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FUNCTIONALS AND THE RANDOM-FORCE METHOD IN TURBULENCE THEORY

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The random-force method previously proposed[1,2] in the Lagrangian description of turbulence (when the motion of fixed fluid particles is being traced) is used to describe the Euler velocity field $v_i(\mathbf{x}, t)$. An equation relating the second- and third-order velocity structural functions with the external correlation function is derived. From this equation it follows, in particular, that the third-order structural function decreases like r^{-4} at distances larger than the external correlation scale L. Further, an equation describing the equilibrium conditions of turbulent flow is derived for the characteristic velocity functional. In the limiting case when $L \to \infty$ a single external parameter, the energy influx ϵ, enters the equation, in accordance with the similarity hypothesis proposed by Kolmogorov.

1. FORMULATION OF THE PROBLEM

THE method of random forces in the Lagrangian description of turbulence (when the motion of a system of fixed liquid particles is traced) was proposed by the author in earlier papers[1,2], in which the analysis was purely statistical and based on the Langevin equations for the velocity of a liquid particle. On going over to the Euler description of turbulence, i.e., to a description of the velocity field $v_i(\mathbf{x}, t)$, it is natural to generalize the method of random forces by including the equations of hydrodynamics.

We write down the equations of motion of a viscous incompressible liquid with random force in the right side

$$\frac{\partial v_i(\mathbf{x}, t)}{\partial t} = - v_k(\mathbf{x}, t)\frac{\partial v_i(\mathbf{x}, t)}{\partial x_k} - \frac{\partial P(\mathbf{x}, t)}{\partial x_i} + \nu\frac{\partial^2 v_i(\mathbf{x}, t)}{\partial x_k^2} + f_i(\mathbf{x}, t); \qquad \frac{\partial v_i(\mathbf{x}, t)}{\partial x_i} = 0. \tag{1.1}$$

Here P—pressure divided by the constant density, and ν—kinematic viscosity; summation from 1 to 3 is implied for the repeated indices. Without loss of generality, the force can be assumed to be solenoidal, since the potential part can be included in the pressure gradient. The pressure is in turn connected with the velocity by the relation

$$\Delta P = -\frac{\partial v_i}{\partial x_k}\frac{\partial v_k}{\partial x_i}, \tag{1.2}$$

which follows from (1.1).

We shall consider the model of homogeneous, isotropic, and statistically stationary turbulent flow of a liquid, the kinetic energy of which is maintained by work done by external forces. The forces will also be assumed to be homogeneous and isotropic random functions of the coordinates and statistically stationary in time. In accordance with the similarity idea advanced by Kolmogorov[3], we shall try to choose the forces in such a way, that the energy influx ϵ will, in scales that are sufficiently small compared with some external turbulence scale L, be the main parameter characterizing the influence of the external forces. In[1,2], when considering the inertial interval of times in the Lagrangian description of the turbulence, we made use of random forces that were δ-correlated in time and had a Gaussian probability distribution. Such forces are characterized only by the value of the energy influx. In the present article we also use Gaussian forces that are δ-correlated in time, but the Euler description of turbulence.

Gaussian forces with zero mean value are defined completely by their second-rank correlation tensor, which in this case is of the form

$$\langle f_i(\mathbf{x} + \mathbf{r}, t + \tau) f_k(\mathbf{x}, t)\rangle = F_{ik}(\mathbf{r})\delta(\tau), \tag{1.3}$$

where the angle brackets denote probability averaging, $\delta(\tau)$—δ-function, and F_{ik}—spatial part of the correlation tensor. The corresponding spectral tensor

$$\mathcal{F}_{ik}(\mathbf{p}) = \frac{1}{(2\pi)^3}\int e^{-i\mathbf{p}\mathbf{x}} F_{ik}(\mathbf{x})\, d^3x \tag{1.4}$$

with account of the isotropy and the solenoidal character of the forces, is written in the form

$$\mathcal{F}_{ik}(\mathbf{p}) = \mathcal{F}(p)(\delta_{ik} - p_i p_k p^{-2}), \tag{1.5}$$

where δ_{ik}—unit tensor and $\mathcal{F}(p)$—unique scalar function characterizing the selected random forces (one can use as the defining function also the function $F_{ii}(r)$, which is connected with $\mathcal{F}(p)$ by a Fourier transformation).

