THE CAUSALITY PRINCIPLE AND THE ASYMPTOTIC BEHAVIOR OF THE SCATTERING AMPLITUDE

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The reaction considered is that of elastic scattering, $p + k \to p' + k'$, where p and p′ are the four-momenta of one of the particles before and after the scattering, and k and k′ are those of the other particle. Two auxiliary amplitudes $A_\infty^{\rm ret}(s, t)$ and $A_\infty^{\rm adv}(s, t)$ are introduced; the first coincides asymptotically with the amplitude of the reaction in the physical region of the reaction, and the second coincides asymptotically with the amplitude of the cross-reaction in the physical region of the cross-reaction. It is shown that $A_\infty^{\rm ret}$ and $A_\infty^{\rm adv}$ are analytic in s in the upper and lower half-planes, respectively, and that at complex infinity they increase more slowly than any linear exponential. The structure of the admissible generalized functions is then derived on the basis of microcausality. The generalized functions can contain derivatives of the δ function to all orders, and need not be generalized functions of temperate growth, as is always assumed.

The Pomeranchuk theorem on asymptotic equality of the total cross sections of particle and antiparticle and on the asymptotic equality of the differential cross sections is proved without assumption as to the analyticity of the scattering amplitude. Therefore experimental demonstration of the equality of the cross sections, if this should be established, cannot be regarded as an unequivocal argument showing that the amplitude is analytic.

1. RETARDED AND ADVANCED AMPLITUDES

WE consider the reaction of elastic scattering of scalar particles a and b with masses μ and M

$$a(k) + b(p) \to a(k') + b(p')$$

and the cross-reaction

$$\bar{a}(k') + b(p) \to \bar{a}(k) + b(p').$$

Here p and p′ are the momenta of particle b, and k and k′ those of particle a, before and after the scattering. For brevity we shall refer to reactions $p + k \to p' + k'$ and $p - k' \to p' - k$.

The amplitude for reaction $p + k \to p' + k'$ is a function of two independent variables, for example the invariant variables s and t, but it is convenient to write it in the form of a function A(s, u, t) of all three invariants:

$$s = (p + k)^2, \quad t = (p - p')^2, \quad u = (p - k')^2;$$

$$s + u + t = 2(\mu^2 + M^2). \qquad (1.1)$$

The meaning of crossing symmetry is that the amplitude A(s, u, t) of the reaction $p + k \to p' + k'$ goes over into that of the cross-reaction when we go over from the s channel to the u channel.

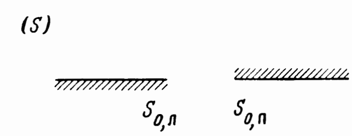

FIG. 1.

In other words, the amplitude for the cross-reaction is the function $A[2(\mu^2 + M^2) - t - s, t]$.

Let us suppose that the system a + b has no bound state with mass less than the sum of the masses μ + M. In this case the threshold of the physical region in the s channel is the point $s_0 = (\mu + M)^2$, and the threshold in the u channel is the point $u_0 = (\mu + M)^2$. Therefore in the s plane (see Fig. 1) the physical region of the reaction is the upper side of the right-hand cut $(s_0, +\infty)$, and the physical region of the cross-reaction is the lower side of the left-hand cut $(-\infty, s_{0L})$, where $s_{0L} = 2(\mu^2 + M^2) - u_0 - t = (M - \mu)^2 - t$.

On our assumption about the masses of intermediate states, the amplitude A(s, u, t) in the physical region of the reaction $p + k \to p' + k'$ coincides with the retarded amplitude (cf. e.g., [1,2])

$$A^{ret}(s,t)=\int d^4x e^{i(k+k')x/2}\left\langle p' \left| \frac{\delta j(-x/2)}{\delta\varphi^+(x/2)} \right| p \right\rangle. \quad (1.2)$$

Here $\varphi(x)$ is the field of particle a in the Heisenberg representation, and $j(x)$ is the current of this field,

$$j(x)=-i\frac{\delta S}{\delta\varphi(x)}S^+,$$
$$j^+(x)=i\frac{\delta S^+}{\delta\varphi^+(x)}S=-i\frac{\delta S}{\delta\varphi^+(x)}S^+, \quad SS^+=1. \quad (1.3)$$

We are here using the local field theory in the Bogolyubov form.

The microcausality condition is now expressed by the formula

$$\frac{\delta}{\varphi^+(y)}\left(\frac{\delta S}{\delta\varphi(x)}S^+\right)=0 \quad \text{for } x\lesssim y. \quad (1.4)$$

The notations ordinarily used are

$$\left\langle p' \left| \frac{\delta j(-x/2)}{\delta\varphi^+(x/2)} \right| p \right\rangle = F^{ret}(x),$$
$$\left\langle p' \left| \frac{\delta j(x/2)}{\delta\varphi^+(-x/2)} \right| p \right\rangle = F^{adv}(x), \quad (1.5)$$

although in reality $F^{ret}(x)$ and $F^{adv}(x)$ depend on px, p′x, and x^2. Owing to microcausality $F^{ret}(x) = 0$ for $x \lesssim 0$, and $F^{adv}(x) = 0$ for $x \gtrsim 0$, and the integration in $A^{ret}(s, t)$ is taken over only the upper half of the light cone.

The advanced amplitude is defined in an analogous way,

$$A^{adv}(s,t)=\int d^4x e^{i(k+k')x/2}F^{adv}(x)\,dx, \quad x\lesssim 0. \quad (1.6)$$

The advanced amplitude coincides with the amplitude of the cross-reaction in its physical region, i.e., on the lower side of the left-hand cut.

$F^{ret}(x)$ and $F^{adv}(x)$ are not ordinary functions; they are generalized functions. This is a very important fact. For example, if $F^{ret}(x)$ and $F^{adv}(x)$ were ordinary integrable functions, then their Fourier transforms $A^{ret}(s, t)$ and $A^{adv}(s, t)$ would increase at complex infinity more slowly than any positive power.

If $A^{ret}(s, t)$ is analytic in the upper half-plane Im s > 0, then $A^{adv}(s, t)$ is analytic in the lower half-plane and $A^{ret}(s, t) = A^{*adv}(s^*, t)$. It is only in this case that crossing symmetry takes on real meaning Since the properties of $A^{adv}(s, t)$ in the lower half-plane are the same as those of $A^{ret}(s, t)$ in the upper half-plane, we can confine ourselves to the examination of $A^{ret}(s, t)$. Every assumption about the structure of $F^{ret}(x)$ and $F^{adv}(x)$ is at the same time an assumption about $A^{ret}(s, t)$ and $A^{adv}(s, t)$.

