ANALYTIC PROPERTIES OF THE AMPLITUDE AS FUNCTION OF MOMENTUM TRANSFER, AND ASYMPTOTIC BEHAVIOR OF SCATTERING PHASE SHIFTS

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The connection between the nearest singularity z_0 of the scattering amplitude $f(E, z)$ and the asymptotic behavior of the partial amplitudes $f_l(E)$ for $l \to \infty$ is considered in general form. Two different cases are investigated: 1) z_0 lies outside the physical region $-1 \le z \le 1$; 2) z_0 is at the edge of the physical region, $z_0 = \pm 1$. The method used consists of a reduction of the series of Legendre polynomials to a power series. The results obtained are applied to nonrelativistic potential scattering. The position and nature of the nearest singularity of the amplitude are determined for scattering by a potential of the asymptotic form $(\mu r)^{-\nu} \exp[-(\mu r)^{\alpha}]$, $0 \le \alpha \le +\infty$. In the concluding part of the paper the method developed is applied to a study of the cross sections for interactions of elementary particles at high energies. A simple derivation is obtained for the Froissart relation which restricts the increase of the total interaction cross sections of elementary particles for $s \to \infty$.

1. INTRODUCTION

It has been shown in a paper by Okun' and Pomeranchuk [1] that the interaction of elementary particles at large distances ($r \gg 1/\mu$, with μ the mass of the π meson) corresponds to the singularities of the scattering amplitude that are nearest to the physical region, and it was pointed out that it is possible to calculate phase shifts δ_l with $l \gg 1$, which correspond to peripheral collisions. The program for calculating the peripheral phase shifts which was indicated in [1] has been carried out in a number of papers, for example for the two-meson phase shifts of nucleon-nucleon scattering,[2,3] for the phase shifts for scattering of π mesons by nucleons,[4] and so on. In this the main attention was given to the most accurate possible determination of δ_l for comparatively large values of l, for the purpose of using calculated peripheral phase shifts in making phase analyses of experimental data. This led to great complexity in the calculations, so that for the values of l considered in [2-4] the asymptotic behavior of δ_l is still not fully revealed.

The present paper is devoted to a treatment of the asymptotic behavior of the phase shifts $\delta_l(E)$ [or the corresponding partial amplitudes $f_l(E)$] in the limiting case $l \to \infty$. Because of the smallness of the δ_l for such values of l it is naturally impossible to compare these phase shifts directly with experimental data. Since, however, the form of the asymptotic behavior of $f_l(E)$ contains within it complete information about the nearest singularities of the scattering amplitude as a function of the momentum transfer, in cases in which $f_l(E)$ can be found for $l \to \infty$ without using the analytic properties of the scattering amplitude $f(E, z)$ the method considered in this paper provides a possibility of determining rather simply the nearest singularity of $f(E, z)$. For example, this is the situation for nonrelativistic potential scattering.

It is known from the theory of functions of a complex variable [5,6] that the asymptotic behavior of the coefficients of the Taylor's series for an analytic function $f(z)$ is uniquely determined by the singularity of $f(z)$ that is nearest to the origin. It is shown in Sections 2 and 3 how the problem of determining the asymptotic behavior of the partial amplitudes $f_l(E)$ for $l \to \infty$ can be reduced to an analogous but much simpler problem for a power series. In this no assumptions are made about the behavior of $f(E, z)$ for $|z| \to \infty$, and in particular it is not assumed that dispersion relations with respect to the momentum transfer exist. In Section 2 the case is considered in which the nearest singularity z_0 of the amplitude $f(E, z)$ lies outside the physical region, and Section 3 deals with the case $z_0 = \pm 1$. The main results of Sections 2 and 3 are collected in the formulas (12) and (13), which give in explict form the connection between the position and nature of the nearest singularity of f(E, z) and the asymptotic behavior of $f_l(E)$. These formulas include all cases in which the asymptotic behavior of $f_l(E)$ is of the form $l^{\alpha}(\ln l)^{\beta}e^{-l\xi}$, with arbitrary values of α, β, ξ.

In Section 4 these results are applied to the investigation of the analytic properties of the amplitude for potential scattering. The singularity nearest to the physical region is found for the amplitude for scattering in a potential which behaves at $r \to \infty$ like $(\mu r)^{-\nu}\exp[-(\mu r)^{\alpha}]$. Section 5 presents a derivation of the well known Froissart inequality [7] for the total interaction cross section which is much simpler than those in the literature, [7,8] and discusses possibilities for further strengthening of the inequality.

2. ASYMPTOTIC BEHAVIOR OF $f_l(E)$ FOR z_0 LYING OUTSIDE THE PHYSICAL REGION

We consider the elastic scattering of spinless particles and introduce the (dimensionless) scattering amplitude[1)]

$$f(E,z) = \sum_{l=0}^{\infty}(2l+1)f_l(E)P_l(z), \quad f_l(E) = \frac{1}{2i}(e^{2i\delta_l}-1). \tag{1}$$

Here E and z are the kinetic energy and the cosine of the angle of scattering in the center-of-mass system of the colliding particles, and are related to the invariant variables s and t by the usual formulas

$$E = s^{1/2} - (m_1+m_2), \qquad z = 1 + t/2k^2,$$
$$k(s) = {}^1/_2[s^{-1}(s^2 - 2(m_1^2+m_2^2)s + (m_1^2-m_2^2)^2)]^{1/2}$$

(m_1, m_2 are the masses of the colliding particles).

The scattering phase shifts $\delta_l(E)$ are in general complex (if there are other open channels besides the elastic-scattering channel) and satisfy the inequality (which follows from the unitarity condition)

$$\operatorname{Im}\delta_l(E) \geqslant 0, \tag{2}$$

which can also be expressed in other forms:

$$\operatorname{Im} f_l \geqslant |f_l|^2 \quad \text{or} \quad |f_l| \leqslant 1. \tag{2a}$$

To elucidate the analytic properties of the amplitude f(E, z) in the complex plane of z (with a fixed value of E) it is convenient to break the infinite sum in (1) up into two terms:

$$f(E,z) = f_1(E,z) + f_2(E,z); \tag{3}$$
$$f_1(E,z) = \sum_{l=0}^{L-1}(2l+1)f_l(E)P_l(z),$$
$$f_2(E,z) = \sum_{l=L}^{\infty}(2l+1)f_l(E)P_l(z). \tag{3a}$$

Since $f_1(E, z)$ is a polynomial in z, all of the singularities of the scattering amplitude in the finite part of the z plane are singularities of its "infinite tail" $f_2(E, z)$ (for any value of L).

