now carry out integration over $dy'dz'$ in the third term of Eq. (1.1). Integrating Eq. (1.1) over the inessential variables v_y and v_z, we get as a result the following equation for the function $f_I(x, v, t) = \int f_I(x, \mathbf{v}, t)\,dv_y dv_z$.

$$\frac{\partial f_I}{\partial t} + v\frac{\partial f_I}{\partial x} - \frac{1}{m}\frac{df_0}{dv}\frac{\partial}{\partial x}\int K(|\mathbf{r}-\mathbf{r}'|)f_I(x', v', t)\,dv'\,d\mathbf{r}'$$

$$= \frac{1}{m}\frac{\partial f_I}{\partial v}\frac{\partial}{\partial x}\quad K(|\mathbf{r}-\mathbf{r}'|)f_I(x', v', t)\,dv'd\mathbf{r}';$$

$$v \equiv v_x, \quad f_0(v) = \int f_0(\mathbf{v})\,dv_y dv_z. \tag{2.2}$$

We shall solve this equation by successive approximations:

$$f_I = f_1 + f_2 + f_3 + \ldots$$

The first approximation satisfies Eq. (2.2) without the right hand side. Since a similar linear equation has been studied by many authors, we shall only briefly recall the general results.

The function f_1 for a plane wave has the form

$$f_1(x, v, t) = g(v)\cos(\omega t - kx) \tag{2.3}$$

with constant ω and k. Substitution gives the following expression for $g(v)$:

$$g(v) = \frac{k}{m}\sigma_k\frac{df_0/dv}{vk-\omega}\delta\rho, \tag{2.4}$$

where $\delta\rho$ is an arbitrary parameter which is assumed to be small in comparison with ρ_0, and

$$\sigma_k = \int K(|\mathbf{r}-\mathbf{r}'|)\cos k(x-x')\,d\mathbf{r}'.$$

We shall assume $\sigma_k > 0$. The meaning of this will be evident from what follows. The resultant solution exists in this case if ω and k are related to each other by the dispersion equation

$$\frac{k}{m}\sigma_k\int_{-\infty}^{+\infty}\frac{df_0/dv}{kv-\omega}\,dv = 1. \tag{2.5}$$

This equation needs further consideration, since the denominator of the integrand vanishes for $v = \omega/k$.[9] A detailed analysis of this question does not enter into our problem; therefore, we shall simply assume in the following that df_0/dv vanishes in some region of change of v which contains the point $v = \omega/k$.[10] Such an assumption by no means eliminates every possibility that the hydrodynamic flow is itself a current of "entrained" particles. Many distributions, among them the Maxwellian, do not satisfy this restriction. But if the phase velocity of the waves ω/k is sufficiently large in comparison with the mean square velocity of the thermal motion of the particles, then these distributions, and also their first derivatives, differ only slightly from zero for $v \sim \omega/k$. This small difference between df_0/dv and zero should not change the final results materially. Therefore, the results we have obtained should be valid for any distribution f_0 if the phase velocities of the waves are sufficiently large.

The explicit dependence of the velocity ω on the wave vector k can be found by expanding the denominator of the integrand in (2.5) in a series. If we assume that $f_0(v)$ is an isotropic function of the velocity, and use three-dimensional notation, we obtain the following result:

$$\frac{\omega^2}{k^2} = \frac{\sigma_k\rho_0}{m} + \langle\mathbf{v}^2\rangle + \frac{m}{\sigma_k\rho_0}(\langle\mathbf{v}^4\rangle - \langle\mathbf{v}^2\rangle^2) + \ldots;$$

$$\langle\mathbf{v}^{2n}\rangle = \frac{1}{\rho_0}\int \mathbf{v}^{2n}f_0(\mathbf{v})\,d\mathbf{v}, \qquad n = 1, 2. \tag{2.6}$$

This expansion is valid for $m\langle\mathbf{v}^2\rangle/\sigma_k\rho_0 \ll 1$. For real ω, it is necessary that $\sigma_k > 0$.

In order to obtain the dependence of the density on the time and the coordinates, it is necessary to integrate Eq. (2.3) over the velocity. With account of the dispersion equation (2.5), this gives

$$\rho_1 = \delta\rho\cos(\omega t - kx).$$

This clarifies the meaning of the quantity $\delta\rho$.