The external turbulence scale is defined by the formula

$$L^{-2} = -[F_{ii}(r)]^{-1} d^2 F_{kk}(r) / dr^2|_{r=0}. \quad (1.6)$$

In the limiting case as $L \to \infty$ there should remain only one parameter ϵ characterizing the external forces. From dimensionality considerations it is clear that in this case

$$F_{ik}(\mathbf{r}) = \frac{2}{3} C\varepsilon\delta_{ik}, \quad \mathcal{F}(p) = C\varepsilon\delta(\mathbf{p}), \quad (1.7)$$

where C is a dimensionless constant, which we shall show in Sec. 3 to be equal to unity.

2. CORRELATION OF GAUSSIAN RANDOM FUNCTIONS WITH FUNCTIONALS THAT ARE DEPENDENT ON THEM

We shall find useful the following formula, which is valid for Gaussian random functions that are δ-correlated in time and homogeneous in space:

$$\langle f_i(\mathbf{x}, t) R[f] \rangle = \int F_{ik}(\mathbf{x} - \mathbf{x}') \left\langle \frac{\delta R[f]}{\delta f_k(\mathbf{x}', t) d^3x' dt} \right\rangle d^3x'. \quad (2.1)$$

Here R—functional of f, on the right side of the angle brackets is the variational derivative of this functional, F_{ik} is the spatial part of the correlation tensor, defined in accordance with (1.3), and the integral is taken over all three-dimensional space.

To prove (2.1) it is simpler technically to consider a more general case of arbitrary Gaussian random functions $f_i(s)$ with zero mean value, and with a correlation tensor

$$\langle f_i(s) f_k(s') \rangle = F_{ik}(s, s'), \quad (2.2)$$

where s—aggregate of arguments on which the random function depends. For such functions we shall prove the formula

$$\langle f_i(s) R[f] \rangle = \int F_{ik}(s, s') \left\langle \frac{\delta R[f]}{\delta f(s') ds'} \right\rangle ds', \quad (2.3)$$

where the integral extends over the region in which the functions are defined. Formula (2.1) is obtained from (2.3) as a particular case when s denotes the aggregate of the spatial coordinates and of the time and the correlation tensor has the special form (1.3).

We represent the functional R in the form of a functional Taylor series in the power-law functionals

$$R[f] = R[0] + \sum_{n=1}^{\infty} \frac{1}{n!} \int \ldots \int R^{(n)}_{i_1 \ldots i_n}(s_1, \ldots, s_n) f_{i_1}(s_1) \ldots f_{i_n}(s_n) ds_1 \ldots ds_n, \quad (2.4)$$

$$R^{(n)}_{i_1 \ldots i_n}(s_1, \ldots, s_n) = \delta^n R[f] / \delta f_{i_1}(s_1) ds_1 \ldots \delta f_{i_n}(s_n) ds_n \Big|_{f=0}. \quad (2.5)$$

The tensor (2.5) is obviously symmetrical in its arguments taken together with the tensor indices. Multiplying (2.4) by $f_i(s)$ and averaging, we obtain

$$\langle f_i(s) R[f] \rangle = \sum_{n=1}^{\infty} \frac{1}{n!} \int \ldots \int R^{(n)}_{i_1 \ldots i_n}(s_1, \ldots, s_n) \times \langle f_i(s) f_{i_1}(s_1) \ldots f_{i_n}(s_n) \rangle ds_1 \ldots ds_n. \quad (2.6)$$

