CONTRIBUTION TO THE THEORY OF THE SKIN EFFECT IN METALS AT LOW TEMPERATURES

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The high-frequency properties of very pure metals at low temperatures are studied for the case in which the collisions between electrons and phonons not accompanied by umklapp processes occur with maximum probability. It is shown that these collisions considerably change the nature of the skin effect. In particular, a rather broad frequency region appears in which the surface impedance depends on the frequency and temperature in an unusual way.

IN a previous paper,[1] it was noted that purely normal collisions between electrons and phonons, unaccompanied by umklapp-processes, should take place with appreciable probability in certain metals, at sufficiently low temperatures, in addition to the scattering of electrons by the various statistical defects of the crystalline lattice or boundaries of the specimen. It was shown that these processes actually lead to an effective interelectronic interaction, although the corresponding free path length l_N is identical in order of magnitude with the electron-phonon length:

$$l_N \approx a(\varepsilon_0/\Theta)(\Theta/T)^5$$

(a is the lattice constant, ϵ_0 is the surface energy, and Θ is the Debye temperature). The effect of these normal collisions on the static electrical conductivity has been investigated in the previous paper. The most significant results were obtained for "thin" samples, the thickness d of which is small in comparison with the electron impurity free path length l_i, but at the same time large in comparison with l_N. The electron moves in the metal like a Brownian particle, and its effective free path length relative to collisions with the boundary is $l_{eff} \approx d^2/l_N$.

It is clear that a similar situation can arise even in the case of the skin effect, but then the role of thickness of the specimen d will be played by the thickness of the skin layer δ.

We begin with the formal solution of the problem, although, as will be seen below, the basic qualitative results could be obtained from intuitive considerations. It is natural to use the hydrodynamic description of electrons in the metal, just as was done in [1].

The complete set of equations has the form

$$i\omega\mathbf{u} - e\mathbf{E}/m = \nu\Delta\mathbf{u} - \mathbf{u}/\tau_V,$$
$$\Delta\mathbf{E} = 4\pi i\omega\mathbf{j}/c^2, \quad \mathbf{j} = ne\mathbf{u}. \tag{1}$$

Here $\mathbf{u}(\mathbf{r})e^{i\omega t}$ is the velocity of ordered motion of the electrons; $\mathbf{E}(\mathbf{r})e^{i\omega t}$ is the electric field intensity; e is the charge, m the mass, n the density of the electrons; $\nu = l_N v_0$ is the kinematic viscosity, v_0 is the velocity on the Fermi surface; according to [1], $l_N = (1/15)l_{ep}$, where l_{ep} is the electron-phonon length which enters into the Bloch theory of electric conductivity; $\tau_V = l_V/v_0$ is the time of flight for collisions with loss of quasimomentum; in what follows, it will be assumed that this is a collision with different microscopic defects of the lattice, of the type of impurity atoms. For simplicity, the dispersion law of the electrons is assumed to be isotropic.

The applicability of the hydrodynamic approach, in addition to the condition $l_N \ll l_V$, which we shall assume to be satisfied, also requires the fulfillment of the inequalities

$$\omega\tau_N \ll 1, \qquad l_N \ll \delta, \tag{2}$$

where $\tau_N = l_N/v_0$ and δ is the skin depth.

As usual, we shall consider the problem of normal incidence of a plane electromagnetic wave on a metallic half-space. Let the z axis be directed into the metal perpendicular to its boundary. After elimination of $\mathbf{u}$ from the system (1), we get the following equation for $E(z)$:

$$E'''' - \alpha E'' + i\beta E = 0,$$

where

$$\alpha^{-1} = l_N l_V(1 + i\omega\tau_V)^{-1},$$
$$\beta^{-1} = \delta_0^2 l_N v_0/\omega, \qquad \delta_0^{-2} = 4\pi ne^2/mc^2.$$

The solution of this equation should satisfy the boundary conditions

$$E(0) = E_0, \quad E(\infty) = E''(\infty) = E''(0) = 0,$$

where E_0 is the amplitude of the field on the surface; the conditions on the second derivatives follow from the requirements of the vanishing of the velocity $u(z)$ on the boundary of the specimen and at infinity.

