NONLINEAR HIGH-FREQUENCY PLASMA CONDUCTIVITY

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We develop a nonlinear theory of the conductivity of a fully ionized plasma in a strong high-frequency field. We find the time-dependent plasma conductivity caused by the electron-ion collisions. We show that, owing to the oscillations of the absolute magnitude of the electron velocity, odd multiple harmonics may arise in the current and lead to the appearance of multiple harmonics of the field. We determine the dependence of the nonlinear current on the polarization and on the constant magnetic field.

1. A situation can often occur in a strong high-frequency field such that the velocity in the electron oscillations under the action of the field is appreciably larger than their thermal velocity. This will be just the case which will interest us in the present paper. When we speak about high frequencies we shall assume that a particle passes through a distance much smaller than its mean free path during a period of oscillation of the field. Under such conditions it is reasonable to talk about free oscillations of a particle under the action of the field (at least, in the zeroth approximation in the ratio of the collision frequency to the field frequency). It is clear that in such a situation there arises an essentially nonlinear field-dependence of the plasma conductivity.1) Indeed, for a field of the form

$$\mathbf{E}(t) = \mathbf{E}_0 \cos(\omega t + \delta) \tag{1.1}$$

for example, the velocity of the oscillations of an electron has the form

$$\mathbf{v}_E = (e\mathbf{E}_0 / m\omega)\sin(\omega t + \delta). \tag{1.2}$$

This is just the velocity which will characterize the effective electron-ion collision frequency which is well known to be inversely proportional to the cube of the absolute magnitude of the velocity.

The formula for the conductivity with a velocity of the form (1.2) is inaccurate when the sine vanishes. We show therefore in the following that there occurs for fields of the form (1.1), apart from the dependence of the conductivity $\sim E_0^{-3}$, a factor containing the logarithm of the ratio of the velocity of the oscillation to the thermal velocity. In the case of a circular polarization of the field, when the absolute magnitude of the velocity remains unchanged, such a logarithm does not occur.

The time-dependence (1.2) of the speed of the oscillations leads to a time-dependence of the conductivity which in turn leads to the occurrence of a field of higher harmonics. The results obtained in this paper enable us to determine the magnitude of such field harmonics. The nonlinear theory of plasma conductivity developed in the following is applicable both to a plasma without a magnetic field and to a strongly magnetized plasma under conditions where the spatial dispersion is unimportant, but where the electron-ion collisions are the determining factor. In other words, we do not take into account effects of the excitation of waves in a plasma or Cerenkov and similar mechanisms for transferring energy from waves to particles. All the more so, since so far there is no theory of waves in such a plasma with an electric field.

We must note that the cause of the nonlinearity of the conductivity considered in the present paper will manifest itself in a whole range of other kinetic characteristics, and not only in the case of a completely ionized plasma. The studies that are possible here may form a new division of the kinetic theory of plasmas.

2. To consider nonlinear effects arising in a completely ionized plasma we use the kinetic equation for a plasma in a strong, uniform electric field

1) An example of such an experimental situation are the experiments on the radiation acceleration of a plasma.[1] Indeed, in these experiments the electron temperature is about 1 eV, and the frequency of the variable field $\omega \approx 2 \times 10^{10}\,\text{sec}^{-1}$. For such parameters the field $\mathcal{E} = mv_T\omega/e$ for which the velocity of the oscillations v_E is comparable to the thermal velocity v_T turns out to be equal to about 300 V/cm. In real devices, however, we are dealing with fields which are larger by orders of magnitude.

which was obtained earlier.[2,3] This equation can be written in the form [2)]

$$\frac{\partial f_a}{\partial t} + e_a \mathbf{E}(t) \frac{\partial f_a}{\partial \mathbf{p}_a} = I_a(\mathbf{p}_a, t). \tag{2.1}$$