In their report in Seattle in 1956 (see [1-3]) Bogolyubov, Medvedev, and Polivanov (BMP) were the first to prove that in the range of momentum transfers $-\sigma\mu^2 \le t \le 0$ the retarded amplitude $A^{ret}(s, t)$ is analytic in the entire upper half-plane Im s > 0 and has two real simple poles at the points $s = M^2$ and $s = 2\mu^2 + M^2 - t$. The amplitudes $A^{ret}(s, t)$ and $A^{adv}(s, t)$ are both real and coincide with each other in the interval between the cuts, and therefore taken together they determine a function which is analytic in the entire cut plane apart from the two real simple poles. Analogous results for the simpler case of forward scattering, i.e., for t = 0, were reported at the same conference by Symanzik and Jost.

In the derivation of one-dimensional dispersion relations the main difficulty is in the analytic continuation of the amplitude in the cut plane. After the proof of BMP the possibility of the continuation was proved by various methods by many authors (cf. the survey [3]). A very important contribution is the integral representation of causal commutators, derived by Jost and Lehmann[4] and generalized in a paper by Dyson.[5]

Despite the depth and importance of the results of these papers they have two essential defects: 1) The analyticity of the amplitude with respect to s is proved only for a certain range of momentum transfers, and the boundary of this range is not distinguished by any physical property (for example, the interval between the cuts in the s plane is still not covered); 2) besides some general physical assumptions, all of the proofs involve a special assumption about the character of the generalized functions $F^{ret}(x)$ and $F^{adv}(x)$. This is a purely mathematical assumption, and does not come out of the physics of the problem.

The meaning and importance of the first restriction are obvious. The second restriction is of a more refined nature, and to formulate it more precisely we recall the definition of generalized functions. Generalized functions are defined as continuous linear functionals over a space of basic functions. In other words, the space of the generalized functions defined over a given space C of basic functions forms an adjoint space C′. We shall denote the elements of the basic space by f(x), and generalized functions by T, ψ. The result of applying the linear functional T to the function f(x) is denoted symbolically by

$$(T,f)=\int f(x)T(x)\,dx. \quad (1.7)$$

The additional mathematical requirement involved in all existing derivations of one-dimensional dispersion relations is that $F^{ret}(x)$ and $F^{adv}(x)$ are generalized functions of moderate growth. This means that as the space of basic functions one takes the space S of infinitely differentiable functions that are defined in the entire x-space and at infinity approach zero together with all their derivatives more rapidly than any negative power of x (cf. e.g., [6]) Generalized functions of temperate growth can have δ-type singularities of only finite order; that is they can contain derivatives of the δ function only of finite orders (cf. [6]). On this assumption $A^{ret}(s, t)$ and $A^{adv}(s, t)$ cannot increase with s more rapidly than polynomials.

The important space of generalized functions of temperate growth is very popular in textbooks, but it is far from being an adequate apparatus for all cases. The study of various cases requires the construction of special spaces. This is the actual difficulty and subtlety of the use of generalized functions.

We shall refrain from the additional mathematical stipulation we have just explained, and shall obtain the structure of the admissible generalized functions from microcausality.

2. AUXILIARY ASYMPTOTIC AMPLITUDES

In the investigation of the analytic properties of $A^{ret}(k', p', k, p)$ with respect to one variable it is advantageous for the generalized function $F^{ret}(x)$ not to depend on this variable. This requirement is satisfied by the Breit coordinate system, defined by the condition $\mathbf{p}' = -\mathbf{p}$. In this system (see Fig. 2)

$$\mathbf{k} = -\mathbf{p} + \lambda\mathbf{e}, \quad \mathbf{k}' = \mathbf{p} + \lambda\mathbf{e},$$
$$p_0 = \sqrt{\mathbf{p}^2 + M^2}, \quad k_0 = \sqrt{\mathbf{p}^2 + \lambda^2 + \mu^2}, \tag{2.1}$$

where $\mathbf{e}$ is a unit vector perpendicular to the vector $\mathbf{p}$, and λ is a scalar, $\lambda = (k_0^2 - p^2 - \mu^2)^{1/2}$.

The system (s, t) is connected with the Breit system by the relations

$$t = (p' - p)^2 = -4\mathbf{p}^2,$$
$$s = (p_0 + k_0)^2 - \lambda^2 = M^2 + \mu^2 - {}^1\!/_2 t + 2p_0k_0,$$
$$k_0 = [s + {}^1\!/_2 t - (M^2 + \mu^2)] / 2\sqrt{M^2 - t/4}. \tag{2.2}$$

In the physical region of the reaction $t \le 0$. In the Breit system the retarded and advanced amplitudes depend on the energy variable k_0 and on the momentum transfer t; the function $F^{ret}(x)$ does not depend on the energy variable k_0.

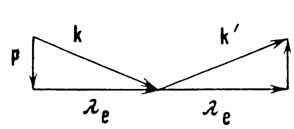

FIG. 2.

It follows from the relation

$$k_0 = [s + {}^1\!/_2 t - (M^2 + \mu^2)] / 2\sqrt{M^2 - t/4},$$

that the charge $s \to u$ corresponds to the change $k_0 \to -k_0$, and therefore the s plane with the cuts $(-\infty, s_{0L})$, $(s_{0R}, +\infty)$ goes over on this change into the plane of k_0 with two symmetrical cuts $(-\infty, -a)$, $(a, +\infty)$. Since $s_{0R} = (M + \mu)^2$, the value of the threshold a is

$$a = (M\mu + t/4) / \sqrt{M^2 - t/4}. \tag{2.3}$$

The upper side of the right-hand cut is the physical region of the reaction $k + p \to k' + p'$ in the k_0 plane, and the lower side of the left-hand cut is the physical region of the cross-reaction. In the Breit system

$$A^{ret}(k_0, t) = \int dx^4 \exp\{i(k_0x_0 - \mathbf{ex}\sqrt{k_0^2 - \mu^2 + t/4})\} F^{ret}(x),$$
$$x \gtrsim 0, \tag{2.4}$$

$$A^{adv}(k_0, t) = \int dx^4 \exp\{i(k_0x_0 - \mathbf{ex}\sqrt{k_0^2 - \mu^2 + t/4})\} F^{adv}(x),$$
$$x \lesssim 0. \tag{2.5}$$

For $k_0 \to +\infty$ in the physical region of the reaction $p + k \to p' + k'$ the retarded amplitude coincides asymptotically with the function

$$A_\infty^{ret}(k_0, t) = \int dx^4 e^{i(k_0x_0 - \mathbf{ex}\,k_0)} F^{ret}(x), \quad x \gtrsim 0. \tag{2.6}$$