We make use of the fact that the mean value of the product of an even number of quantities with a joint Gaussian probability distribution is equal to the sum of the products of the mean values of all possible pairwise combinations. The mean value of the product of an odd number of such quantities is equal to zero. It is easy to see that in this case

$$\langle f_i(s) f_{i_1}(s_1) \ldots f_{i_n}(s_n) \rangle = \sum_{\alpha=1}^{n} \langle f_i(s) f_{i_\alpha}(s_\alpha) \rangle \times \langle f_{i_1}(s_1) \ldots f_{i_{\alpha-1}}(s_{\alpha-1}) f_{i_{\alpha+1}}(s_{\alpha+1}) \ldots f_{i_n}(s_n) \rangle. \quad (2.7)$$

Substituting (2.7) in (2.6) we obtain, with account of the symmetry of the tensor (2.5),

$$\langle f_i(s) R[f] \rangle = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int F_{ii_1}(s, s_1) \times \left[\int \ldots \int R^{(n)}_{i_1 i_2 \ldots i_n}(s_1, s_2, \ldots, s_n) \langle f_{i_2}(s) \ldots f_{i_n}(s_n) \rangle ds_2 \ldots ds_n \right] ds_1. \quad (2.8)$$

On the other hand, from (2.4), again taking into account the symmetry of the tensor (2.5), we have

$$\frac{\delta R[f]}{\delta f_k(s') ds'} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int \ldots \int R^{(n)}_{k i_2 \ldots i_n}(s', s_2, \ldots, s_n) \times f_{i_2}(s_2) \ldots f_{i_n}(s_n) ds_2 \ldots ds_n \cdot \quad (2.9)$$

Substituting (2.9) in the right side of (2.3), we see that the resultant expression coincides with (2.8). This proves by the same token (2.3), and consequently also (2.1).

Using (2.3), we can easily obtain additional formulas for the correlation of the power-law functional with the arbitrary functional, and also for the correlation of two arbitrary functionals. We shall not write out these formulas, which we do not need in this article.

3. CORRELATION BETWEEN THE FORCE AND THE VELOCITY, AND STRUCTURAL VELOCITY FUNCTIONS

We multiply the first equation of (1.1) by $v_i(\mathbf{x}, t)$ and average. Taking into account stationarity, homogeneity, and incompressibility we obtain

$$\langle f_i(\mathbf{x}, t) v_i(\mathbf{x}, t)\rangle = \nu\langle \partial v_i(\mathbf{x}, t) / \partial x_k)^2\rangle \equiv \varepsilon. \qquad (3.1)$$

Hence, using (2.1) we get

$$\int F_{ik}(\mathbf{x}-\mathbf{x}') \left\langle \frac{\delta v_i(\mathbf{x}, t)}{\delta f_k(\mathbf{x}', t)\, d^3x'\, dt} \right\rangle d^3x' = \varepsilon. \qquad (3.2)$$

To calculate the variational derivative of the velocity with respect to the force we write down (1.1) in the form

$$v_i(\mathbf{x}, t) = v_i(\mathbf{x}, 0) + \int_0^t A_i[v(\tau), \mathbf{x}]\, d\tau + \int_0^t f_i(\mathbf{x}, \tau)\, d\ , \qquad (3.3)$$

where A is the operator of the Navier-Stokes equation, and the pressure is assumed eliminated with the aid of (1.2). Taking the variational derivative of each term in (3.3), we get

$$\frac{\delta v_i(\mathbf{x}, t)}{\delta f_k(\mathbf{x}', t')\, d^3x'\, dt'} = \int_{t'}^{t} \frac{\delta A_i[v(\tau), \mathbf{x}]}{\delta f_k(\mathbf{x}', t')\, d^3x'\, dt'}$$

$$+ \gamma(t-t')\,\delta_{ik}\,\delta(\mathbf{x}-\mathbf{x}'). \qquad (3.4)$$

Here $0 < t' \leq t$, $\gamma(\tau)$—unit function equal to unity when $\tau > 0$, 1/2 when $\tau = 0$, and zero when $\tau < 0$. It is shown in (3.4) that the velocity cannot depend on the force taken at a later instant of time. As $t' \to t$, the first term on the right side of (3.4) drops out, and consequently

$$\delta v_i(\mathbf{x}, t)/\delta f_k(\mathbf{x}', t)\, d^3x'\, dt = \frac{1}{2}\delta_{ik}\delta(\mathbf{x}-\mathbf{x}'). \qquad (3.5)$$

Substitution of (3.5) in (3.2) yields

$$F_{ii}(0) = 2\varepsilon, \qquad (3.6)$$

from which it follows that the constant in formulas (1.7) is equal to unity.