3. SECOND APPROXIMATION. HYDRODYNAMIC FLOW

The approximation considered above for the solution of Eq. (2.2) must take into account possible corrections to the frequency ω, which are determined from the dispersion equation (2.5). The exact value of the frequency Ω will have the form

$$\Omega = \omega + \omega^{(1)} + \omega^{(2)} + \ldots,$$

where $\omega^{(1)}, \omega^{(2)}, \ldots$ are terms of higher order of smallness in comparison with ω. Therefore, in the investigation of terms of higher order of smallness in Eq. (2.2), it is necessary to consider that the time derivative possesses a term of second order proportional to $\omega^{(1)}$, in addition to the term $\partial f_2/\partial t$.

Using (2.3), we obtain the following equation for the function f_2:

$$\frac{\partial f_2}{\partial t}+v\frac{\partial f_2}{\partial x}-\frac{1}{m}\frac{df_0}{dv}\frac{\partial}{\partial x}\int K(|\mathbf{r}-\mathbf{r}'|)f_2(x',v',t)\,dv'\,d\mathbf{r}'$$
$$=\frac{k}{2m}\frac{dg}{dv}\sigma_k\delta\rho\sin 2(\Omega t-kx)+\omega^{(1)}g(v)\sin(\Omega t-kx). \quad (3.1)$$

We seek a solution of this equation in the form

$$f_2(x,v,t)=\varphi_1(v)+\varphi_2(v)\cos(\Omega t-kx)+\varphi_3(v)\cos 2(\Omega t-kx). \quad (3.2)$$

Thus, there is a stationary contribution to the equilibrium distribution function in the second approximation, a contribution generated by the wave, and also a second harmonic and a term analogous to f_1. Knowing the stationary term φ_1, we can compute the hydrodynamic flow δj according to the formula

$$\delta j=\int_{-\infty}^{+\infty}v\varphi_1(v)\,dv. \quad (3.3)$$

The stationary term φ_1 is considered in this section. If the expression (3.2) is substituted directly in Eq. (3.1), then φ_1 is not determined. This is connected with the fact that one cannot take as the distribution function a completely arbitrary function of the velocity, which falls off sufficiently rapidly as $|\mathbf{v}|\to\infty$. Therefore, one can divide the distribution $f(\mathbf{r},\mathbf{v},t)$ into the two components (2.1) only with an accuracy to within an arbitrary function of the velocity. When $f_I(\mathbf{r},\mathbf{v},t)$ is considered in first approximation, the problem of this arbitrary function does not arise, since we limit ourselves to solutions of the form (2.3). But in the second approximation the division of the distribution into $f_0(\mathbf{v})$ and $f_I(\mathbf{r},\mathbf{v},t)$, is complicated by the appearance of a function of the velocity $\varphi_1(\mathbf{v})$ brought about by the presence of the wave in the first approximation. Moreover, it is also necessary to take into account the remark on the indeterminacy in the choice of a system of coordinates, made in the introduction.

In order to determine the function $\varphi_1(\mathbf{v})$, we proceed in the following fashion: we assume that the amplitude of the wave is a slowly varying function of the time. Then φ_1 must also depend on the time, and is thus separated from f_0. The dependence of the wave amplitude on the time must be such that the linearized equation (2.2) is satisfied. Such a dependence must be obtained if we assume that the frequency ω has a small imaginary part $i\gamma$; subsequently we let γ approach zero. The actual solution of the linearized equation (2.2) can be written here in the form

$$f_1(x,v,t)=\frac{1}{2}g(v)e^{ipt-ikx}+\frac{1}{2}g^*(v)e^{-ip^*t+ikx}, \quad (3.4)$$

$$p=\omega+i\gamma,\qquad g(v)=\frac{k}{m}\sigma_k\frac{df_0/dv}{vk-p}\delta\rho. \quad (3.5)$$

If $\gamma\to 0$, then these formulas transform to the formulas (2.3) and (2.4).