It is not difficult to show that

$$E(z) = E_0[-s_2^2 e^{s_1 z} + s_1^2 e^{s_2 z}] / (s_1^2 - s_2^2),$$

where s_1 and s_2 are the roots of the equation $s^4 - \alpha s^2 + i\beta = 0$, which possess a negative real part. The surface impedance is then

$$\zeta = -i\omega c^{-1} E(0) / E'(0) = -i\omega c^{-1}(1/s_1 + 1/s_2).$$

Proceeding to the study of these expressions, we first note that $\omega\tau_V \ll 1$. As becomes clear from the following, the reverse inequality is not possible in practice in the range of frequencies considered. We remark that in this case we do not need to trouble about the first of the conditions (2), since, by assumption, $\tau_N \ll \tau_V$.

We consider possible limiting cases.

a) For $\alpha^2 \gg \beta$ one of the roots, say s_2, is much larger than the other and its effect can be neglected. Then

$$E(z) \cong E_0 \exp(s_1 z),$$

$$s_1 = -(1+i)\delta_n^{-1}, \quad \delta_n = \delta_0(2/\omega\tau_V)^{1/2},$$

δ_n is the skin depth for the normal skin effect. This result is easily understood, since the inequality $\alpha^2 \gg \beta$, rewritten in the form $l_V \ll \delta_n^2/l_N$, means that an electron diffusing in the limits of the skin layer makes many collisions with loss of momentum. We note that the second of conditions (2) is satisfied since, for $l_N \ll l_V$, it follows from the given inequality that $l_N \ll \delta_n$.

b) For $\alpha^2 \ll \beta$, as is easy to show,

$$s_1 = -is_2 = -\beta^{1/4} e^{-i\pi/8}, \quad E(z) = {}^1\!/_2 E_0(e^{s_1 z} + e^{is_1 z}).$$

Thus, the electric field falls off inside the metal, roughly speaking, like $\exp(-z/\delta^*)$, where the effective thickness of the skin layer is

$$\delta^* \approx \beta^{-1/4} = (\delta_0^2 v_0 \omega^{-1} l_N)^{1/4}.$$

The last expression can be obtained by an intuitive method similar to that proposed by Pippard [2] for the interpretation of the results of the theory of the anomalous skin effect. We note that the skin depth can always be written in order of magnitude in the form

$$\delta \sim \delta_0/\sqrt{\omega\tau} \approx (mv_0/n^* e^2 \omega l^*)^{1/2},$$

if by n^* is meant the density of those electrons which interact appreciably with the field, and by l^* the smaller of the following two quantities: the path length of the electron in the skin layer and the mean free path with loss of momentum l_V. In the limits of the hydrodynamic approach we have $l_N \ll \delta$, and therefore all electrons play essentially the same role, i.e., $n^* \approx n$. However, it follows from the inequality $\alpha^2 \ll \beta$ that the path followed by the electron in the skin layer is $(\delta^*)^2/l_N \ll l_V$, and therefore one must take $l^* \approx \delta^2/l_N$. Finally we get the equation

$$\delta \approx \delta_0 (v_0 l_N / \omega\delta^2)^{1/2},$$

from which follows the desired result: $\delta^* < \delta_{an}$.

The inequalities $l_N \ll \delta^*$ and $\alpha^2 \ll \beta$ lead to the following limits on the value of l_N:

$$\delta_n^2 / l_V \ll l_N \ll \delta_{an},$$

where $\delta_{an} \approx (\delta_0^2 v_0/\omega)^{1/3}$ is the skin depth for the anomalous skin effect.

It is easy to establish the fact that $\delta^*/\delta_{an} \approx (l_N/\delta_{an})^{1/4}$ i.e., $\delta^* < \delta_{an}$. However, upon increase in frequency, δ_{an} falls off more rapidly than δ^*, and for $l_N \approx \delta_{an}$, these quantities coincide. Thus, the region considered is directly adjacent to the region of the anomalous skin effect (in which the hydrodynamic description is obviously unsuitable).