Here

$$I_a(\mathbf{p}_a, t) = \sum_b \frac{\partial}{\partial p_a^i} \int d\mathbf{p}_b d\mathbf{r}_b \frac{\partial U_{ab}(|\mathbf{r}_a - \mathbf{r}_b|)}{\partial r_a^i}$$
$$\times \int_{-\infty}^{0} d\tau \frac{\partial}{\partial r_a^j} U_{ab}(|\mathbf{R}_a - \mathbf{R}_b|) \left(\frac{\partial}{\partial P_a^j} - \frac{\partial}{\partial P_b^j} \right)$$
$$\times f_a(\mathbf{P}_a, t+\tau) f_b(\mathbf{P}_b, t+\tau), \tag{2.2}$$

where

$$\mathbf{P}_a \equiv \mathbf{P}_a[t+\tau, t, \mathbf{p}_a] = \mathbf{p}_a + e_a \int_t^{t+\tau} dt' \mathbf{E}(t'), \tag{2.3}$$

$$\mathbf{R}_a \equiv \mathbf{R}_a[t+\tau, t, \mathbf{p}_a, \mathbf{r}_a] = \mathbf{r}_a + \mathbf{v}_a \tau + \frac{e_a}{m_a} \int_t^{t+\tau} dt' \int_t^{t'} dt'' \mathbf{E}(t''), \tag{2.4}$$

while $\mathbf{E}(t)$ is the high-frequency electric field, e_a the charge of the a-th kind of particle, m_a their mass, $\mathbf{v}_a$ their velocity, $\mathbf{p}_a$ their momentum, and $\mathbf{r}_a$ their coordinate. The distribution function f_a is normalized to the number of particles, N_a, in unit volume of the plasma. Of course, we must note that to apply perturbation theory with respect to the Coulomb interaction U_{ab}, in the framework of which the kinetic equation from [2,3] given here was obtained, we must have in mind a field that is screened at large distances and we must also cut off the integration at small impact parameters. This procedure is conventional when we use the Fokker-Planck equation for particles with a Coulomb interaction.[4] We note that in the recently published book by Balescu[5] Eq. (2.1) was obtained by means of a peculiar diagram technique, verifying the direct perturbation theory method which was the basis of the earlier papers[2,3].

To solve Eq. (2.1) we use the fact that the field is a high-frequency one. Accordingly we neglect—as is usually done when the role played by the collisions is small—in zeroth approximation the collision integral. The solution of the equation in zeroth approximation,

$$\partial f_a^0 / \partial t + e_a \mathbf{E}(t) \partial f_a^0 / \partial \mathbf{p}_a = 0, \tag{2.5}$$

can then, clearly, be written in the form

$$f_a^0(\mathbf{p}_a, t) = f_{a0}\left(\mathbf{p}_a - e_a \int_{-\infty}^{t} dt' \mathbf{E}(t') \right). \tag{2.6}$$

Equation (2.6) enables us to write down the following zeroth approximation expression for the current density in the plasma:

$$\mathbf{j}^{(0)}(t) = \sum_a \frac{e_a^2 N_a}{m_a} \int_{-\infty}^{t} dt' \mathbf{E}(t'). \tag{2.7}$$

It is clear that the zeroth approximation current does not describe any nonlinear effects. On the other hand, the zeroth approximation function (2.6) depends essentially non-linearly on the field. In particular, when f_{a0} is a Maxwell function with temperature T_a, we get the following expression for the average energy of the particle:

$$\left\langle \frac{m_a v_a^2}{2} \right\rangle = \frac{3}{2} \varkappa T_a + \frac{e_a^2}{2m_a} \left(\int_{-\infty}^{t} dt' \mathbf{E}(t') \right)^2. \tag{2.8}$$

It is clear that nonlinear effects will be particularly distinct when the second term on the right-hand side of (2.8) turns out to be larger than the first one.

The solution of the first-approximation equation,

$$\frac{\partial f_a^1}{dt} + e_a \mathbf{E}(t) \frac{\partial f_a^1}{\partial \mathbf{p}_a} = I_a^0(\mathbf{p}_a, t) \tag{2.9}$$

can clearly be written in the form

$$f_a^1(\mathbf{p}_a, t) = \int_{-\infty}^{t} dt' I_a^0\left(\mathbf{p}_a - e_a \int_t^{t'} dt'' \mathbf{E}(t''), t' \right), \tag{2.10}$$

where $I_a^{(0)}$ is the collision integral (2.2) in which we have substituted an expression of the form (2.6) for the distribution function. As a result we get for the first-approximation current density

$$\mathbf{j}^{(1)} = i \sum_{ab} N_a N_b \frac{e_a}{m_a} \left(\frac{1}{m_a} + \frac{1}{m_b} \right) \frac{(4\pi e_a e_b)^2}{(2\pi)^3} \int_{-\infty}^{t} dt' \int_{-\infty}^{0} d\tau \cdot \tau \int d\mathbf{k} \frac{\mathbf{k}}{k^2}$$
$$\times \int d\mathbf{p}_a d\mathbf{p}_b f_{a0}(\mathbf{p}_a) f_{b0}(\mathbf{p}_b)$$
$$\times \exp\left\{ i\mathbf{k}, (\mathbf{v}_a - \mathbf{v}_b)\tau + \left(\frac{e_a}{m_a} - \frac{e_b}{m_b} \right) \right.$$
$$\left. \times \left[\int_{t'}^{t'+\tau} dt''(t' - t'') \mathbf{E}(t'') + \tau \int_{-\infty}^{t'+\tau} dt'' \mathbf{E}(t'') \right] \right\}. \tag{2.11}$$