Now, using (3.4), we obtain the following equation for the function f_2:

$$\frac{\partial f_2}{\partial t}+v\frac{\partial f_2}{\partial x}-\frac{1}{m}\frac{df_0}{dv}\frac{\partial}{\partial x}\int K(|\mathbf{r}-\mathbf{r}'|)f_2(x',v',t)\,dv'\,d\mathbf{r}'$$
$$=\frac{ik}{4m}\sigma_k\left[\frac{dg}{dv}\int g^*(v')\,dv'-\frac{dg^*}{dv}\int g(v')\,dv'\right]e^{-2\gamma t}$$
$$+Q(x,v,t),$$

where $Q(x,v,t)$ are oscillating terms similar to the terms on the right hand side of Eq. (3.1). The solution of this equation will be sought in the form of an expression analogous to (3.2). But now the first term must be written in the form $\varphi_1(v)\exp(-2\gamma t)$; the other terms must also undergo changes associated with the fact that $\gamma\neq 0$. Substitution gives the following formula for $\varphi_1(v)$:

$$\varphi_1(v)=-\frac{ik}{8m}\sigma_k\frac{1}{\gamma}\left[\frac{dg}{dv}\int g^*(v')\,dv'-\frac{dg^*}{dv}\int g(v')\,dv'\right].$$

If we transform this formula with the help of (3.5), and then let γ approach zero, we get

$$\varphi_1(v)=\frac{k^2}{4m^2}\sigma_k^2(\delta\rho)^2\frac{d}{dv}\frac{df_0/dv}{(vk-\omega)^2}. \quad (3.6)$$

We now compute the hydrodynamic flow. Substituting (3.6) and (3.3), and integrating by parts, we find

$$\delta j=-\frac{k^2}{4m^2}\sigma_k^2(\delta\rho)^2\int_{-\infty}^{+\infty}\frac{df_0/dv}{(vk-\omega)^2}dv.$$

By expanding the denominator in a series with the aid of the same assumptions as in the derivation of the relation (2.6), this formula can be written in the form

$$\delta j=\frac{1}{2}\frac{(\delta\rho)^2}{\rho_0}\left(\frac{\sigma_k\rho_0}{m}\right)^{1/2}$$
$$\times\left[1+\frac{1}{2}\frac{m^2}{\sigma_k\rho_0}\langle v^2\rangle+\frac{3}{2}\frac{m^2}{\sigma_k^2\rho_0^2}\left(\langle v^4\rangle-\frac{13}{12}\langle v^2\rangle^2\right)+\ldots\right].$$

If the thermal motion of the particles is neglected, one then gets a formula obtained by Andreev[8] on the basis of a quantum-mechanical consideration for small k.

4. THE DISPERSION RELATION IN HIGHER APPROXIMATIONS

We consider the second and third terms in the functions $f_2(x, v, t)$, (3.2). Substituting (3.2) in (3.1) and equating the coefficients for $\sin(\Omega t - kx)$ and $\sin 2(\Omega t - kx)$ on the right and left sides, we get equations for $\varphi_2(v)$ and $\varphi_3(v)$ (we retain only terms of second order of smallness):

$$(vk-\omega)\varphi_2(v) - \frac{k}{m}\sigma_k \frac{df_0}{dv}\int_{-\infty}^{+\infty}\varphi_2(v')\,dv' = \omega^{(1)} g(v), \quad (4.1)$$

$$(vk-\omega)\varphi_3(v) - \frac{k}{m}\sigma_{2k}\frac{df_0}{dv}\int_{-\infty}^{+\infty}\varphi_3(v')\,dv' = \frac{k}{4m}\sigma_k \frac{dg}{dv}\delta\rho. \quad (4.2)$$

We divide the first of these equations by $vk - \omega$ and integrate over v from $-\infty$ to $+\infty$:

$$\left[1 - \frac{k}{m}\sigma_k \int_{-\infty}^{+\infty}\frac{df_0/dv}{vk-\omega}dv\right]\int_{-\infty}^{+\infty}\varphi_2(v')\,dv' = \omega^{(1)}\int_{-\infty}^{+\infty}\frac{g(v)}{vk-\omega}dv.$$

The first factor on the left side is equal to zero as a result of the dispersion equation (2.5). But since the integral on the right hand side is not equal to zero in the general case, it follows that

$$\omega^{(1)} = 0.$$

Therefore, Eq. (4.1) yields

$$\varphi_2(v) = C_2 g(v),$$

where C_2 is an arbitrary constant.