It is assumed that for $k_0 \to +\infty$ in the physical region of the reaction the amplitude $A^{ret}(k_0, t)$ neither increases too rapidly nor oscillates rapidly. More exactly, if c is an arbitrary real constant, then at least one of the two conditions

$$\lim_{k_0 \to \infty} A^{ret}\left(k_0 + \frac{c}{k_0}, t\right) \Big/ A^{ret}(k_0, t) = 1,$$
$$\lim \left[A^{ret}\left(k_0 + \frac{c}{k_0}, t\right) - A^{ret}(k_0, t)\right] = 0$$

is satisfied. In the former case it is assumed that $A^{ret}(k_0, t)$ has a limited number of zeros in the physical region. The statement that $A_\infty^{ret}(k_0, t)$ and $A^{ret}(k_0, t)$ coincide asymptotically means in the former case that $\lim A_\infty^{ret}(k_0, t)/A^{ret}(k_0, t) = 1$, and in the latter case that the difference $A_\infty^{ret}(k_0, t) - A^{ret}(k_0, t)$ approaches zero. This statement is essentially a condition on $F^{ret}(x)$. It will become clear as we go on that this condition is satisfied in the class of admissible generalized functions with which we are concerned here. Similarly, for $k_0 \to -\infty$ in the physical region of the reaction $p - k' \to p' - k$ the advanced amplitude coincides asymptotically with the function

$$A_\infty^{adv}(k_0, t) = \int dx^4 e^{i(k_0 x_0 - \mathbf{ex} k_0)} F^{adv}(x), \quad x \lessgtr 0. \quad (2.7)$$

These properties justify our calling the functions $A_\infty^{ret}(k_0, t)$ and $A^{adv}(k_0, t)$ the asymptotic retarded and asymptotic advanced amplitudes (this definition differs somewhat from that given in [7]). Let us introduce the variable $\tau = x_0 - \mathbf{ex}$; then

$$A_\infty^{ret}(k_0, t) = \int_0^\infty d\tau\, e^{ik_0\tau}\psi(\tau, t), \quad (2.8)$$

where $\psi(\tau, t)$ is the result of integrating the generalized function $F^{ret}(x)$ over the intersection of the light cone with the hypersurface $x_0 - \mathbf{ex} = \tau$

3. CAUSALITY AND THE STRUCTURE OF ADMISSIBLE GENERALIZED FUNCTIONS

The principle of causality finds its expression in the fact that in the expression (1.2) for the retarded amplitude $A^{ret}(s, t)$ the integration is taken over the upper region of the light cone. When we go over to the asymptotic retarded amplitude causality is expressed by the fact that in the expression

$$A_\infty^{ret}(k_0, t) = \int_0^\infty d\tau\, e^{ik_0\tau}\psi(\tau, t) \quad (3.1)$$

the integration is taken over the positive semiaxis. Here it is tacitly assumed that the integral (3.1) communicates signals only from points of the positive semiaxis, because otherwise causality would lose its meaning. For ordinary functions $\psi(\tau)$ this holds automatically, but for generalized functions this requirement is a genuine restriction.

From causality and the absence of action at a distance it follows that the generalized functions $\psi(\tau)$ must satisfy the following condition: for arbitrary τ_1 and τ_2 the integral

$$\int_{\tau_1}^{\tau_2} d\tau\, e^{ik_0\tau}\psi(\tau) \quad (3.2)$$

must not be equivalent to a superposition of signals whose sources lie outside the interval (τ_1, τ_2). Here it is necessary to give the following clarification. As we shall see below, the integral (3.2) is often identically equal to an integral

$$\int_L d\tau\, e^{ik_0\tau} u_\psi(\tau), \quad (3.3)$$

where $u_\psi(\tau)$ is an analytic function in some neighborhood of the interval (τ_1, τ_2) and L is an arbitrary curve in this neighborhood which passes around the interval (τ_1, τ_2) (see Fig. 3). The line L can be contracted continuously onto the cut (τ_1, τ_2) so that in the limit the integration is taken along the upper and lower sides of the interval (τ_1, τ_2). It is not excluded that the integral (3.3) be equivalent to an integral (3.3) in this sense. Integrals in which the paths of integration can be changed continuously into one another are regarded as the same integral. It is forbidden for (3.2) to be equivalent to an integral (3.3) if the contour L cannot be continuously contracted onto (τ_1, τ_2).

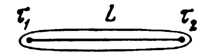

FIG. 3.

The necessity of this stipulation is illustrated by the identity

$$\int_{\tau_0-\varepsilon}^{\tau_0+\varepsilon} e^{ik_0\tau}\delta(\tau-\tau_0)\,d\tau = \frac{1}{2\pi i}\int_{|\tau-\tau_0|=\varepsilon} e^{ik_0\tau}\frac{d\tau}{\tau-\tau_0}.$$

Because of its importance we call this requirement the locality principle for generalized functions. In a physics which satisfies microcausality and has no action at a distance we can admit only generalized functions which satisfy the locality principle.

Let us consider some examples of generalized functions which violate the principle of locality.

I. Let

$$\psi(\tau) = \sum_\nu \delta^\nu(\tau)/\nu!.$$

Obviously for $0 < \epsilon < 1$

$$\int_{-\varepsilon}^{\infty} e^{ik_0\tau}\psi(\tau)\,d\tau = \sum_\nu \frac{(-ik_0)^\nu}{\nu!} = e^{-ik_0},$$

i.e., the integral transmits a signal from the point $\tau = -1$, and there is a direct violation of the causality principle.

II. Let

$$\psi(\tau) = \sum_\nu \frac{(-c)^\nu}{\nu!}\delta(\tau-\tau_0),$$

where $\tau_1 < \tau_0 < \tau_2$, and the real or complex number c is such that the point $\tau_0 + c$ lies outside the interval (τ_1, τ_2). It is obvious that

$$\int_{\tau_1}^{\tau_2} e^{ik_0\tau}\psi(\tau)\,d\tau = \sum_\nu \frac{c^\nu}{\nu!}\frac{d^\nu}{d\tau^\nu}(e^{ik_0\tau})\Big|_{\tau=\tau_0} = e^{ik_0(\tau_0+c)},$$

i.e., the integral (3.2) is equivalent to a signal from the point $\tau_0 + c$, which lies outside (τ_1, τ_2).