We first assume that all of the singularities of f(E, z) as function of z lie outside an ellipse $\mathscr{E}$ with its foci at the points $z = \pm 1$ and with the semiaxis major $a = 1 + \epsilon$ ($\epsilon > 0$ by an arbitrarily small but finite amount. Taking $L \gg 1$, we substitute in $f_2(E, z)$ the asymptotic formula[2)] for $P_l(z)$, which holds for points z outside the ellipse $\mathscr{E}$:

$$P_l(z) \approx \left[\frac{1}{2\pi l}\left(1+\frac{z}{\sqrt{z^2-1}}\right)\right]^{1/2}(z+\sqrt{z^2-1})^l. \tag{4}$$

The conformal transformation $w = z + (z^2 - 1)^{1/2}$ takes the ellipse $\mathscr{E}$ into the circle $|w| = a + (a^2-1)^{1/2} > 1$. Changing to the variable w, we get for the "tail" $f_2(E, z)$ of the amplitude

$$f_2(E,z) = \frac{2w}{[\pi(w^2-1)]^{1/2}}\sum_{l=L}^{\infty} l^{1/2} f_l(E)w^l\left[1 + O\left(\frac{1}{l\sqrt{\varepsilon}}, \frac{1}{l}\right)\right] \tag{5}$$

for $|w| \geq a + (a^2-1)^{1/2}$. For sufficiently large L the correction terms not written out explicitly in (5) are arbitrarily small, and have no effect at all on the asymptotic behavior of the coefficients of the power series in (5) for $l \to \infty$.

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1)The true scattering amplitude [the coefficient of the outgoing wave $r^{-1}\exp(ikr)$] in nonrelativistic quantum mechanics differs from (1) by a factor k^{-1}. In relativistic theory it is convenient to use a dimensionless scattering amplitude (cf. [7,9]) which differs from (1) by a further factor $s^{1/2}/2k(s)$. For $s \gg m_1^2$, m_2^2 this factor approaches unity.

2)This formula, without indication of the error, is given in [10]. To obtain an accurate estimate of the error we can start from the following representation of $P_\nu(z)$ (see [11], page 142):

$$\frac{\Gamma(\nu+{}^3/_2)}{\Gamma(\nu+1)}\left[\pi\left(w-\frac{1}{w}\right)\right]^{1/2}P_\nu(z) = w^{\nu+1/2}F\left(\frac{1}{2};\frac{1}{2};\nu+\frac{3}{2};\frac{w^2}{w^2-1}\right) + iw^{-(\nu+1/2)}F\left(\frac{1}{2};\frac{1}{2};\nu+\frac{3}{2};-\frac{1}{w^2-1}\right),$$
$$w = z + (z^2-1)^{1/2}.$$

For $\nu = l \to \infty$ and $|w| > 1$ we can replace the hypergeometric functions by unity and neglect the second term; the terms dropped are of the respective orders of magnitude $|4l(w^2-1)|^{-1}$ and $\exp(-2l\ln|w|)$. For z outside $\mathcal{E}$, $|w| > 1 + (2\epsilon)^{1/2}$, and the error of Eq. (4) does not exceed $\max\{(8l2^{1/2}\epsilon^{1/2})^{-1}, \exp(-2l2^{1/2}\epsilon^{1/2})\}$. The condition $L \gg 0.1\epsilon^{-1/2}$ assures that the error in (4) is not larger than $(8l2^{1/2}\epsilon^{1/2})^{-1}$, uniformly in z everywhere outside the ellipse $\mathcal{E}$.

It follows from (5) that the analytic properties of the function $f_2(E, z)$ are determined by a power series (in the variable w). As is well known however, the region of convergence of a power series is a circle with its center at $w = 0$ and having on its circumference the singular point of the function nearest to $w = 0$ (or several such points), and the asymptotic behavior of the coefficients of the power series is uniquely determined by the position and nature of this nearest singularity.[5,6] To put this connection in explicit form, we let w_0 be the nearest singularity of the function $f_2(E, z)$ in the w plane. Going over to the variable $\zeta = w/w_0$, we find that the power series

$$\varphi(\zeta) = \sum_{n=L}^{\infty} a_n \zeta^n, \qquad a_n = 2\left[\frac{n}{\pi(w_0^2-1)}\right]^{1/2} w_0^{n+1} f_n(E) \tag{6}$$

has its nearest singularity at the point $\zeta = 1$, and its nature is uniquely determined by the nature of the singularity of f (E, z) at $z = z_0$. According to the Cauchy-Hadamard formula (cf.[6]),

$$\overline{\lim_{n\to\infty}} |a_n|^{1/n} = 1,$$

and therefore the asymptotic behavior of the coefficients a_n is subject to the restriction

$$\overline{\lim_{n\to\infty}} \frac{\ln|a_n|}{n} = 0. \tag{7}$$

It follows from this that the possible "asymptotic states" for the behavior of the a_n for $n \to \infty$ are as follows:

A. $a_n \sim n^\alpha$, $n^\alpha(\ln n)^\beta$ (α, β arbitrary numbers).

B. $a_n \sim \chi(n)\exp(an^\alpha)$, where $0 < \alpha < 1$, a can be of either sign, and $\chi(n)$ is a slower (at $n \to \infty$) function (in comparison with an exponential), for example any function of type A.

The treatment of an asymptotic behavior of the a_n of type A is an elementary matter.[3)] We give only the results:

$$a_n \sim n^\alpha(\ln n)^\beta, \quad \alpha \neq -m \quad (m = 1, 2, 3, \ldots),$$

$$\varphi(\zeta) \sim \frac{\Gamma(\alpha+1)}{(1-\zeta)^{\alpha+1}}\left(\ln\frac{1}{1-\zeta}\right)^\beta; \tag{8a}$$

$$a_n \sim (\ln n)^\beta / n^m, \quad \beta \neq -1,$$

$$\varphi(\zeta) \sim \frac{(-)^{m-1}}{(m-1)!(\beta+1)}(1-\zeta)^{m-1}\left(\ln\frac{1}{1-\zeta}\right)^{\beta+1}; \tag{8b}$$

$$a_n \sim 1/n^m \ln n, \quad \beta = -1,$$

$$\varphi(\zeta) \sim \frac{(-)^{m-1}}{(m-1)!}(1-\zeta)^{m-1}\ln\ln\frac{1}{1-\zeta}. \tag{8c}$$

These three possibilities exaust all cases of asymptotic behavior of the type $a_n \sim n^\alpha(\ln n)^\beta$ with arbitrary α and β. We emphasize that here only the main terms are given of the expansion of the function $\varphi(\zeta)$ at the point $\zeta = 1$, those which have the highest degree of singularity.

Passing to asymptotic forms of the a_n of type B, we note at once that the nature of the singularity of $\varphi(\zeta)$ at $\zeta = 1$ is different according to the sign of a.

1. $a > 0$. A very large number of terms of the power series for $\varphi(\zeta)$ are important for $\zeta \to 1$, and therefore the series can be replaced by an integral and calculated by the method of steepest descents:

$$\varphi(\zeta) = \sum_{n=0}^{\infty} \chi(n) e^{an^\alpha}\zeta^n \sim \int_0^\infty \chi(n)\exp(an^\alpha + n\ln\zeta)\,dn$$

$$= C(\alpha, a)(1-\zeta)^{-(2-\alpha)/2(1-\alpha)}\chi\left\{\left(\frac{\alpha a}{1-\zeta}\right)^{1/(1-\alpha)}\right\}$$

$$\times \exp\left[(1-\alpha)a\left(\frac{\alpha a}{1-\zeta}\right)^{\alpha/(1-\alpha)}\right]. \tag{9}$$

It can be seen from this that $\varphi(\zeta)$ has an essential singularity at the point $\zeta = 1$, whose nature (i.e., the index of the exponential) does not depend on the form of the "weak" function $\chi(n)$.