We now consider Eq. (4.2). We divide it by $vk - \omega$ and integrate over v from $-\infty$ to $+\infty$. Then, by using the dispersion equation, the resultant expression can be represented in the form

$$\int_{-\infty}^{+\infty}\varphi_3(v)\,dv = \frac{k}{4m}\frac{\sigma_k^2}{\sigma_k - \sigma_{2k}}\delta\rho\int_{-\infty}^{+\infty}\frac{dg/dv}{vk-\omega}dv \quad (4.3)$$

under the condition $\sigma_k \neq \sigma_{2k}$. Substituting this expression in Eq. (4.2), we find

$$\varphi_3(v) = \frac{k}{4m}\frac{\sigma_k\sigma_{2k}}{\sigma_k-\sigma_{2k}}g(v)\int_{-\infty}^{+\infty}\frac{dg/dv'}{v'k-\omega}dv' + \frac{k}{4m}\sigma_k\frac{dg/dv}{vk-\omega}\delta\rho. \quad (4.4)$$

As we have just seen, the correction to the frequency is equal to zero in this approximation. In order to find the first nonvanishing correction, let us consider the third approximation. Retaining the terms of third order of smallness in Eq. (2.2), and using Eqs. (2.3) and (3.2), we obtain the following equation:

$$\frac{\partial f_3}{\partial t} + v\frac{\partial f_3}{\partial x} - \frac{1}{m}\frac{df_0}{dv}\frac{\partial}{\partial x}\int K(|\mathbf{r}-\mathbf{r}'|) f_3(x', v', t)\,dv'd\mathbf{r}'$$
$$= \left[\frac{k}{m}\sigma_k\frac{d\varphi_1}{dv}\delta\rho + \frac{k}{m}\sigma_{2k}\frac{dg}{dv}\int_{-\infty}^{+\infty}\varphi_3(v')\,dv' - \frac{k}{2m}\sigma_k\frac{d\varphi_3}{dv}\delta\rho + \omega^{(2)}g\right]\sin(\Omega t - kx)$$
$$+ \alpha(v)\sin 2(\Omega t - kx) + \beta(v)\sin 3(\Omega t - kx),$$

where $\alpha(v)$ and $\beta(v)$ are certain functions of v, the specific form of which is not of interest. The solution of this equation has the form

$$f_3(x, v, t) = \psi_1(v) + \psi_2(v)\cos(\Omega t - kx) + \psi_3(v)\cos 2(\Omega t - kx) + \psi_4(v)\cos 3(\Omega t - kx).$$

Of this entire expression, let us consider only the second term. Substitution yields an equation for $\psi_2(v)$:

$$(vk-\omega)\psi_2 - \frac{k}{m}\sigma_k\frac{df_0}{dv}\int_{-\infty}^{+\infty}\psi_2(v')\,dv' = \frac{k}{m}\sigma_k\frac{d\varphi_1}{dv}\delta\rho$$
$$+ \frac{k}{m}\sigma_{2k}\frac{dg}{dv}\int_{-\infty}^{+\infty}\varphi_3(v')\,dv' - \frac{k}{2m}\sigma_k\frac{d\varphi_3}{dv}\delta\rho + \omega^{(2)}g(v).$$

For the determination of $\omega^{(2)}$, we use the same method which we applied in the determination of $\omega^{(1)}$, i.e., we divide this equation by $vk - \omega$ and integrate over v from $-\infty$ to $+\infty$, using the dispersion equation (2.5). As a result, we obtain the formula

$$\omega^{(2)} = k\left[\frac{1}{2}\sigma_k\delta\rho\int_{-\infty}^{+\infty}\frac{d\varphi_3/dv}{vk-\omega}dv - \sigma_k\delta\rho\int_{-\infty}^{+\infty}\frac{d\varphi_1/dv}{vk-\omega}dv - \sigma_{2k}\int_{-\infty}^{+\infty}\varphi_3(v')\,dv'\int_{-\infty}^{+\infty}\frac{dg/dv}{vk-\omega}dv\right]\Big/ m\int_{-\infty}^{+\infty}\frac{g(v)}{vk-\omega}dv.$$

Substitution here of Eqs. (2.4), (3.6), (4.3), and (4.4) enables us to put this formula in the form

$$\omega^{(2)} = -\frac{k^4\sigma_k^2}{8m^2}(\delta\rho)^2\left[\int_{-\infty}^{+\infty}\frac{df_0/dv}{(vk-\omega)^5}dv + \frac{k}{m}\frac{\sigma_k\sigma_{2k}}{\sigma_k-\sigma_{2k}}\left(\int_{-\infty}^{+\infty}\frac{df_0/dv}{(vk-\omega)^3}dv\right)^2\right]\left(\int_{-\infty}^{+\infty}\frac{df_0/dv}{(vk-\omega)^2}dv\right)^{-1}.$$