Let us consider generalized functions of the form

$$\psi(\tau)=\sum_{\nu}\frac{(-1)^{\nu}}{\nu!}c_{\nu}\delta^{(\nu)}(\tau-\tau_0) \qquad (3.4)$$

with arbitrary coefficients c_ν. There are two essentially different possibilities:

$$\text{a) } \overline{\lim}\,|c_\nu|^{1/\nu}=\rho>0, \qquad (3.5)$$

$$\text{b) } \overline{\lim}\,|c_\nu|^{1/\nu}=0. \qquad (3.6)$$

In the first case for any $\epsilon > 0$ there exists a representation

$$c_\nu=\iint\limits_{|\zeta|\leqslant\rho+\varepsilon}\zeta^{\nu}\,d\sigma_{\varepsilon}(\zeta), \qquad \nu=0,1,2\ldots \qquad (3.7)$$

with density $d\sigma_\epsilon(\zeta)$ which satisfies the inequality

$$\iint\limits_{|\zeta|\leqslant\rho+\varepsilon}|d\sigma_{\varepsilon}(\zeta)|<M_{\varepsilon}<+\infty. \qquad (3.8)$$

There is no circle of radius smaller than ρ in which such a representation is possible, since from such a representation it would follow that $\overline{\lim}\,(|c_\nu|)^{1/\nu}<\rho$.

To prove (3.7) we introduce the function

$$\Phi(\zeta)=\sum_{\nu}c_{\nu}\zeta^{-\nu-1}.$$

It follows from (3.5) that the function $\Phi(\zeta)$ is analytic for $|\zeta|>\rho$. Therefore

$$c_\nu=\int\limits_{|\zeta|=\rho+\varepsilon}\zeta^{\nu}\frac{1}{2\pi i}\Phi(\zeta)\,d\zeta. \qquad (3.9)$$

Let us take an arbitrary positive $\eta<\rho$. Substituting the expression for c_ν from (3.7), we get

$$\int\limits_{\tau_0-\eta}^{\tau_0+\eta}e^{ik_0\tau}\psi(\tau)\,d\tau=\iint\limits_{|\zeta|\leqslant\rho+\varepsilon}e^{ik_0(\tau+\zeta)}\,d\sigma_{\varepsilon}(\zeta).$$

Since $\eta<\rho$ and no representation of the type of (3.7) and (3.8) is possible in a circle of radius less than ρ, the generalized function (3.4) with the condition (3.5) does not satisfy the locality principle. Functions (3.4) with the condition (3.6) do satisfy this principle.

We have arrived at the following conclusion, which is fundamental for our further work: we must take as the space of basic functions a space on which all continuous linear functionals of the form (3.4) with the condition (3.6) are defined, and on which functionals of the form (3.4) with the condition (3.5) do not exist.

We shall prove the following properties of the basic space (b. s.): 1) Every function $f(\tau)$ of the b. s. is analytic in some neighborhood of the semiaxis $\tau \geq 0$; 2) there does not exist any fixed neighborhood $|\tau-\tau_0| \leq d$ of any point $\tau_0 \geq 0$ in which all of the functions of the b. s. can be analytic.

Proof. 1) Let $f(\tau)$ belong to the b. s. and suppose $\tau_0 \geq 0$. It is to be proved that the series

$$\sum_{\nu}\frac{1}{\nu!}f^{(\nu)}(\tau_0)(\tau-\tau_0)^{\nu}$$

has a positive radius of convergence. Let us assume the opposite; then

$$\overline{\lim}\left(\frac{1}{\nu!}f^{(\nu)}(\tau_0)\right)^{1/\nu}=\infty,$$

i.e., there exists a sequence $\{\nu'\}$ of indices such that for $\nu'\to\infty$ this limit is infinite. Set

$$c_\nu=[f^{(\nu)}(\tau_0)\,/\,\nu!]^{-1}$$

if ν belongs to the sequence $\{\nu'\}$ and $c_\nu = 0$ otherwise. It is obvious that $|c_\nu|^{1/\nu}\to 0$, and by hypothesis the functional

$$\psi(\tau)=\sum_{\nu}\frac{(-1)^{\nu}}{\nu!}c_{\nu}\,\delta^{(\nu)}(\tau-\tau_0)$$

is admissible, but this is in contradiction with the fact that

$$(\psi,f)=\sum_{\nu}c_{\nu}\frac{1}{\nu!}f^{(\nu)}(\tau_0)=\infty.$$

2) If all of the functions $f(\tau)$ of the b. s. were analytic in a fixed circle $|\tau-\tau_0|<d$, then for any function of the b. s. the radius of convergence of the series

$$\sum_{\nu}\frac{1}{\nu!}f^{(\nu)}(\tau_0)\,(\tau-\tau_0)^{\nu}$$

would be not smaller than d. It would follow from this that the functional (3.4) would have meaning not under the condition $|c_\nu|^{1/\nu}\to 0$, but under the condition $\overline{\lim}\,|c_\nu|^{1/\nu}<d$.

These arguments lead naturally to the following formal definition of the space C_0 of basic functions:

1. Each function $f(\tau)$ of C_0 is analytic in a certain half-strip O_f: $\operatorname{Re}\tau \geq -d_f$, $|\operatorname{Im}\tau| \leq d_f$ (see Fig. 4). The positive constant d_f depends on the function $f(\tau)$.

2. The function $f(\tau)$ and all of its derivatives are bounded in O_f: $|f^{(k)}(\tau)|<M_{k,f}$.

3. For $\tau\to\infty$ in O_f the function $f(\tau)$ goes to a limit $f(+\infty)$, and the difference $|f(+\infty)$

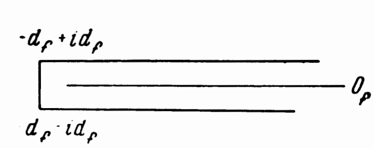

FIG. 4.

$- f(\tau)|$ decreases more rapidly than $|\tau|^{-\alpha_f}$, where α_f is a positive constant.

4. The space C_0 contains all functions that satisfy the conditions 1, 2, and 3.

A sequence of functions $f_n(\tau)$ belonging to C_0 converges to zero if there exists a half-strip O_f in which all of the functions of the sequence are analytic and approach zero uniformly.

The narrower the space of the basic functions, the broader the conjugate space of generalized functions. If we compare the spaces C_0 and S, we see that the requirement of analyticity decidedly narrows C_0 as compared with S. On the other hand, the functions of S are required to decrease rapidly at ∞. Condition 3, which defines the behavior of functions of C_0 at ∞, greatly simplifies the investigation. It will be shown below that the removal of the condition 3 has little effect on the final results.