We now multiply (1.1) by $v_j(\mathbf{x}', t)$ and average. Symmetrizing the obtained equation and taking stationarity, homogeneity, isotropy, and incompressibility into account, we obtain

$$2\frac{\partial}{\partial x_k}\langle v_k(\mathbf{x}, t) v_i(\mathbf{x}, t) v_j(\mathbf{x}', t)\rangle - 2\nu\frac{\partial^2}{\partial x_k^2}\langle v_i(\mathbf{x}, t) v_j(\mathbf{x}', t)\rangle$$

$$= \langle f_i(\mathbf{x}, t) v_j(\mathbf{x}', t)\rangle + \langle f_j(\mathbf{x}', t) v_i(\mathbf{x}, t)\rangle. \qquad (3.7)$$

We have used here, in particular, the fact that in a homogeneous, isotropic, and incompressible stream the pressure does not correlate with the velocity[4]. Taking (2.1) and (3.5) into account, we have

$$\langle f_i(\mathbf{x}, t) v_j(\mathbf{x}', t)\rangle = \frac{1}{2}F_{ij}(\mathbf{x}-\mathbf{x}'). \qquad (3.8)$$

We transform the left side of (3.7) in analogy with the procedure used in[4]. We obtain ultimately the following equation:

$$D_3(r) - 6\nu\frac{dD_2(r)}{dr} = -\frac{2}{r^4}\int_0^r \rho^4 F_{ii}(\rho)\, d\rho, \qquad (3.9)$$

where

$$D_n(r) = \langle [v_r(\mathbf{x}+\mathbf{r}) - v_r(\mathbf{x})]^n\rangle$$

—structural functions of the velocity field (the index r denotes projection on the $\mathbf{r}$ direction).

Taking account of (3.6) and (1.6), we write

$$F_{ii}(r) = 2\varepsilon\psi(r/L), \quad \psi(0) = 1, \quad \psi''(0) = -1, \qquad (3.10)$$

where $\psi(x)$—dimensionless function. Expanding this function in a series and taking parity into consideration, we get from (3.9)

$$D_3(r) - 6\nu\frac{dD_2(r)}{dr} = -\frac{4}{5}\varepsilon r\left[1 - \frac{5}{14}\left(\frac{r}{L}\right)^2 + O\left(\frac{r}{L}\right)^4\right]. \qquad (3.11)$$

When $r \ll L$, only the first term remains in the right side of (3.11), which now goes over into the Kolmogorov equation[5].

For distances that are large compared with the internal turbulent scale $l_0 = \nu^{3/4}\epsilon^{-1/4}$[3], the second term in the right side of (3.9) is small, and consequently

$$D_3(r) = -\frac{2}{r^4}\int_0^r \rho^4 F_{ii}(\rho)\, d\rho. \qquad (3.12)$$

The turbulent stream can be homogeneous and isotropic in scales that are larger than the external correlation scale (for example, the turbulence behind a screen whose dimensions are large compared with the dimensions of each individual mesh). If we assume that $F_{ii}(\rho)$ decreases with increasing ρ sufficiently rapidly, so that the integral in (3.12) converges as $r \to \infty$, then we get at large distances

$$D_3(r) = -\alpha\varepsilon L(L/r)^4, \quad \alpha = 4\int_0^\infty x^4\psi(x)\, dx, \qquad (3.13)$$

where α—dimensionless constant.