2. $a < 0$. In this case $a_n \to 0$ more rapidly than any power of n; therefore the function $\varphi(\zeta)$ is infinitely differentiable in the closed circle $|\zeta| \le 1$, and in particular at all points of the circle of convergence. Nevertheless, $\zeta = 1$ is an essential singular point for $\varphi(\zeta)$.

This can be seen in the following way. Since the nature of the singularity is entirely determined by the "tail" of the power series, the singularity of $\varphi(\zeta)$ at $\zeta = 1$ is the same as that of the integral

$$\int_0^\infty \exp(an^\alpha + n\ln\zeta)\chi(n)\,dn = \psi(\zeta).$$

For $\zeta \to 1$ it is not hard to get the following expansion:

$$\psi(\zeta) = \frac{1}{1-\zeta}\sum_{n=0}^{\infty}\chi\left(\frac{\alpha n}{1-\zeta}\right)\frac{\Gamma(\alpha n+1)}{\Gamma(n+1)}(-x)^n$$

$$x = |a|/(1-\zeta)^\alpha. \tag{10}$$

If $\chi(n)$ has an asymptotic behavior of type A, the sum in (10) is an entire function of x of the order $\rho = (1-\alpha)^{-1}$ and the type $\sigma = (1-\alpha)\alpha^{\alpha/(1-\alpha)}$.

[3)]For example, the case of a power-law behavior $a_n \approx n^\alpha$ can be analyzed by means of the following expansions:

$$(1-\zeta)^{-\alpha} = \sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}\zeta^n, \qquad \ln\frac{1}{1-\zeta} = \sum_{n=1}^{\infty}\frac{\zeta^n}{n}.$$

Since the neighborhood of the point $\zeta = 1$ corresponds to a large circle in the complex plane of x, it is clear that for $\zeta \to 1$ directions can be found along which $|\psi(\zeta)|$ increases like $\exp(\sigma |x|^{\rho})$, that is, more rapidly than any power $(1-\zeta)^{-n}$.

The entire function of the type (10) [with $\chi(n) = 1$] has been studied by Barnes.[12] Using his results, we find: a) on the physical sheet of the ζ plane $\varphi(\zeta)$ has a finite limit in all directions at the point $\zeta = 1$; b) nevertheless, $\zeta = 1$ is a branch point, and on those sheets of the Riemann surface where the condition

$$\mathrm{Re}\left\{\left[-\left(\frac{\alpha|a|}{1-\zeta}\right)^{\alpha}\right]^{1/(1-\alpha)}\right\} > 0$$

holds, $\varphi(\zeta)$ increases for $\zeta \to 1$ in the following way:

$$\varphi(\zeta) \sim \exp\left\{(1-\alpha)|a|\left[-\left(\frac{\alpha|a|}{1-\zeta}\right)^{\alpha}\right]^{1/(1-\alpha)}\right\} \tag{11}$$

[here slower factors of the type $(1-\zeta)^{\beta}$ have been omitted].

It is now easy to find the connection between the nearest singularity of the amplitude in the z plane and the asymptotic behavior of the partial amplitudes $f_l(E)$; for this we must substitute in (6) the asymptotic form of the coefficients a_n which corresponds to the chosen type of singularity of f(E, z). The results are shown below [Eqs. (12), (13)]; we have confined ourselves to the cases in which the coefficient of the exponential in $f_l(E)$ is of the form $l^{\alpha}(\ln l)^{\beta}$, so that it belongs to class A. By means of (9), (11) it is not hard to examine also cases in which the coefficient of the exponential belongs to class B, but we shall not do this in general form; we confine ourselves to selecting, in Section 4, one specific example of this sort of case of potential scattering.

Accordingly, if the nearest singularity of f(E, z) is outside the physical region we have

$$f_l(E) \sim l^{\alpha-3/2}(\ln l)^{\beta} e^{-l\xi}, \qquad \alpha \neq -m \quad (m = 0, 1, 2\ldots),$$

$$f^{(s)}(E, z) = A\Gamma(\alpha)(z_0^2-1)^{\alpha/2}(z_0-z)^{-\alpha}\left(\ln\frac{1}{z_0-z}\right)^{\beta}; \tag{12a}$$

$$f_l(E) \sim \frac{(\ln l)^{\beta}}{l^{m+1/2}} e^{-l\xi} \quad (\beta \neq -1),$$

$$f^{(s)}(E, z) = \frac{A(-)^m}{m!(\beta+1)(z_0^2-1)^{m/2}}(z_0-z)^m\left(\ln\frac{1}{z_0-z}\right)^{\beta+1}. \tag{12b}$$

$$f_l(E) \sim \frac{1}{l^{m+1/2}\ln l} e^{-l\xi},$$

$$f^{(s)}(E, z) = \frac{A(-)^m}{m!(z_0^2-1)^{m/2}}(z_0-z)^m \ln\ln\frac{1}{z_0-z}. \tag{12c}$$

If the nearest singularity of the amplitude is located on the edge of the physical region ($z_0 = 1$), then the formula (4) for $P_l(z)$ cannot be applied in its neighborhood and $f_2(E, z)$ does not reduce to a power series in the variable w. Therefore the method we have used does not hold. In Sec. 3 a different method will be presented for finding the connection between the nearest singularity of f(E, z) and the asymptotic behavior of $f_l(E)$, which holds also for $z_0 = \pm 1$. For convenience we give here the results obtained with this method

$$f_l(E) \sim (\ln l)^{\beta}/l^{\alpha}, \qquad \alpha > 0, \qquad \alpha \neq 2m,$$

$$f^{(s)}(E, z) = \frac{2\pi(1-z)^{\alpha/2-1}}{2^{\alpha/2+\beta}\sin(\pi\alpha/2)\Gamma^2(\alpha/2)}\left(\ln\frac{1}{1-z}\right)^{\beta}; \tag{13a}$$

$$f_l(E) \sim (\ln l)^{\beta}/l^{2m}, \quad \beta \neq -1, \quad m = 1, 2, 3, \ldots,$$

$$f^{(s)}(E, z) = \frac{(-)^{m-1}(1-z)^{m-1}}{2^{m+\beta-2}[(m-1)!]^2(\beta+1)}\left(\ln\frac{1}{1-z}\right)^{\beta+1}; \tag{13b}$$

$$f_l(E) \sim 1/l^{2m}\ln l,$$

$$f^{(s)}(E, z) = \frac{(-)^{m-1}(1-z)^{m-1}}{2^{m-2}[(m-1)!]^2}\ln\ln\frac{1}{1-z} \tag{13c}$$

Here z_0 is the singularity of the amplitude which is nearest to the physical region;

$$A = \left[\frac{2}{\pi}\left(1 + z_0/\sqrt{z_0^2-1}\right)\right]^{1/2};$$

$$z_0 = \cosh\xi, \quad \xi = \ln[z_0 + \sqrt{z_0^2-1}];$$

$f^{(s)}(E, z)$ is the most singular part of the scattering amplitude at the point z_0; and the sign $\sim$ denotes asymptotic equality—that is, $f(l) \sim g(l)$ if

$$\lim [f(l)/g(l)] = 1.$$

Now let f(E, z) be the amplitude corresponding to a Feynman diagram of arbitrary order. In order not to complicate matters by including spin effects, we assume that all internal lines of the diagram correspond to scalar particles. The singularity corresponding to the given diagram is of the form [13,14]

$$f(E, z) \sim \begin{cases} C_0(z_0-z)^{\varkappa}, & \varkappa \neq 0, 1, 2, \ldots \\ -C_0(z_0-z)^{\varkappa}\ln(z_0-z), & \varkappa = 0, 1, 2, \ldots \end{cases}$$

Here $\kappa = \frac{1}{2}(3n - 4v + 3)/2$, n is the number of internal lines, and v is the number of vertices in the diagram.