Let us denote by C_0' the space of generalized functions defined over C_0. We shall verify that C_0' contains all of the generalized functions (3.4) with the condition $|c_\nu|^{1/\nu} \to 0$ and does not contain any $\psi(\tau)$ for which $\overline{\lim}\, |c_\nu|^{1/\nu} \to \rho > 0$. Let $f(\tau)$ be a function of C_0; then the series

$$\sum_\nu \frac{1}{\nu!} f^{(\nu)}(\tau_0)(\tau-\tau_0)^\nu$$

converges absolutely for $|\tau - \tau_0| < \alpha_f$, and the series

$$(\psi, f) = \sum_\nu \frac{1}{\nu!} f^{(\nu)}(\tau_0) c_\nu$$

clearly converges for $|c_\nu|^{1/\nu} \to 0$.

If the functional $\psi(\tau)$ belongs to C_0', then this series must converge for all $f(\tau)$ of C_0, and $f^{(\nu)}(\tau_0) c_\nu/\nu! \to 0$. Let us take as the $f(\tau)$ the function

$$\frac{2i}{\rho}\left(\tau - \tau_0 - i\frac{\rho}{2}\right)^{-1} = \sum_\nu \left(\frac{2}{i\rho}\right)^\nu (\tau-\tau_0)^\nu,$$

so that $f^{(\nu)}(\tau_0)/\nu! = (2/i\rho)^\nu$. If $\overline{\lim}\, |c_\nu|^{1/\nu} = \rho$, then $\overline{\lim}\, |(2/i\rho)^\nu c_\nu| = \infty$, and not zero; i.e., the series (ψ, f) diverges.

The general form of a linear functional T of the space C_0' is:

$$(T, f(\tau)) = \sum_\nu \int_0^\infty \frac{f^{(\nu)}(\tau)}{\nu!} d\sigma(\tau),$$

$$T(\tau) = \sum_\nu \int_0^\infty (-1)^\nu \delta^{(\nu)}(\tau - t) \frac{d\sigma_\nu(t)}{\nu!}, \tag{3.10}$$

where the measures $\sigma_\nu(\tau)$ satisfy the condition

$$s_\nu = \int_0^\infty |d\sigma_\nu(\tau)| < +\infty, \quad \lim_{\nu\to\infty} s_\nu^{1/\nu} = 0. \tag{3.11}$$

That the condition (3.11) is necessary can be seen already from the case we have considered of the functionals (3.4), for which all of the measures $d\sigma_\nu(\tau)$ are concentrated at the point τ_0; the condition is obviously sufficient.

If $d\sigma_\nu(\tau) = 0$ for $\nu \geq 1$, and $d\sigma_0(\tau) = T(\tau)\, d\tau$, where $T(\tau)$ is an ordinary integrable function, then (T, f) reduces to an ordinary integral.

4. REPRESENTATION OF GENERALIZED FUNCTIONS AS CONTOUR INTEGRALS

Let T be a functional belonging to C_0', and let ζ be a point outside the semiaxis $\tau \geq 0$. Since $(\zeta - \tau)^{-1}/2\pi i$, as a function of τ, belongs to C_0,

$$(T, 1/2\pi i(\zeta - \tau)) = u_T(\zeta) \tag{4.1}$$

is a function of ζ. It follows from the linearity and continuity of T that

$$\frac{d}{d\zeta} u_T(\zeta) = -\frac{1}{2\pi i}(T, (\zeta-\tau)^{-2}), \tag{4.2}$$

i.e., the function $u_T(\zeta)$ is holomorphic outside the positive semiaxis $\zeta \geq 0$.

From the representation (3.10) for T it follows that

$$u_T(\zeta) = \sum_\nu \frac{1}{2\pi i} \int_0^\infty (\zeta-\tau)^{-\nu-1} d\sigma_\nu(\tau) = -\frac{1}{2\pi i} \sum_\nu \frac{d\sigma_\nu(0)}{\zeta^{\nu+1}}$$

$$+ \sum_\nu \frac{\nu+1}{2\pi i} \int_0^\infty \frac{\sigma_\nu(\tau)}{(\zeta-\tau)^{\nu+2}}. \tag{4.3}$$

At infinity the function $u_T(\zeta)$ falls off at least as ζ^{-1}.

Let $f(\tau)$ be a function of the space C_0, and let Γ_f be the boundary of the half-strip O_f; then

$$f(\tau) = \int_{\Gamma_f} [f(\zeta) - f(+\infty)] \frac{1}{2\pi i} \frac{1}{\zeta - \tau} d\zeta$$

$$+ f(+\infty) \int_{\Gamma_f} \frac{1}{2\pi i} \frac{d\zeta}{\zeta-\tau}.$$

We put the last integral in the following form:

$$\int_{\Gamma_f} \frac{d\zeta}{\zeta-\tau} = \int_{-d_f+id_f}^{-d_f-id_f} \frac{d\zeta}{\zeta-\tau} + \int_{-d_f} \left(\frac{1}{\zeta - id_f - \tau} - \frac{1}{\zeta + id_f - \tau}\right) d\zeta.$$

When broken up in this way all of the integrals involved in the representation of $f(\tau)$ converge absolutely, and the operations of integration and application of the functional T can be interchanged; i.e.,

$$(T, f(\tau)) = \int_{\Gamma_f} f(\zeta) u_T(\zeta) d\zeta. \quad (4.4)$$

Equation (4.4) gives the general form of a continuous linear functional over the space C_0.[1)] The contour Γ_f can be replaced by any contour in O_f which goes around the semiaxis $\zeta \geq 0$.

5. FOURIER TRANSFORMATION OF THE GENERALIZED FUNCTIONS OF THE SPACE

The retarded asymptotic amplitude $A_\infty^{ret}(k_0, t)$ is the Fourier transform of a generalized function $\psi(\tau, t)$ of the space C_0'. If the point $z = x + iy$ lies in the upper half-plane $y > 0$, then the function $e^{iz\tau}$ belongs to C_0, and the Fourier transform of a generalized function T of C_0' is simply the value of the functional T of the function $e^{iz\tau}$; that is,

$$\tilde{T}(z) = (T, e^{iz\tau}/2\pi). \quad (5.1)$$

According to (4.4) and (4.3)

$$\tilde{T}(z) = \frac{1}{2\pi} \int_\Gamma u_T(\zeta) e^{iz\zeta} d\zeta,$$

$$\tilde{T}(z) = \frac{1}{2\pi} \int_\Gamma \left[\sum_\nu \frac{1}{2\pi i} \int_0^\infty \frac{d\sigma_\nu(\tau)}{(\zeta - \tau)^{\nu+1}} \right] e^{iz\zeta} d\zeta. \quad (5.2)$$