We note that Batchelor and Proudman[6] obtained an asymptotic expression analogous to (3.13) for the problem concerning time-attenuating turbulence, under the condition that at the initial instant of time the cumulants of the velocity field decrease at large distances more rapidly than any power of the distance.

4. GENERALIZED HOPF EQUATION

Gaussian random forces δ-correlated in time were used recently by Edwards[7], who wrote down some equation for the probability distribution density of a turbulent velocity field. However, the probability distribution density in functional space, as well as the volume in functional space, has no clearcut mathematical meaning, so that the entire analysis in[7] has a heuristic character (which does not detract from the value of this interesting paper). The probability distribution in functional space is conveniently described with the aid of a characteristic functional

$$\Phi_t[y] = \langle \exp\{i(y, v(t))\}\rangle,$$
$$(y, v(t)) = \int y_i(\mathbf{x}) v_i(\mathbf{x}, t)\, d^3x \tag{4.1}$$

($y_i(\mathbf{x})$—real functions that fall off sufficiently rapidly at infinity). Different correlation moments of the velocity field are expressed in terms of variational derivatives of the functional (4.1), taken at y = 0.

The idea of using a characteristic functional in turbulence theory belongs to Hopf[8], who obtained from the Navier-Stokes equation a certain linear variational-differential equation

$$\partial\Phi_t[y]/\partial t = (\mathcal{L}_2 + \nu\mathcal{L}_1)\Phi_t[y], \tag{4.2}$$

where

$$\mathcal{L}_2\Phi = i\int \widetilde{y}_k(\mathbf{x}) \frac{\partial}{\partial x_l}\left(\frac{\delta^2\Phi}{\delta y_l(\mathbf{x})\, d^3x \delta y_k(\mathbf{x})\, d^3x}\right) d^3x, \tag{4.3}$$

$$\mathcal{L}_1\Phi = \int y_k(\mathbf{x}) \frac{\partial^2}{\partial x_l^2}\left(\frac{\delta\Phi}{\delta y_k(\mathbf{x})\, d^3x}\right) d^3x \tag{4.4}$$

($\widetilde{y}_k(\mathbf{x})$—divergence-free part of the field $y_k(\mathbf{x})$).

To investigate the stationary turbulence mode, Hopf proposed to seek that solution of his stationary equation [Eq. (4.2) without the left side], which describes the structure of the small-scale turbulence and corresponds to the Kolmogorov similarity hypotheses. However, one might think that the stationary Hopf equation does not contain such a solution, since it does not take into account the energy transfer from the large-scale to the small-scale motion. In particular, from the stationary Hopf equation we obtain, by variational differentiation, Eq. (3.9) without the right side, which, as can be readily seen, can correspond only to the quiescent state. In this connection it is advantageous to generalize the Hopf equation with account of the external forces that supply energy to the turbulent flow and assume the role of large-scale motions.

Differentiating (4.1) with respect to the time, we have with account of (1.1)

$$\frac{\partial\Phi_t[y]}{\partial t} = i\int y_k(\mathbf{x})\Big\langle\Big\{-v_l(\mathbf{x}, t)\frac{\partial v_k(\mathbf{x}, t)}{\partial x_l} - \frac{\partial P(\mathbf{x}, t)}{\partial x_k} + \nu\frac{\partial^2 v_k(\mathbf{x}, t)}{\partial x_l^2}\Big\}$$
$$\times \exp\{i(y, v(t))\}\Big\rangle d^3x + i\int y_k(\mathbf{x})\langle f_k(\mathbf{x}, t)\exp\{i(y, v(t))\}\rangle d^3x. \tag{4.5}$$