In the case in which this singularity is the one nearest to the physical region,[4)] the asymptotic behavior of $f_l(E)$ is given by

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4)We recall that besides its own singularities a Feynman diagram has the singularities of all the "narrower" diagrams.

$$f_l(E) \sim C_0\left[\frac{2}{\pi}\left(1+z_0/\sqrt{z_0^2-1}\right)\right]^{-1/2}(z_0^2-1)^{\varkappa/2}$$

$$\times\frac{e^{-l\xi}}{l^{\varkappa+3/2}}\begin{cases}[\Gamma(-\varkappa)]^{-1}, & \varkappa\neq 0,1,2,\ldots\\ (-)^{\varkappa}\varkappa!, & \varkappa=0,1,2,\ldots\end{cases},$$

where $\xi = \ln[z_0 + (z_0^2 - 1)^{1/2}]$.

We note that for partial amplitudes of Feynman diagrams the coefficient of the exponential is always simply a power ($\ln l$ does not appear). The values corresponding to a pole diagram in the t channel (pole at $t = \mu^2$) are as follows:

$$C_0 = g^2/2k^2(s), \quad z_0 = 1 + \mu^2/2k^2(s), \quad \varkappa = -1;$$

the pole phase shifts fall off at $l \to \infty$ as $l^{-1/2}e^{-l\xi}$. Since the value of κ increases as the diagram is made more complicated,[15] according to the foregoing formula this leads to a more rapid decrease of the corresponding phase shifts δ_l with increase of l.

3. THE ASYMPTOTIC BEHAVIOR OF $f_l(E)$ FOR $z_0 = \pm 1$

If the scattering amplitude has a singularity at one end of the physical region ($z_0 = \pm 1$), we cannot use formula (4) for $P_l(z)$, and the method given in Sec. 2 for finding the asymptotic behavior of $f_l(E)$ obviously does not apply. We take a circuitous approach, trying again to reduce the problem to a simpler one—the study of the asymptotic behavior of the coefficients of a power series.

We set up, alongside Eq. (1), a power series with the coefficients

$$g(E, w) = \sum_{n=0}^{\infty} f_n(E) w^n, \qquad (14)$$

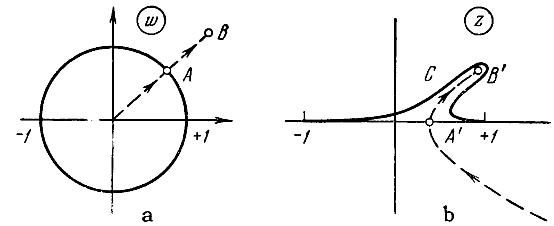
where w is an auxiliary complex variable. We establish a connection between $g(E, w)$ and $F(E, z)$. At first let $|w| < 1$.[5] Substituting in Eq. (14)

$$f_n(E) = \frac{1}{2}\int_{-1}^{1} f(E, z) P_n(z)\, dz$$

and changing the order of summation and integration, we have

$$g(E, w) = \frac{1}{2}\int_{-1} \frac{f(E, z)\, dz}{(1 - 2wz + w^2)^{1/2}}, \quad |w| < 1. \qquad (15)$$

This expression is to be continued analytically into the region $|w| \geq 1$. To do this we note that the singular points of $(1 - 2wz + w^2)^{1/2}$ with respect to the variable z are $z_1(w) = (w + w^{-1})/2$ and $z_2(w) = \infty$. When w moves along a radius from 0 to ∞, then $z_1(w)$ moves along a hyperbola which intersects the segment $-1 \leq z \leq 1$ at the instant when $|w| = 1$ (see figure). This means that the path of integration is pushed along by the moving singularity $z_1(w)$, and for $|w| > 1$ the integration in (15) is taken along the path C shown in the figure.



Path of integration C in Eq. (15) for $|w| > 1$. The dashed lines show the trajectories of the point w and the corresponding singularity $z_1(w)$. The unit circle $|w| = 1$ corresponds to the segment $-1 \leqslant z \leqslant 1$ in the z plane. Points A, B in the w plane correspond to points A′, B′ in the z plane.

A singularity of the function $g(E, w)$ arises[13] when the path C is pinched between a singularity z_0 of the amplitude $f(E, z)$ and the point $z_1(w)$. This gives $z_1(w) = z_0$, from which we have

$$w = w_0 = z_0 \pm (z_0^2 - 1)^{1/2}. \qquad (16)$$

(Here the sign of the root must be chosen so that $|w_0| > 1$.)

For the discontinuity of $g(E, w)$ on the cut $w_0 \leq w \leq +\infty$ we can derive the formula

$$\operatorname{Im} g(E, w) = \int_{z_0}^{z_1(w)} \frac{A(E, z)\, dz}{(1 - 2wz + w^2)^{1/2}} \qquad (17)$$

$$= \left[\frac{z_1(w) - z_0}{2w}\right]^{1/2} \int_0 \frac{A(z_0 + [z_1(w) - z_0]x)}{(1 - x)^{1/2}}\, dx,$$

where

$$A(E, z) = [f(E, z + i\varepsilon) - f(E, z - i\varepsilon)]/2i.$$

This formula essentially solves the problem in hand, by reducing it to elementary calculations: from the nature of the singularity of $f(E, z)$ at the point z_0 one determines the behavior of $A(E, z)$ for z close to z_0, then finds $\operatorname{Im} g(E, w)$ from (17), and thus the actual singularity of $g(E, w)$ for $w \to w_0$. Since $g(E, w)$ is a power series with the coefficients $f_n(E)$, to find the asymptotic behavior of $f_n(E)$ for $n \to \infty$ we then have only to apply the formulas (8) and (9) or (11).

This method for finding the asymptotic behavior of $f_l(E)$ is very general and can be applied independently of the location of the nearest singularity z_0. The method of Sec. 2 is simpler, however, if z_0 lies outside the physical region. For $z_0 = 1$ the quantity w_0 is also equal to unity, and we must use (17). The results obtained in this way

[5]It follows from the unitarity condition [cf. (2a)] that the function g(E, w) has no singularities for $|w| < 1$.

are collected in the formulas (13) [for the asymptotic behavior of $f_l(E)$ of the form $l^{\alpha}(\ln l)^{\beta}$ with arbitrary α, β]. If $z_0 = -1$, there is a factor $(-)^l$ in the partial amplitudes $f_l(E)$.