It follows from the condition (3.11) that it is legitimate to change the order of the integrations. Since

$$\frac{1}{2\pi i} \int_\Gamma \frac{e^{iz\zeta}}{(\zeta - \tau)^{\nu+1}} = \frac{(iz)^\nu}{\nu!} e^{iz\tau},$$

we have

$$\tilde{T}(z) = \sum_\nu c_\nu(z) \frac{(iz)^\nu}{\nu!}, \quad c_\nu(z) = \frac{1}{2\pi} \int_0^\infty e^{iz\tau} d\sigma_\nu(\tau). \quad (5.3)$$

By (3.11), $|c_\nu(z)|^{1/\nu} \to 0$ for Im $z \geq 0$; therefore the increase of $\tilde{T}(z)$ in the upper half-plane (including the real axis) is slower than that of any linear exponential: for any $\epsilon > 0$ a number $R(\epsilon)$ can be found so that for $|z| \geq R(\epsilon)$, Im $z \geq 0$,

$$|\tilde{T}(z)| < e^{\varepsilon |z|}. \quad (5.4)$$

If Im $z > 0$, then $c_\nu(z)$ is an analytic function of z, and $\tilde{T}(z)$, being the sum of a uniformly converging series of analytic functions, is also an analytic function of z in the upper half-plane Im $z > 0$.

[1)]For analogous formulas for other spaces of analytic functions see [8-10].

If the generalized function is generated by a function $T(\tau)$ which is Lebesgue integrable on the semiaxis $(0, +\infty)$—that is, $d\sigma_0(\tau) = T(\tau) d\tau$ and $d\sigma_\nu(\tau) = 0$ for $\nu \geq 1$—then

$$u_T(\zeta) = \frac{1}{2\pi i} \int_0^\infty \frac{T(\tau)}{\zeta - \tau} d\tau.$$

At points of continuity of the function $T(\tau)$ it is obvious that $u_T(\tau + i0) - u_T(\tau - i0) = -T(\tau)$, and the formula (5.2) reduces to the usual formula for Fourier transformation of a function $T(\tau)$ which is equal to zero for $\tau < 0$:

$$\tilde{T}(z) = \frac{1}{2\pi} \int_0 T(\tau) e^{iz\tau} d\tau.$$

We have established that the Fourier transform of a generalized function belonging to C_0' is analytic in the upper half-plane and increases in the closed upper half-plane more slowly than any linear exponential. It is very important to find out how these properties change if there is some change of the space of basic functions. What happens, for instance, if we replace conditions 2 and 3 in the definition of C_0 by the requirement that $f(\tau)$ and all of its derivatives go to zero at infinity not more slowly than τ^{-1}? In this case the general form of the functional $T(z)$ is somewhat changed, namely

$$T(\tau) = \sum_{\nu=0}^\infty \int_0^\infty (-1)^\nu \delta^{(\nu)}(\tau - t) \frac{1+t}{\nu!} d\sigma_\nu(t). \quad (5.5)$$

The condition (3.11) still holds here. There is a corresponding change in the expression for $\tilde{T}(z)$:

$$\tilde{T}(z) = \sum_\nu c_\nu(z) \frac{(iz)^\nu}{\nu!},$$

$$c_\nu(z) = \frac{1}{2\pi} \int_0^\infty e^{iz\tau}(1+\tau) d\sigma_\nu(\tau). \quad (5.6)$$

In this case the boundary $R(\epsilon)$ beyond which the estimate (5.4) holds would depend not only on ϵ, but also on Im z, and for Im $z \to 0$ the boundary $R(\epsilon) \to \infty$, so that on the real axis (5.4) would not hold. It can be shown—and this is very important —that in this case also the limiting relation

$$\lim_{R \to \infty} \frac{1}{R} \int_0^{2\pi} \ln^+ |\tilde{T}(Re^{i\theta})| |\sin\theta| d\theta = 0$$

$(\ln^+|a| = \ln|a|$ for $|a| \geqslant 1$ and is zero for $|a| < 1)$. (5.7)

The function $\tilde{T}(z)$ is also still analytic in the upper half-plane Im $z > 0$. Moreover, the relation (5.7) and the analyticity for Im $z > 0$ remain valid even if we require that the functions of the basic space vanish just as rapidly at infinity as the functions of the space S.

6. INVERSION OF THE FOURIER TRANSFORMATION

According to (4.4) the action of the functional T is determined by an analytic function $u_T(\zeta)$. By the inverse Fourier transformation we mean the representation of the function $u_T(\zeta)$ in terms of the function $\tilde{T}(z)$. We shall show that the inverse transformation is the Borel transformation

$$u_T(\zeta) = \int_L e^{-i\zeta z} \tilde{T}(z)\,dz, \qquad (6.1)$$

where L is a certain ray $\arg z = \theta$, $0 \le \theta \le \pi$.

Because of the inequality (5.4) the integral (6.1) converges absolutely in the half-plane of ζ defined by the inequality $\operatorname{Re} \zeta e^{i(\theta+\pi/2)} > \epsilon > 0$. The boundary line of this half-plane is perpendicular to the ray $e^{-i(\theta+\pi/2)}$ and cuts off on this ray a segment of length ϵ (see Fig. 5). If $0 < \theta_2 - \theta_1 < \pi$, then the half-planes in which the integrals (6.1) taken over the rays L_1 and L_2 are regular overlap over a certain angle and are equal at points in this angle; therefore with different choices of L the integrals (6.1) are each other's analytic continuations. Any point ζ not on the positive semiaxis falls in some half-plane of regularity; i.e., $u_T(\zeta)$ is regular outside the semiaxis $\zeta \ge 0$.

We shall show that the transformations (5.2) and (6.1) are mutually inverse, i.e., that

$$\int_L e^{-i\zeta z}\,dz \cdot \frac{1}{2\pi} \int_\Gamma u_T(\zeta_1)\, e^{i\zeta_1 z}\, d\zeta_1 = u_T(\zeta). \qquad (6.2)$$

Here the meaning of $u_T(\zeta)$ is $(T, 1/2\pi i(\zeta - \tau))$. The decrease of the function $u_T(\zeta)$ at infinity is not slower than ζ^{-1} [cf. (4.3)], and therefore in the region external to the semiaxis the Cauchy formula

$$u_T(\zeta) = -\frac{1}{2\pi i} \int_\Gamma \frac{u_T(\zeta_1)}{\zeta_1 - \zeta}\, d\zeta_1$$

holds. We can take as the ray L any ray $\arg z = \theta$, $0 \le \theta \le \pi$ for which $\operatorname{Re} \zeta e^{i(\theta+\pi/2)} > 0$. If besides this the condition $\operatorname{Re}(\zeta - \zeta_1)\, e^{i(\theta+\pi/2)} > 0$