From the very procedure of the derivation of the Hopf equation[8] it follows that the first term in the right side of (4.5) coincides with the right side of (4.2). We transform the second term on the right side of (4.5), with allowance for (2.1) and (3.5):

$$\langle f_k(\mathbf{x}, t)\exp\{i(y, v(t))\}\rangle = \int F_{kl}(\mathbf{x}-\mathbf{x}')\Big\langle\frac{\delta}{\delta f_l(\mathbf{x}', t)\, d^3x'\, dt} \times [\exp\{i(y, v(t))\}]\Big\rangle d^3x' = i\int F_{kl}(\mathbf{x}-\mathbf{x}')$$
$$\times\left[\int y_m(\mathbf{x}'')\Big\langle\frac{\delta v_m(\mathbf{x}'', t)}{\delta f_l(\mathbf{x}', t)\, d^3x'\, dt}\exp\{i(y, v(t))\}\Big\rangle d^3x''\right] d^3x'$$
$$= \frac{i}{2}\left(\int F_{kl}(\mathbf{x}-\mathbf{x}')\, y_l(\mathbf{x}')\, d^3x'\right)\Phi_t[y]. \tag{4.6}$$

Ultimately we obtain

$$\partial\Phi_t[y]/\partial t = (\mathcal{L}_2 + \nu\mathcal{L}_1 + \mathcal{L}_0)\Phi_t[y], \tag{4.7}$$

$$\mathcal{L}_0 = -\frac{1}{2}\iint F_{kl}(\mathbf{x}-\mathbf{x}')\, y_k(\mathbf{x})\, y_l(\mathbf{x}')\, d^3x\, d^3x'. \tag{4.8}$$

It is natural to call (4.7) the generalized Hopf equation. The supplementary term describes the influence of the external forces, and does not depend on the concrete form of the operator of the Navier-Stokes equation. In the spectral representation we have

$$\mathcal{L}_0 = -\frac{1}{2}\int \mathcal{F}_{kl}(\mathbf{p})\, z_k(\mathbf{p})\, z_l(-\mathbf{p})\, d^3p,$$
$$z_k(\mathbf{p}) = \int e^{i\mathbf{p}\mathbf{x}}\, y_k(\mathbf{x})\, d^3x. \tag{4.9}$$

Expressions for the operators $\mathcal{L}_2$ and $\mathcal{L}_1$ in the spectral representation are given in the paper of Hopf[8].

An analogy can be drawn between Eq. (4.7) and the continual generalization of the diffusion equation in velocity space. The role of the diffusion coefficient, which is different for different wave components of the velocity field, is played by the spectral force tensor $\mathcal{F}_{ik}(\mathbf{p})$.

The stationary turbulence mode is defined by the equation [1)]

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1)We note that in the present article the concept of probabilistic averaging is taken here to have a somewhat different meaning than used by Hopf[8], who took averaging to mean averaging over the initial velocity field. In the present paper, in the case of the nonstationary problem, averaging is taken to mean over the external forces and over the initial velocity field, assumed to be independent of the external forces. In the stationary problem it remains only to average over the external forces, since the information concerning the initial velocity field drops out (ergodicity!). Actually, we are studying a stationary mode established by the action of statistically time-stationary external forces, if the liquid was at rest at t = $-\infty$.

$$(\mathscr{L}_2 + \nu\mathscr{L}_1 + \mathscr{L}_0)\,\Phi[y] = 0. \qquad (4.10)$$

It is easy to verify that (3.9) is obtained from (4.10) by variational differentiation. If we are interested in sufficiently large scales, where the effect of viscosity does not yet come into play, then the second term on the left in (4.10) can be dropped. In the limiting case, when $L \to \infty$, we have

$$\mathscr{L}_0 = -\frac{\varepsilon}{3}\left(\int y_k(\mathbf{x})\,d^3x\right)^2 = -\frac{\varepsilon}{3}\,z_k^2(0). \qquad (4.11)$$

In this case Eq. (4.10) contains only two dimensional parameters, ϵ and ν, which, in accordance with Kolmogorov's hypothesis[3], define the small-scale turbulence mode. Equation (4.10) can be used to investigate the intermittence of turbulent flow, but the approximation (5.1) is no longer applicable in this case, since intermittence is characterized not only by the magnitude of the flux ϵ but by additional parameters[9].

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