Finally, we write out the expression for the surface impedance in the different frequency regions. We denote by $\omega_1 = \tau_V^{-1}(\delta_0/l_V)^2$ the frequency at which a transition takes place from the normal to the anomalous skin effect in the ordinary case $(l_V \approx \delta_n(\omega_1) \approx \delta_{an}(\omega_1))$.

For $\omega/\omega_1 \ll l_V/l_N$, the skin effect remains normal,

$$\zeta_n \approx (\omega/\omega_0^2\tau_V)^{1/2} e^{i\pi/4}, \quad \omega_0 = c/\delta_0.$$

Furthermore, in the region[1)] $l_V/l_N \ll \omega/\omega_1 \ll (l_V/l_N)^3$, we have

$$\zeta^* \approx (v_0/c)^{1/2} (\omega^3\tau_N/\omega_0^2)^{1/4} e^{i3\pi/8}. \quad (3)$$

Finally, for $\omega/\omega_1 \gg (l_V/l_N)^3$ (or $l_N \gg \delta_{an}$), the anomalous skin effect takes place:

$$\zeta_{an} \approx (v_0/c)^{1/3} (\omega/\omega_0)^{2/3} e^{i\pi/3}.$$

In these formulas, numerical coefficients of the order of unity have been omitted since their values are connected with the assumption of iso-

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[1)] We note that this region can be rather broad. Thus, for $\delta_0 = 3 \times 10^{-6}$ cm, $l_V = 0.1$ cm, $l_N = 10^{-3}$ cm and $v_0 = 10^8$ cm/sec, $10^2 \ll \omega \ll 10^6$ sec^{-1}. From this estimate, it is evident that the inequality $\omega\tau_V \ll 1$ is virtually always satisfied.

tropy of the dispersion laws of electrons and phonons. It is clear that the order of magnitude of the momentum and its dependence on the fundamental physical parameters remain valid for any dispersion law. This assertion also applies to the relation between the real and imaginary parts of the impedance in the cases of normal and anomalous skin effects. In the most interesting case (3), unfortunately, the ratio of Re ζ^* to Im ζ^* depends on the form of the dispersion law of electrons and phonons in a very complicated way, and therefore the phase factor $\exp(i3\pi/8)$ apparently has no physical meaning. We note that, in the general case, the viscosity ν is a tensor of fourth rank and therefore even for a cubic lattice, say, the surface impedance should have tensor properties.

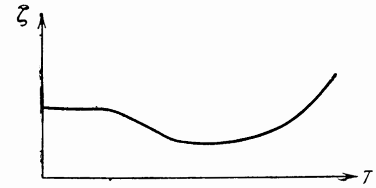
It is evident from what has been pointed out above that the normal electron-phonon collisions, in the case in which they take place more frequently than collisions with loss of momentum, considerably change the character of the skin effect in metals.

In conclusion, we reiterate the fundamental results.

1. The skin effect remains normal up to much higher frequencies than usual. On the other hand, the region of the anomalous skin effect is sharply cut off in the direction of low frequencies (by $(l_V/l_N)^3$).

2. There is a broad intermediate region of frequencies in which the surface impedance depends on the frequency in an unusual way, and is chiefly a function of the temperature ($\zeta^* \sim \omega^{3/4} T^{-5/4}$).

3. The dependence of $\zeta(T)$ has the form shown in the drawing.



For very low temperatures, $\zeta = \zeta_{an}$ and is constant. After transition to the intermediate region, the impedance falls off in proportion to $T^{-5/4}$. With subsequent increase in temperature, the skin effect becomes normal. In this region the surface impedance is at first constant and then increases with temperature after the phonon-phonon umklapp processes are turned on.

The described dependence will be observed for frequencies ω satisfying the inequalities

$$1 \ll \omega / \omega_1 \ll (l_V / l^U(T_1))^2,$$

where $l^U(T)$ is the free path length of the electrons, a length associated with the phonon-phonon umklapp processes (see [1]); the temperature T_1 is determined by the relation $l^U(T_1) \approx l_N(T_1)$.

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[1] R. N. Gurzhi, JETP 47, 1415 (1964); Soviet Phys. JETP **20**, 953 (1965).

[2] A. B. Pippard, Proc. Roy. Soc. (London) **A191**, 370, 385 (1947).

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