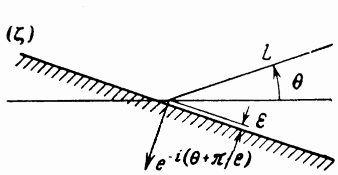

FIG. 5.

is satisfied as the point ζ_1 traverses the contour Γ, then we can change the order of the integrations in the left member of (6.2), and

$$\int_L e^{-i\zeta z}dz \cdot \frac{1}{2\pi}\int_\Gamma u_T(\zeta)\, e^{i\zeta_1 z}\, d\zeta_1 = \frac{1}{2\pi}\int_\Gamma u_T(\zeta_1)\, d\zeta_1 \int_L e^{-i(\zeta-\zeta_1)z}dz = u_T(\zeta).$$

The whole question reduces to satisfying the inequalities in question. Let $\operatorname{Im} \zeta > 0$; then for a Γ sufficiently close to the positive semiaxis $\operatorname{Im}(\zeta - \zeta_1) > 0$ and the inequalities are satisfied for $\theta = \pi$. If $\operatorname{Im} \zeta = 0$, then $\zeta < 0$, and the inequalities are satisfied for $\theta = \pi/2$.

We formulate the final result: The space C_0' of generalized functions and the space of functions analytic in the upper half-plane $\operatorname{Im} z > 0$ whose increase in the closed upper half-plane is slower than that of any exponential are two dual spaces in the sense of Fourier transformation. Equations (5.2) and (6.1) are the direct and inverse Fourier transformations that carry these spaces into each other.

If the space C_0' were replaced by another space of generalized functions in which the locality principle holds, then the elements of the dual space would as before be analytic functions in the upper half-plane, and for $\operatorname{Im} z > \delta > 0$ would increase more slowly than any exponential, but nothing could be said about the increase along the real axis. As is shown at the end of Sec. 5, the limiting equation (5.7) holds in this case also.

Example. Let $\tilde{T}(z) = iz \ln z$; then

$$u_T(\zeta) = i\int_0^{+\infty} e^{-i\zeta z}\, z \ln z\, dz = i(\gamma - i + i\pi/2 + \ln \zeta)/\zeta^2,$$

where γ is the Euler constant. It follows from (4 4) that

$$(T, f) = \int_\Gamma f(\zeta)\, u_T(\zeta)\, d\zeta = 2\pi f(1) - (\gamma + 3\pi^2) f'(0) - 2\pi \int_1^\infty f(\zeta)\,\frac{d\zeta}{\zeta^2} + 2\pi \int_0^1 f''(\zeta) \ln \zeta d\zeta,$$

$$T(\tau) = 2\pi\delta(\tau - 1) + (\gamma + 3\pi^2 i)\,\delta'(\tau) - 2\pi\int_1^\infty \delta(\tau - \zeta)\, d\zeta/\zeta^2 + 2\pi\int_0^1 \delta''(\tau - \zeta) \ln \zeta d\zeta.$$

The last term in $T(\tau)$ determines the asymptotic behavior of $\tilde{T}(z)$.

7. THE ASYMPTOTIC AMPLITUDES, AND RELATIONS BETWEEN THE CROSS SECTIONS OF PARTICLES AND ANTI-PARTICLES

Let us list the properties of $A_\infty^{ret}(k_0, t)$ and $A_\infty^{adv}(k_0, t)$.

1. The function $A_\infty^{ret}(k_0, t)$ is analytic in the upper half-plane $\operatorname{Im} k_0 > 0$. In any fixed half-plane $\operatorname{Im} k_0 \geq \delta > 0$ the function $A(k_0, t)$ increases more slowly than any linear exponential, and satisfies the limiting equation

$$\lim \frac{1}{R}\int_0^\pi \ln^+|A_\infty{}^{ret}(Re^{i\theta}, t)| \sin\theta d\theta = 0. \qquad (7.1)$$

It has been shown by R. and F. Nevanlinna[11] that the relation (7.1) is the most general condition for the validity of the generalized maximum principle of Phragmen and Lindelöf. If (7.1) holds and along the real axis $|A(k_0)| \leq M$, then in the entire upper half-plane $|A(k_0)| < M$. According to Lindelöf's theorem (cf. e.g.,[12,13]), if for $k_0 \to \mp\infty$ the function $A(k_0)$ goes to finite limits $A(-\infty)$ and $A(+\infty)$ and $A(-\infty) \neq A(+\infty)$, then the function $A(k_0)$ is unbounded in the upper half-plane, and consequently cannot satisfy the condition (7.1). This theorem remains valid for any region obtainable from the upper half-plane by an arbitrary finite deformation.

The function $A_\infty^{adv}(k_0, t)$ has analogous properties in the lower half-plane.

2. There is crossing symmetry: $A_\infty^{ret}(k_0, t) = A_\infty^{*adv}(k_0^*, t)$.

3. The amplitudes $A_\infty^{ret}(k_0, t)$ and $A^{ret}(k_0, t)$ are asymptotically equal in the physical region of the reaction $p + k \to p' + k'$, and the amplitudes $A_\infty^{adv}(k_0, t)$ and $A^{adv}(k_0, t)$ are asymptotically equal in the physical region of the cross-reaction. The meaning of this statement is explained in Sec. 2.

We note, though it is not important for what follows, that if the amplitude $A(k_0, t)$ is analytic and

$$\lim \frac{1}{R}\int_{-\pi}^{\pi} \ln^+|A(Re^{i\theta}, t)|\,|\sin\theta|\, d\theta = 0, \qquad (7.2)$$

then A_∞^{ret} and A_∞^{adv} are also asymptotically equal to A^{ret} and A^{adv} at complex infinity.

It follows from property 3 that in the determination of asymptotic relations between the cross sections of particle and antiparticle at high energies we can replace the actual scattering amplitudes by $A_\infty^{ret}(k_0, t)$ and $A_\infty^{adv}(k_0, t)$.

The significance of this is that the asymptotic amplitudes are analytic and satisfy the generalized maximum principle of Phragmén and Lindelöf and the crossing symmetry $A_\infty^{ret}(k_0, t) = A_\infty^{*adv}(k_0^*, t)$. As has been shown in[12,13], precisely these properties are needed for the proof of Pomeranchuk's theorem and its various sharper forms.

We present the simplest argument of this type. We assume that the amplitudes $A^{ret}(k_0, t)$ and $A^{adv}(k_0, t)$ are analytic and satisfy (7.1), i.e., they satisfy the generalized maximum principle.