4. APPLICATION TO POTENTIAL SCATTERING

Let us turn to nonrelativistic potential scattering. There are two methods for determining the asymptotic behavior of the scattering phase shifts $\delta_l(E)$:

1) for large l the motion is quasi-classical, and we can use the quasi-classical approximation for the phase shifts [16]:

$$\delta_l(E) = -\frac{m}{\hbar^2 k}\int_{r_0}^{\infty}\frac{rV(r)}{(r^2-r_0^2)^{1/2}}dr, \qquad r_0 = \frac{l+{}^1/_2}{k}; \quad (18)$$

2) for $l \to +\infty$ the centrifugal barrier $l(l+1)r^{-2}$ increases without limit, and all of the scattering occurs at the "tail" of the potential, i.e., in the region where the potential is weak. Therefore we can use the perturbation-theory formula [16]

$$\delta_l(E) = -\frac{\pi m}{\hbar^2}\int_0^{\infty}[J_{l+1/2}(kr)]^2 V(r) r\, dr. \quad (19)$$

Whereas (19) always gives the correct asymptotic behavior of δ_l, the quasi-classical formula (18) leads to incorrect results if $V(r)$ falls off too rapidly for $r \to \infty$.[6)] It can be verified that for a potential $V(r)$ with asymptotic behavior $\exp[-(\mu r)^{\alpha}]$ the formula (18) for δ_l is no longer valid for $\alpha > 1$, and for $\alpha = 1$ (a potential with an exponential tail of the Yukawa type) it holds only under the additional condition $k \gg \mu$.

Let us consider various forms of the potential $V(r)$.

Long-range Potentials[7)]:

$$V(r) \sim G/r(\mu r)^{\nu} \quad (r\to\infty), \qquad \nu > 0. \quad (20)$$

For $l \to \infty$ calculations by (18) and (19) lead to the same result:

$$\delta_l = -\frac{\varkappa}{k}\frac{\sqrt{\pi}\,\Gamma(\nu/2)}{2\Gamma((1+\nu)/2)}\left(\frac{l}{l_0}\right)^{-\nu}, \qquad l_0 = \frac{k}{\mu}. \quad (21)$$

Here we have introduced the notation $\varkappa = mG/\hbar^2$; $\varkappa$ has the dimensions of momentum. Turning to (13), by means of (21) we find the connection between the "tail" of the potential $V(r)$ and the nature of the nearest singularity of the scattering amplitude. The results are given below in Eqs. (28) and (29). We see that the scattering amplitude for a potential with power-law decrease has a singularity precisely on the boundary of the physical region ($z_0 = 1$).

Let us consider examples of potentials for which the exact solution of the scattering problem is known:

A. $V(r) = Gr^{-1}$ (Coulomb potential). In this case we cannot directly apply the formula (28). As is well known,[16]

$$\delta_l(E) = \arg\Gamma(l+1+i\eta), \qquad \eta = \frac{mG}{\hbar^2 k} = \frac{\varkappa}{k};$$

$$f_l(E) = \frac{1}{2i}\frac{\Gamma(l+1+i\eta)}{\Gamma(l+1-i\eta)} \sim \frac{l^{2i\eta}}{2i}.$$

Using (13a), we get from this

$$f(E,z) \sim -{}^1/_2\eta e^{2i\delta_0}[(1-z)/2]^{-(1+i\eta)}, \qquad z\to 1. \quad (22)$$

We note that the expression obtained for $f(E, z)$ is valid not only near the singularity $z_0 = 1$, but also in the entire z plane. This is a specific characteristic of the Coulomb potential and does not happen with other potentials. Another characteristic feature of the Coulomb potential is that the index of the singularity (22) depends on the energy.

B. $V(r) = G/\mu r^2$, $G > 0$ (we shall consider only the repulsive potential, since in an attractive potential $\sim r^{-2}$ there is collapse into the center[16] and difficulties arise in normalizing the wave functions). The exact solution is of the form [19]

$$\delta_l(E) = -\frac{\pi}{2}\frac{\lambda_0^2}{\lambda+\sqrt{\lambda^2+\lambda_0^2}} \sim -\frac{\pi\varkappa}{2\mu}l^{-1}$$

$$(\lambda = l + {}^1/_2, \qquad \lambda_0 = \sqrt{2\varkappa/\mu})$$

and corresponds to a square-root type of singularity:

$$f(E,z) \sim -\pi\frac{\varkappa}{\mu}[2(1-z)]^{-1/2}.$$

C. $V(r) = \alpha r^{-1} + \beta r^{-2}$ (the foregoing potentials are special cases of this one). The partial amplitudes are given by the formula cf. e.g., [19])

$$f_l(E) = \frac{1}{2i}\exp\left[-\frac{i\pi\lambda_0^2}{\lambda+\sqrt{\lambda^2+\lambda_0^2}}\right]\frac{\Gamma(\sqrt{\lambda^2+\lambda_0^2}+{}^1/_2+i\eta)}{\Gamma(\sqrt{\lambda^2+\lambda_0^2}+{}^1/_2-i\eta)};$$

$$\lambda = l + {}^1/_2, \qquad \lambda_0 = (2m\beta/\hbar^2)^{1/2}, \qquad \eta = m\alpha/\hbar^2 k.$$

For $l \to \infty$ we have: $f_l(E) \sim (2i)^{-1} l^{2i\eta}$, from which it follows that the most singular part of $f(E, z)$ at the point $z = 1$ is of the form (22) and

6)This is already clear from the consideration of potentials that are zero for $r > R$. For potentials of the type of superpositions of Yukawa potentials it is shown in [17,18] that the exact phase shift δ_l approaches the Born approximation (19) for $l \to \infty$ not only along the real axis, but also on a large semicircle in the right half-plane.

7)The condition $\nu > 0$ is necessary in order for the incident and scattered waves to have the usual asymptotic ($r \to \infty$) forms. For $\nu = 0$ we get the Coulomb potential, which leads to a distortion of the asymptotic form of the wave function.[16]

is determined (for $\alpha \neq 0$) only by the "Coulomb" part of the potential, which predominates for $r \to \infty$.

Potential of the Type

$$V(r) \sim \frac{G\exp[-(\mu r)^\alpha]}{r(\mu r)^\nu}, \quad 0<\alpha<1, \quad \nu \text{ arbitrary}. \tag{23}$$

In this case it is convenient to use the quasi-classical formula (18). We have for $l \gg l_0$ ($l_0 = k/\mu$)

$$\delta_l = -\frac{\varkappa}{k}\left(\frac{\pi}{2\alpha}\right)^{1/2}\left(\frac{l}{l_0}\right)^{-(\nu+\alpha/2)}\exp\left[-\left(\frac{l}{l_0}\right)^\alpha\right]. \tag{24}$$

According to Sec. 2, this sort of singularity of $f_l(E)$ corresponds to an essential singularity at $z_0 = 1$, which manifests itself in an exponential increase of the amplitude as we approach $z = 1$ on nonphysical sheets. For example, for $\alpha = \frac{1}{2}[V(r) = \exp[-(\mu r)^{1/2}]$ the Riemann surface of $f(E, z)$ has two sheets which branch at the point $z = 1$. Here the function $f(E, z)$ is finite at $z = 1$ on the physical sheet, and on the nonphysical sheet it increases as $\exp[(1/4\, l_0) \times (1-z)^{-1}]$.