If the same assumptions are used as in the derivation of the relation (2.6), and if we expand the denominator of the integrals in a series, then this formula takes the form

$$\omega^{(2)} = \frac{k}{16}\left(\frac{\sigma_k\rho_0}{m}\right)^{1/2}\left(\frac{\delta\rho}{\rho_0}\right)^2\left[5 + \frac{9\sigma_{2k}}{\sigma_k - \sigma_{2k}}\right.$$

$$+\frac{3}{2}\left(35+\frac{39\sigma_{2k}}{\sigma_k-\sigma_{2k}}\right)\frac{\varkappa T}{\sigma_k\rho_0}+\ldots\Big].$$

Here, in place of the mean square velocity, we introduce the temperature $3\kappa T/m = \langle v^2\rangle$.

5. ELECTRON PLASMA

A particular but important case of systems described by Eqs. (1.1) is a plasma consisting of electrons interacting according to Coulomb's law, in which the collisions between particles can be neglected. For the existence of an equilibrium distribution of the electrons in space, it is necessary to assume that there are positive ions which neutralize the negative charge. In studying the propagation of waves in a plasma, we neglect the motion of the ions in view of their large mass. Therefore the role of ions in this approximation is reduced only to the compensation of the stationary negative charges. As is easy to see, the function $\varphi_1(v)$ does not make any contribution to the density and therefore there are no additional negative charges in the second approximation.

For particles interacting by Coulomb's law,

$$\sigma_k = 4\pi e^2 / k^2,$$

where e is the charge on the particle. If we substitute this expression for σ_k in (2.6), the dispersion relation then takes the form

$$\omega^2=\omega_0^2\Big[1+\frac{k^2}{\omega_0^2}\langle v^2\rangle+\frac{k^4}{\omega_0^4}(\langle v^4\rangle-\langle v^2\rangle^2)+\ldots\Big];$$
$$\omega_0^2=4\pi e^2\rho_0/m.$$

The hydrodynamic flow is given in this case by the expression

$$\delta j=\frac{1}{2}\frac{(\delta\rho)^2}{\rho_0}\frac{\omega_0}{k}\Big[1+\frac{1}{2}\frac{k^2}{\omega_0^2}\langle v^2\rangle+\frac{3}{2}\frac{k^4}{\omega_0^4}(\langle v^4\rangle-\frac{13}{12}\langle v^2\rangle^2)+\ldots\Big]. \tag{5.1}$$

The amplitude-dependent contribution to the frequency is equal to

$$\omega^{(2)}=\frac{\omega_0}{2}\left(\frac{\delta\rho}{\rho_0}\right)^2\Big[1+9\frac{\varkappa T}{m}\frac{k^2}{\omega_0^2}+\ldots\Big]. \tag{5.2}$$

Equations (5.1) and (5.2) reveal the following feature. We transform to a system of coordinates in which the stationary macroscopic flows are absent (in second approximation). This set of coordinates is moved relative to that considered above with the velocity $u_0 = \delta j/\rho_0$ parallel to the propagation of the wave. Thanks to the Doppler effect, the wave frequency in this system of coordinates is equal to

$$\Omega'=\Omega-u_0k.$$

If we now substitute (5.1) and (5.2) here, we get

$$\Omega'=\omega_0\Big[1+\frac{15}{4}\frac{k^2}{\omega_0^2}\frac{\varkappa T}{m}\left(\frac{\delta\rho}{\rho_0}\right)^2+\ldots\Big].$$

Thus, in the given set of coordinates, for $T = 0$, the frequency of the wave is equal to ω_0 even in second approximation. But even though stationary macroscopic flows are absent in the given set of coordinates, the "noise" is not equal to $f_0(v)$; it is equal to $f_0(v + u_0) + \varphi_1(v)$ with accuracy to quantities of third order of smallness.

It should be noted that in the general case of arbitrary media, systems of coordinates do not exist in which stationary macroscopic flows would be absent, and the contribution to the frequency would be equal to zero in the second approximation.

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