For a Maxwell distribution, and also assuming that there are ions of one kind only, and neglecting small quantities of the order of the ratio of the electron mass to the ionic mass and of the order of the ionic thermal velocity to the electronic thermal velocity, we get

$$\mathbf{j}^{(1)} = -\frac{2}{\pi} \frac{e^4 e_i^2 N_e N_i}{m^3 v_T^2} \int_{-\infty}^{t} dt' \int_{-\infty}^{0} d\tau \int \frac{d\mathbf{k}}{k^4} \mathbf{k} \int_{-\infty}^{t'+\tau} dt'' (\mathbf{k}\mathbf{E}(t''))$$

[2)] We completely neglected in Eq. (2.2) the variable magnetic field and also the non-uniformity of the particle distribution in the plasma. These effects are small for a nonrelativistic plasma and, for instance, for the case of waves with a phase velocity close to the velocity of light.

$$\times \exp\left\{-\frac{1}{2}k^2 v_T^{2}\tau^2 + i\mathbf{k}\frac{e}{m}\left[\int_{t'}^{t'+\tau} dt_1 (t'-t_1)\mathbf{E}(t_1) + \tau \int_{-\infty}^{t'+\tau} dt_1 \mathbf{E}(t_1)\right]\right\}. \quad (2.12)$$

Here $v_T = (\kappa T_e/m)^{1/2}$ is the electronic thermal velocity. The integration over k in Eqs. (2.11) and (2.12) and in subsequent equations is from k_{min} to k_{max}. We must note that to determine k_{max} by estimating the limits of the applicability of classical mechanics or of perturbation theory we must use for the energy the expression defined by Eq. (2.8). We shall use for k_{min} the reciprocal of the ionic Debye radius assuming that the ionic velocity in the electric field is not larger than the thermal velocity.

3. Equation (2.12) gives a description of nonlinear effects occurring under conditions when the electronic velocity in the electric field appreciably exceeds the thermal velocity. Of course, the nonlinear formula (2.12) will give different effects for different time dependences of the field. However, we can understand a whole series of regularities by considering a field of the form (1.1). We get in that case from (2.12)

$$\frac{d\mathbf{j}^{(1)}}{dt} = -\frac{e^2 N_e}{m\omega}\mathbf{E}_0 \frac{4N_i e^2 e_i^2}{m^2 v_T^3}\int_{-1}^{+1} dx \cdot x^2 \int_0^{\infty} dy \int_{k_{min}}^{k_{max}} \frac{dk}{k} \times \sin\left(\omega t + \delta - \frac{\omega y}{k v_T}\right) \times \exp\left\{-\frac{1}{2}y^2 + ikx\frac{2eE_0}{m\omega^2}\sin\frac{\omega y}{2kv_T}\sin\left(\omega t + \delta - \frac{\omega y}{2kv_T}\right)\right\}. \quad (3.1)$$

Equation (3.1) becomes especially simple when the frequencies satisfy the condition

$$\omega \ll v_T k_{min}. \quad (3.2)$$

We then get

$$\frac{d\mathbf{j}^{(1)}}{dt} = -\frac{e^2 N_e}{m\omega}\mathbf{E}_0 \sin(\omega t + \delta)\,\frac{4}{3}\,\frac{\sqrt{2\pi}N_i e^2 e_i^2}{m^2 v_T^3}\,\psi \ln\frac{k_{max}}{k_{min}}. \quad (3.3)$$

Here

$$\psi = 3\frac{v_T^3}{v_E^3}\left\{\sqrt{\frac{\pi}{2}}\Phi\left(\frac{v_E}{\sqrt{2}v_T}\right) - \frac{v_E}{v_T}\exp\left(-\frac{v_E^2}{2v_T^2}\right)\right\}, \quad (3.4)$$

where

$$\Phi(x) = \frac{2}{\sqrt{\pi}}\int_0^x dt e^{-t^2}$$

is the probability integral.