According to the optical theorem the total cross sections $\sigma_+(k_0)$ and $\sigma_-(k_0)$ of the reaction and the cross-reaction can be expressed in terms of the amplitudes and the three-dimensional momenta

$$\sigma_+(k_0) = \operatorname{Im} A^{ret}(k_0, 0) \,/\, \sqrt{k_0^2 - \mu^2},$$
$$\sigma_-(k_0) = \operatorname{Im} A^{adv}(-k_0, 0) \,/\, \sqrt{k_0^2 - \mu^2}.$$

The first of these expressions is taken at the point $k_0 + i0$ of the upper side of the right-hand cut, and the second at the point $-k_0 - i0$ of the lower side of the left-hand cut. It follows from the crossing symmetry, $A^{adv}(-k_0 - i0) = A^{*ret}(-k_0 + i0)$, that when we go from the lower side of the left-hand cut to the upper both numerator and denominator change sign, and therefore

$$\sigma_-(k_0) = \operatorname{Im} A^{ret}(-k_0) \,/\, \sqrt{k_0^2 - \mu^2}.$$

Thus the total cross sections $\sigma_\pm(k_0)$ are the imaginary parts of the same analytic function

$$\Phi(k_0) = A^{ret}(k_0) \,/\, \sqrt{k_0^2 - \mu^2}$$

at points $k_0 + i0$ and $-k_0 + i0$ on the upper sides of the cuts. We assume that finite limits $\Phi(-\infty)$ and $\Phi(+\infty)$ exist; then $\lim \sigma_\pm(k_0) = \operatorname{Im} \Phi(\mp\infty)$.

The function $\Phi(k_0)$ satisfies the condition (7.1), and therefore according to the Lindelöf theorem (see Point 1 above) $\Phi(-\infty) = \Phi(\infty)$, and the Pomeranchuk theorem is proved. Logunov, Todorov, Nguyen Van Hieu, and Khrustalev[14] have remarked that this at the same time proves that the differential cross sections for forward scattering are equal, since these cross sections are proportional to $|\Phi(k_0 + i0)|^2$ and $|\Phi(-k_0 + i0)|^2$. For further details see[12-16].

We can consider the function $\Phi(k_0)$ not in the entire upper half-plane but only in the neighborhood of upper infinity, so that a finite number of singularities of the amplitude $A(k_0)$ is without importance. This is particularly important in extending the Pomeranchuk theorem to cover inelastic processes.

Actually it has not been established that

$A^{ret}(k_0, t)$ and $A^{adv}(k_0, t)$ are analytic and satisfy Eq. (7.1) [analyticity for small t has been proved only with special assumptions about the character of $F^{ret}(x)$ and $F^{adv}(x)$], but, as we have shown above, in proving the asymptotic relations between these amplitudes we can replace them by the amplitudes $A_\infty^{ret}(k_0, t)$ and $A_\infty^{adv}(k_0, t)$.

Accordingly the Pomeranchuk theorem and analogous theorems for the differential cross sections are true without any assumptions about the analyticity of the scattering amplitudes. In particular, they hold for any value $t \leq 0$ of the momentum transfer.

8. DISPERSION RELATIONS

Suppose it is known that for a given fixed value of the momentum transfer t the actual amplitude $A(k_0, t)$ is, apart from two pole terms, an analytic function in the plane with the cuts $(-\infty, -a)$, $(a, +\infty)$. The question arises as to the conditions under which we can write for the amplitude $A(k_0, t)$ a dispersion relation with a given number of subtractions. The pole terms are unimportant, and the subtractive procedure reduces to division by a polynomial of appropriate degree; therefore it is sufficient to derive the condition for writing a dispersion relation without subtraction for a function $A(k_0)$ which is analytic in the plane with the cuts and has the symmetry $A(k_0^*) = A^*(k_0)$. The usual requirement for this is that $A(k_0)$ go to zero with a power law at all complex infinity. The actual requirements are much less demanding.

By a dispersion relation we mean a relation

$$A(k_0) = \frac{1}{\pi}\int_{-\infty}^{-a} \frac{\operatorname{Im} A(k_0')}{k_0' - k_0} dk_0' + \frac{1}{\pi}\int_{a}^{\infty} \frac{\operatorname{Im} A(k_0')}{k_0' - k_0} dk_0'. \quad (8.1)$$

The integral in this formula can be interpreted in different ways. For example, we can take it to mean the principal value, i.e., the limit of the integral

$$\frac{1}{\pi}\left[\int_{-R}^{-a} + \int_{a}^{R}\right]$$

for $R \to \infty$. We adopt a simpler assumption, namely that the integrals in the dispersion relation (8.1) converge absolutely.

If the Cauchy formula holds for the function $A(k_0)$, then

$$A(k_0) = \frac{1}{2\pi i}\int_{\Gamma} \frac{A(k_0')}{k_0' - k} dk_0', \quad (8.2)$$

where Γ is the complete boundary of the cut plane, i.e., the integral is taken over the upper and lower edges of the cuts. Since $A(k_0' - i0) = A^*(k_0' + i0)$, the dispersion formula follows from the Cauchy formula. The same question arises now as before: in what sense are we to understand the integral in the formula (8.2)? It is simplest to assume that the integral in the Cauchy formula converges absolutely. As has been shown earlier,[16] the necessary and sufficient condition for the validity of the Cauchy formula so understood is that (7.2) hold.

Thus from the absolute convergence of the integral along the boundary

$$\int_{\Gamma} \frac{|A(k_0')|}{|k_0'|} |dk_0'| < +\infty \quad (8.3)$$

and the equalities (7.2) there follow the Cauchy formula and the dispersion relation (8.1) But the dispersion relation (8.1), and indeed also with the absolutely converging integral

$$\int_{-\infty}^{-a} \frac{|\operatorname{Im} A(k_0')|}{|k_0'|} dk_0' + \int_{0}^{\infty} \frac{|\operatorname{Im} A(k_0')|}{|k_0'|} dk_0' < +\infty \quad (8.4)$$

can also hold when the integral (8.3) diverges. It can be shown, however, that from the dispersion relation (8.1) with the condition (8.4) it follows that the integral

$$\int_{\Gamma} \frac{|A(k_0')|^{1-\varepsilon}}{|k_0|^{1+\varepsilon}} |dk_0'| < +\infty \quad (8.5)$$

converges for arbitrarily small $\epsilon > 0$. Equation (7.2) then holds also.

Thus the conditions (7.2) and (8.3) are sufficient conditions, and (7.2) and (8.5) are necessary conditions, for the validity of the dispersion formula (8.1). There is very little spread between these conditions. It is clear that the conditions (7.2) and (8.4) do not guarantee the validity of (8.1). This follows from the fact that these conditions are not spoiled by adding to $A(k_0)$ an arbitrary polynomial with real coefficients.

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