Potentials with Exponential Asymptotic Behavior

A. Let us consider a potential which is a continuous superposition of Yukawa potentials:

$$rV(r) = G\int_\mu^\infty \sigma(\mu')e^{-\mu' r}d\mu'. \tag{25}$$

From (19) we have

$$\delta_l(E) = -\frac{\varkappa}{k}\int_\mu^\infty \sigma(\mu')Q_l\left(1+\frac{\mu'^2}{2k^2}\right)d\mu'. \tag{25a}$$

From this it is clear that the "tail" of the potential and the asymptotic behavior of the distant phases are determined by the behavior of $\sigma(\mu')$ for $\mu' \to \mu$. Setting

$$\sigma(\mu') = \frac{x^{\nu-1}}{\mu\Gamma(\nu)}\left(\ln\frac{1}{x}\right)^\lambda \quad \text{for } x\to 0 \quad \left(x = \frac{\mu'-\mu}{\mu}\right)$$

[where $\nu > 0$ for the convergence of the integral (25)], we get the results given in Eqs. (30), (31). Using the fact that the application of the operator $\mu\partial/\partial\mu$ to the potential $V(r)$ leads to multiplication of the asymptotic form of $V(r)$ for $r \to \infty$ by $(-\mu r)$, we can easily extend the results to the case of arbitrary sign of ν.

We now note that the requirement we have used, that the potential be capable of representation in the form (25), is too restrictive. Indeed, it is not satisfied by the Woods-Saxon potential, which is widely used in nuclear physics. In fact, the connection between the "tail" of $V(r)$ and the nearest singularity of $f(E, z)$ is of a more general character and does not depend on the assumption (25).

B. Let us consider a potential of the form

$$V(r) = G/r\{1+\exp[\mu(r-R)]\},$$

which differs from the Woods-Saxon potential by an unimportant[8] factor r^{-1}. This potential has the expansion (25) only for $R = 0$, since for $R > 0$ the series

$$V(r) = G\sum_{n=1}^\infty (-)^{n+1}e^{n\mu R}r^{-1}e^{-n\mu r}$$

converges only for $r > R$, and the coefficients increase exponentially with increasing n. Nevertheless, by means of (19) we can get for $\delta_l(E)$ an expression in the form of an asymptotic series,

$$\delta_l(E) = \sum_{n=1}^{N-1}(-)^n\frac{\varkappa}{k}e^{n\mu R}Q_l\left(1+\frac{n^2\mu^2}{2k^2}\right) + O\left(\frac{\exp N\mu R}{N^{2l+2}}\right),$$

valid for $l \gg \mu RN(\ln N)^{-1}$. It follows from this that the nearest singularity of $f(E, z)$ is a pole at $z_0 = 1+\mu^2/2k^2$ with the residue $-(\kappa/k)e^{\mu R}$, which is in agreement with the "tail" of the potential in question.

Potentials with Finite Radius:

$$V(r) \sim \frac{G\exp[-(\mu r)^\alpha]}{r(\mu r)^\nu}, \quad \alpha > 1. \tag{26}$$

In this case we can use in the formula (19) the following asymptotic representation for the Bessel function

$$J_\lambda(x) \sim \frac{1}{\Gamma(\lambda+1)}\left(\frac{x}{2}\right)^\lambda\exp\left(-\frac{x^2}{4\lambda}\right), \quad \lambda \gg |x|.$$

Using the method of steepest descent to calculate δ_l, we get

$$\delta_l(E) = C(l)\left(\frac{e\rho\sigma}{l}\right)^{l/\rho}, \quad \rho = \frac{\alpha}{2(\alpha-1)},$$

$$\sigma = (\alpha-1)\left(\frac{k}{\alpha\mu}\right)^{\alpha/(\alpha-1)}, \quad e = 2.718\ldots, \tag{27}$$

where $C(l)$ is a slower function of l than those written out (for example, a power law for $\alpha \geq 2$). It follows from the form of $\delta_l(E)$ that the amplitude $f(E, z)$ is an entire function of z of order

8)It is more convenient to work with this potential, since it leads to singularities of f(E, z) of the simple-pole type, whereas the Woods-Saxon potential leads to second-order poles in the scattering amplitude. In principle this makes no difference to the course of the argument.

ρ and type σ (cf. e.g., [6]). The only singular point of f(E, z) is an essential singularity at infinity, and on a large circle f(E, z) satisfies the inequality

$$|f(E, z)| < \exp\{(\alpha - 1 + \varepsilon)\,|t/2\alpha^2\mu^2|^{\alpha/2(\alpha-1)}\}. \quad (27a)$$
$$t = 2k^2(1 - z), \quad \varepsilon > 0.$$

For $\alpha \to \infty$ (26) goes over into a potential which has a finite cut-off radius and is zero for $r > R = \mu^{-1}$. Then $\rho = 1/2$, $\sigma = k/\mu$, and the inequality (27a) takes the form

$$|f(E, z)| < \exp\{(1 + \varepsilon)\,|t/2\mu^2|^{1/2}\}.$$

Summarizing the argument, we can say that for the potential

$$V(r) \sim \frac{G\exp[-(\mu r)^\alpha]}{r(\mu r)^\nu} \quad (r \to \infty)$$

the nearest singularity of the scattering amplitude is as follows: 1) for $\alpha < 1$, it is the point $z_0 = 1$, which for $\alpha = 0$ is a power or logarithmic singularity of a nature depending on ν, and for $0 < \alpha < 1$ is an essential singular point; 2) for $\alpha = 1$, it is the point $z_0 = 1 + \mu^2/2k^2$, and the nature of the singularity depends on ν; 3) for $\alpha > 1$ the function f(E, z) has no singularity in the finite part of the z plane, but $z = \infty$ is an essential singularity.

We now present the formulas which give in explicit form the connection between the asymptotic behavior of the potential V(r) for $r \to \infty$ and the nature of the nearest singularity of the scattering amplitude:

$$V(r) \sim G/r(\mu r)^\nu \quad (\nu > 0,\ \nu \neq 2n),$$

$$f^{(s)}(E, z) \sim -\frac{\pi}{\Gamma(\nu)\sin(\pi\nu/2)}\,\frac{\varkappa}{k}\left(\frac{k}{\mu}\right)^\nu [2(1-z)]^{\nu/2-1}; \quad (28)$$

$$V(r) \sim G/r(\mu r)^{2n} \quad (n = 1, 2, 3\ldots),$$

$$f^{(s)}(E, z) \sim \frac{(-)^n}{(2n-1)!}\,\frac{\varkappa}{k}\left(\frac{k}{\mu}\right)^{2n} [2(1-z)]^{n-1}\ln\frac{1}{1-z}; \quad (29)$$

$$V(r) \sim Ge^{-\mu r}/r(\mu r)^\nu \quad (\nu \neq n),$$

$$f^{(s)}(E, z) \sim -\frac{\pi}{\Gamma(\nu)\sin\pi\nu}\,\frac{\varkappa}{k}\left(\frac{k}{\mu}\right)^{2\nu}(z_0 - z)^{\nu-1}; \quad (30)$$

$$V(r) \sim Ge^{-\mu r}/r(\mu r)^n \quad (n = 1, 2, 3, \ldots),$$

$$f^{(s)}(E, z) \sim \frac{(-)^n}{(n-1)!}\,\frac{\varkappa}{k}\left(\frac{k}{\mu}\right)^{2n}(z_0 - z)^{n-1}\ln\frac{1}{z_0 - z}; \quad (31)$$

$$V(r) \sim G\exp[-(\mu r)^\alpha]/r(\mu r)^\nu \quad (\alpha > 1), \quad (32)$$

f(E, z) is an entire function of z of order $\rho = \alpha/2(\alpha - 1)$ and type $\sigma = (\alpha - 1) \times (k/\alpha\mu)^{\alpha/(\alpha-1)}$. The constant factor here is $G = \pm g^2$, where g is the interaction constant; the sign + (−) corresponds to repulsion (attraction) at large distances; $\kappa = mG/\hbar^2$, where m is the reduced mass of the particle being scattered by the potential; $z_0 = 1 + \mu^2/2k^2$; the other notations are as in (12) and (13).