In the limit $v_E^2 \gg v_T^2$ we have

$$\psi = 3\sqrt{\frac{\pi}{2}}\frac{v_T^3}{v_E^3} \equiv 3\sqrt{\frac{\pi}{2}}\left|\frac{m\omega v_T}{eE_0\sin(\omega t+\delta)}\right|^3 \quad (3.5)$$

and hence

$$\frac{d\mathbf{j}^{(1)}}{dt} = -\frac{e^2 N_e}{m\omega}\mathbf{E}_0 \sin(\omega t + \delta)\,\frac{4\pi N_i e_i^2 m}{|eE_0^3 \sin^3(\omega t+\delta)|}\ln\frac{k_{max}}{k_{min}}. \quad (3.6)$$

Such an asymptotic formula could be obtained using Landau's collision integral[4] if in it we change in the proper way the minimum impact parameter. The reason for this lies in the fact that in the case considered the kinetic energy of the oscillations turns out to be appreciably larger than the change in the particle energy under the action of the field during a collision time.

Using Eq. (3.1) we can answer the question about the heat given off in the plasma under the action of the field E_0. To do this we average, over a period of the oscillations of the field, the product of the current density and the electrical field strength. As a result we get

$$\langle \mathbf{E}\mathbf{j}\rangle = \frac{1}{2}E_0^2\sigma_0' \equiv \frac{1}{2}E_0^2\frac{e^2 N_e}{m\omega^2}\nu, \quad (3.7)$$

where

$$\nu = \frac{16 N_i e_i^2 \omega^3 m}{eE_0^3}\left[R\left(\frac{eE_0}{m\omega v_T}, \frac{v_T k_{max}}{\omega}\right) - R\left(\frac{eE_0}{m\omega v_T}, \frac{v_T k_{min}}{\omega}\right)\right], \quad (3.8)$$

$$R(\rho, x) = \rho\int_0^x dz \int_0^{\infty} dy J_0\left(2\rho z \sin\frac{y}{2}\right) \times \left[e^{-x^2y^2/2} + \frac{1}{2}\mathrm{Ei}\left(-\frac{x^2y^2}{2}\right) - e^{-z^2y^2/2} - \frac{1}{2}\mathrm{Ei}\left(-\frac{z^2y^2}{2}\right)\right]. \quad (3.9)$$

If condition (3.2) is satisfied, Eq. (3.8) can be written in the form

$$\nu = \frac{32\sqrt{2\pi}N_i e_i^2 e^2}{m^2 v_T^3}\left(\frac{m\omega v_T}{eE_0}\right)^3 Q\left(\frac{eE_0}{2m\omega v_T}\right)\ln\frac{k_{max}}{k_{min}}, \quad (3.10)$$

where

$$Q(r) = \int_0^r dz z^2 e^{-z^2}[I_0(z^2) - I_1(z^2)]. \quad (3.11)$$

In the limit of weak fields we get from Eq. (3.10) the usual expression for the effective collision frequency

$$\nu_{\text{eff}} = \frac{4}{3}\frac{\sqrt{2\pi}N_i e_i^2 e^2}{m^2 v_T^3}\ln\frac{k_{max}}{k_{min}}, \qquad \frac{eE_0}{m\omega} \ll v_T. \quad (3.12)$$

For strong fields we have

$$\nu_{as} = \frac{16 N_i e_i^2 m\omega^3}{eE_0^3}\left\{\ln\frac{eE_0}{2m\omega v_T} + 1\right\}\ln\frac{k_{max}}{k_{min}}, \qquad \frac{eE_0}{m\omega} \gg v_T. \quad (3.13)$$

In this last formula, the 1 occurring within the braces is the result of the numerical integration.[3)] See the table for intermediate fields.

3) $Q(r) \cong \{\ln r + 1\}/2\sqrt{2\pi}$ when $r \gg 1$.