The analytic properties (as function of momentum transfer) of the amplitude for scattering by a potential of the type of a superposition of Yukawa potentials has been treated in a number of papers.[17,20,21] For these potentials it has been shown that all of the singularities of f(E, z) as function of z lie on the real axis; the nearest singular point of the scattering amplitude coincides with the singularity of the Born term and is located at $z = 1 + \mu^2/2k^2$, and the next singularity is at $z = 1 + 2\mu^2/k^2$. Since to a complex singularity z_0 of the scattering amplitude there corresponds a contribution $\sim e^{-l\xi}$ in the phase shift δ_l for $l \to \infty$ (here $\xi = \ln[z_0 + (z_0^2 - 1)^{1/2}]$, $\mathrm{Im}\,\xi \neq 0$), it is clear that the amplitude for scattering by a real potential has no complex singularities. The fact that the next-nearest singularity of f(E, z) is located at $t = 4\mu^2$ is a specific characteristic of Yukawa potentials (25), and does not happen for a potential of general form. At the same time the results we have obtained on the position and nature of the nearest singularity of f(E, z) are valid for arbitrary potentials.

5. THE BEHAVIOR OF THE INTERACTION CROSS SECTIONS OF ELEMENTARY PARTICLES AT HIGH ENERGIES

It is very interesting to see what information about the behavior of the interaction cross sections of elementary particles at high energies can be obtained on the basis solely of the analyticity and unitarity properties of the scattering amplitude. The first rigorous result in this direction is due to Froissart,[7] who showed that if the scattering amplitude f(s, t) satisfies the Mandelstam double representation with a finite number of subtractions, then for $s \to \infty$

$$\sigma_{\mathrm{tot}} < c_1(\ln s)^2,$$
$$d\sigma_{\mathrm{elas}}/d\Omega < c_2 s^{1/2}(\ln s)^3, \quad \theta \neq 0, \pi, \quad (33)$$

where c_1 and c_2 are constants which do not depend on s.

Greenberg and Low [8] have remarked that the restrictions (33) follow already from the assumption that $f(s, z)$ is analytic in an ellipse [9)] with foci at the points $z = \pm 1$ and semiaxis major $a = 1 + c/s$ (c is a constant with the dimensions mass squared). The derivations of the inequalities (33) given in [7,8] are both rather cumbersome.

By using the results of Sec. 3 we shall give a simple derivation of the Froissart inequalities.

We start from the following assumptions about the analytic properties of the scattering amplitude:

1) For physical s the function $f(s, z)$ is analytic inside the ellipse $\mathscr{E}(s)$ with the foci $z = \pm 1$ and semiaxis major $a = 1 + m^2/s$.

2) In this ellipse $f(s, z)$ does not increase for $s \to \infty$ more rapidly than a polynomial: $|f(s, z)| < Cs^N$, where C and N are independent of s.

The last assumption corresponds to the stipulation that for fixed t the amplitude $f(s, t)$ satisfies a dispersion relation with respect to s with a finite number of subtractions. It is convenient to generalize the second condition somewhat, by assuming that there exists a monotonically increasing (for $s \to \infty$) function $H(s)$ such that for z inside the ellipse $\mathscr{E}(s)$

$$|f(s, z)| < H(s) \tag{34}$$

[for $H(s) = Cs^N$ we come back to the second condition].

Let us see what restrictions on the function $g(s, w)$ [cf. (14)] are imposed by our stated conditions 1) and 2). It follows from (2a) and (16) that $g(s, w)$ is analytic in the circle $|w| \le 1 + (2m^2/s)^{1/2}$. We shall show that it is bounded in this circle by the same function $H(s)$ as $f(s, z)$. Let

$$1 \leqslant |w| \leqslant 1 + \sqrt{2m^2/s}.$$

From (15) we have

$$|g(s, w)| \leqslant \frac{H(s)}{4|w|} \int\limits_C \frac{|dz|}{|z_1(w) - z|^{1/2}}. \tag{35}$$

Taking as the path of integration C two segments connecting the point $z_1(w)$ with $z = \pm 1$, we get

$$|g(s, w)| \leqslant \frac{H(s)}{2|w|} \{|z_1(w) - 1|^{1/2} + |z_1(w) + 1|^{1/2}\}$$
$$\leqslant \frac{1}{2}\left(1 + \frac{1}{|w|^2}\right) H(s),$$

from which we have for any value of s

$$|g(s, w)| < H(s) \quad \text{for} \quad 1 \leqslant |w| \leqslant 1 + \sqrt{2m^2/s}. \tag{36}$$

To estimate $f_l(s)$ we now apply the Cauchy inequalities for the coefficients of a power series (cf. [6]): if

$$f(z) = \sum_{n=0}^{\infty} f_n z^n$$

is analytic in the circle $|z| \le \rho$ and $M(\rho) = \max\limits_{|z|=\rho} (f(z)^{n=0}$, then $f_n \le M(\rho)\rho^{-n}$. Choosing $\rho = \exp[(m^2/s)^{1/2}]$ and using (36), we have

$$|f_l(s)| < H(s) \exp\{-l\sqrt{m^2/s}\}. \tag{37}$$

The derivation of this inequality in [7,8] is the most cumbersome part of the whole proof. We can proceed further in the usual way: we define L so that for $l \ge L$ the estimate (37) will be stronger than (2a); for $l \ge L$ we use (37), and for $0 \le l \le L - 1$ we use the estimate (2a). This gives

$$L = \sqrt{s/m^2} \ln H(s),$$

$$|f(s, 1)| \leqslant \sum_{l=0}^{\infty} (2l+1)|f_l(s)| < L^2\left[1 + O\left(\frac{1}{\ln H(s)}\right)\right] \tag{38}$$

[the main contribution to the sum for $s \to \infty$ comes from the first L terms; the terms with $l \ge L$ give an amount smaller by a factor $\ln H(s)$]. The final results are

$$\sigma_{\text{tot}} < \frac{16\pi}{s} L^2 = c_1 [\ln H(s)]^2,$$

$$\frac{d\sigma_{\text{elas}}}{d\Omega} < \frac{c_2}{\sin\theta} s^{1/2} [\ln H(s)]^3 \text{ when } 0 < \theta < \pi \tag{39}$$

(in getting the last estimate we have used the inequality

$$|P_l(\cos\theta)| < (2/\pi l \sin\theta)^{1/2}, \quad 0 < \theta < \pi;$$

see [10], page 172). For $H(s) = Cs^N$ the conditions (39) become the Froissart inequalities (33).

It is interesting to note that even if $f(s, z)$ were to increase inside the ellipse as $\exp(as^\alpha)$, so that there could not be even any question of having dispersion relations with a finite number of subtractions, nevertheless σ_{tot} could increase for $s \to \infty$ only by a power law: $\sigma_{\text{tot}} < c_1 s^{2\alpha}$.

The experimental data show that for $s \to \infty$ the total cross sections of all processes approach constant values. This behavior of σ_{tot} does not follow directly from the inequalities (33), and at present it is unknown whether it is a consequence of the analyticity and unitarity conditions alone. Various authors [9,23,24] have attempted to obtain stronger forms of the Froissart inequalities.

The paper by Kinoshita, Loeffel, and Martin [9]

[9)]Strictly speaking, we cannot regard it as proved that the amplitude f(s, z) is analytic in this ellipse. Starting from the general principles of quantum field theory, Lehmann [22] has proved that f(s, z) is analytic only in a much smaller (for $s \to \infty$) ellipse with the semiaxis $a' = 1 + c'/s^2$. Naturally the use of the analyticity in the Lehmann ellipse led Greenberg and Low [8] to restrictions on the increase of σ_{tot} and $d\sigma_{\text{elas}}/d\Omega$ which are much weaker than (33).

starts from the assumption that $|f(s, z)|$ is bounded by a polynomial in s in a part of the z plane much larger than the ellipse $\mathscr{E}(s)$ considered here, and gets a stronger inequality than (33) for $d\sigma_{elas}/d\Omega$ for $0 < \theta < \pi$, but has no success in improving the inequality for σ_{tot}. We shall give an example which proves that the requirements 1) and 2) which we have formulated, together with the condition of unitarity in the s channel, still do not exclude the possibility of increase of σ_{tot} for $s \to \infty$.

Let us consider a function $f(s, z)$ which corresponds to a condensation of an infinite number of singularities of the power-law type $\sim (z_0 - z)^{-\nu+p/2}$, $p = 0, 1, 2, \ldots$, at the point $z_0 = 1 + M^2/s$:

$$f(s,z) = F(s)\left(\frac{z_0-1}{z_0-z}\right)^{\nu} \times \exp\left\{\alpha\left[1-\left(\frac{z_0-z}{z_0-1}\right)^{1/2}\right]\ln\frac{\beta H(s)}{|F(s)|}\right\},$$

$$\alpha = \left(1+\sqrt{1-\frac{m^2}{M^2}}\right)^{-1},$$

$$\beta = \left(1-\frac{m^2}{M^2}\right)^{\nu/2}, \quad 0 < m < M. \tag{40}$$

For $\nu = 1$ the strongest singularity at the point $z = z_0$ is a pole, and the function $f(s, z)$ can serve, for example, as a model of the amplitude for NN scattering. For $\nu = 0$ the strongest singularity $\sim (z_0 - z)^{1/2}$, and we get a model for the absorptive part of the amplitude $A(s, z)$.[10)]

[10)]An analogous model is mentioned in [24]. We note that the partial amplitudes $a_l(s)$ that correspond to the absorptive part $A(s, z)$ must satisfy a more severe restriction than (2a): $0 \leqslant a_l(s) \leqslant 1$ for all l. A verification of this condition can be made in the following way. From (40), setting $\nu = 0$, and also for simplicity $m = M$, we get without difficulty that

$$a_0(s) = F(s)\left[\frac{s}{M^2}\ln\frac{H(s)}{F(s)}\right]^{-1}, \quad a_l(s) \approx a_0(s) \quad \text{for } l \ll l_0,$$

where $l_0 \sim [(s/M^2)\ln\{H(s)/F(s)\}]^{1/2}$. On the other hand, to find $a_l(s)$ for $l << 1$ (sic) we can use the asymptotic form of $P_l(z)$ and get

$$a_l(s) = \frac{1}{s}\int A\left(s, 1-\frac{2\tau}{s}\right) J_0\left(2l\sqrt{\frac{\tau}{s}}\right)d\tau = a_0(s)\left(1+\frac{l^2}{\Lambda^2}\right)^{-1}\exp\left\{-\left(\sqrt{1+\frac{l^2}{\Lambda^2}}-1\right)\ln\frac{H(s)}{F(s)}\right\},$$

where $\Lambda(s) = \frac{1}{2}(s/M^2)^{1/2}\ln[H(s)/F(s)]$. It can be seen from this that as l increases $a_l(s)$ decreases monotonically and remains positive, and therefore it suffices to choose F(s) so as to satisfy the inequality $a_0(s) \leqslant 1$; this is assured by the inequality (4l). Thus we have an example of an absorptive part with respect to the variable s,

$$A(s, t) = F(s)\exp\left\{(1-\sqrt{1-t/t_0})\ln\frac{H(s)}{F(s)}\right\}, \quad t_0 > 0,$$

which is compatible with conditions 1), 2) and the unitarity condition $0 \leqslant a_l(s) \leqslant 1$. It is, however, in contradiction with the condition of crossing symmetry for the spectral function $\rho(s, t)$, so that it cannot be excluded that it may be possible to get a stronger form of the Froissart inequalities (when the requirement of crossing symmetry is taken into account).

From (40) we have: a) $f(s, 1) = F(s)$; b) for all points z in the ellipse $\mathscr{E}(s)$ we have $f(s, z) \le H(s)$, with equality only at the point $z = 1 + m^2/s$. Thus (40) satisfies the conditions 1) and 2) for any function $F(s)$ that satisfies the inequality $|F(s)| \le H(s)$.

Let us now use the unitarity condition (2a). Since $f(s, z)/f(s, 1) > 0$ in the physical region $-1 \le z \le 1$, we have $|f_l(s)| \le |f_0(s)|$, and it suffices to verify the condition (2a) for $l = 0$. For $s \to \infty$ we have $f_0(s) \sim F(s)/s \ln H(s)$, and from this we get

$$|F(s)| \lesssim s\ln H(s). \tag{41}$$

Setting $H(s) = Cs^N$, $N > 1$, we get the following "manner of increase" of the total cross section, compatible with the conditions 1), 2), and Eq. (2a):

$$\mathrm{Im} f(s, 1) = F(s)/s = f\ln s,$$

$$f \leqslant f_{max} = (N-1)[1-\sqrt{1-m^2/M^2}]^{-1};$$

$$\sigma_{tot} \underset{s\to\infty}{\longrightarrow} 16\pi f\ln s, \quad \sigma_{elas}/\sigma_{tot} \underset{s\to\infty}{\longrightarrow} f/2f_{max} \leqslant 1/2. \tag{42}$$

A comparison with (33) shows that the increase of σ_{tot} in this example is not the strongest possible. The reason for this is that the part of the ellipse $\mathscr{E}(s)$ in which $|f(s, z)|$ increases in proportion to $H(s) = Cs^N$ decreases without limit as $s \to \infty$.

From the point of view of the complex j plane the behavior of an amplitude of the type of (40) corresponds to a double pole which passes through the point $j = 1$ at $t = 0$: $f_j(t) \sim (j - 1 - \gamma t)^{-2}$ (I. Ya. Pomeranchuk called the writer's attention to this).

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