Values of the function $R(\rho, x)$

ρ \ x	0.5	1.5	5	15
0.5		0.01	0.04	0.07
1	0.01	0.076	0.27	0.49
2	0.068	0.49	1.6	2.6
3	0.19	1.2	3.1	4.8
4	0.35	1.9	4.3	6.5
5	0.49	2.5	5.3	7.8
6	0.57	2.9	6.0	8.8
7	0.65	3.2	6.8	9.8
8	0.70	3.5	7.0	10
10	0.76	3.8	7.6	11
14	0.88	4.4	8.8	13
18	0.96	4.8	9.5	14
24	1.0	5.2	10.5	15
30	1.1	5.5	11.1	16

4. The nonlinear dependence of the current density on the electric field strength leads to the possibility of the occurrence of new harmonics. Expanding the right-hand side of Eq. (3.1) in a Fourier series we have

$$\mathbf{j}^{(1)}(t) = \mathbf{E}_0 \sum_{n=0}^{\infty} [\sigma_n' \cos\{(2n+1)(\omega t+\delta)\} + \sigma_n'' \sin\{(2n+1)(\omega t+\delta)\}], \tag{4.1}$$

$$\sigma_n' = \frac{e^2N_e}{m\omega^2}\frac{1}{2n+1}\frac{4N_ie^2e_i^2}{m^2v_T^3}\int_{-1}^{+1} dx\cdot x^2 \int_0^\infty dz e^{-z^2/2}\int_{k_{min}}^{k_{max}}\frac{dk}{k}$$
$$\times\left\{\cos\left[\frac{(n+1)\omega z}{kv_T}\right] J_{2n}\left(2x\frac{eE_0k}{m\omega^2}\sin\frac{\omega z}{2kv_T}\right) - \cos\frac{n\omega z}{kv_T}\, J_{2n+2}\left(2x\frac{eE_0k}{m\omega^2}\sin\frac{\omega z}{2kv_T}\right)\right\}; \tag{4.2}$$

σ_n'' differs by having sines instead of cosines.

For frequencies satisfying condition (3.2) we get thus

$$\mathbf{j}^{(1)} = \frac{e^2N_e}{m\omega^2}\mathbf{E}_0\frac{4\sqrt{2\pi}\,N_ie^2e_i^2}{m^2v_T^3}\ln\frac{k_{max}}{k_{min}}\sum_{n=0}^{\infty}\frac{1}{2n+1}$$
$$\times\cos[(2n+1)(\omega t+\delta)]\int_0^1 dx\cdot x^2\exp\left(-\frac{x^2e^2E_0^2}{4m^2\omega^2v_T^2}\right)$$
$$\times\left[I_n\left(\frac{x^2e^2E_0^2}{4m^2\omega^2v_T^2}\right) - I_{n+1}\left(\frac{x^2e^2E_0^2}{4m^2\omega^2v_T^2}\right)\right]. \tag{4.3}$$

In the limit of strong fields Eq. (4.3) becomes

$$\mathbf{j}^{(1)} = \sigma_{as}\sum_{n=0}^{\infty}\cos[(2n+1)(\omega t+\delta)]\mathbf{E}_0, \tag{4.4}$$

where

$$\sigma_{as} = \frac{e^2N_e}{m\omega^2}\frac{16N_ie_i^2m\omega^3}{eE_0^3}\ln\frac{eE_0}{m\omega v_T}\ln\frac{k_{max}}{k_{min}}. \tag{4.5}$$

The sum over n in Eq. (4.4) is extended up to a maximum value of the order of magnitude of $eE_0/m\omega v_T$.

In the opposite limit of weak fields we get from Eq. (4.3)

$$\mathbf{j}^{(1)} = \mathbf{E}_0\frac{e^2N_e}{m\omega^2}\nu_{\text{eff}}\sum_{n=0}^{\infty}\left(\frac{eE_0}{2m\omega v_T}\right)^{2n}\frac{3\cos[(2n+1)(\omega t+\delta)]}{(2n+1)(2n+3)n!}. \tag{4.6}$$

For high frequencies, which violate condition (3.2), we must use instead of Eq. (4.6) for weak fields an expression for the effective collision frequency in which we must substitute ω/v_T for k_{min}.

As an example we turn to the problem of the excitation of new harmonics by the field of a traveling wave. E_0 is then a constant and the dependence of the nonlinear current on the coordinates occurs through $\delta = -k_0z$, where $k_0^2c^2 = \omega^2 - \omega_{Le}^2$. The field of the new harmonics can be looked for in the form

$$\mathbf{E}_n \exp\{i(2n+1)(k_0z-\omega t)\}.$$

In that case

$$\mathbf{E}_n = \mathbf{E}_0\frac{\pi\omega(\sigma_n''-i\sigma_n')(2n+1)}{2\omega_{Le}^2n(n+1)}. \tag{4.7}$$

An estimate of the order of magnitude of the field of the harmonics for a frequency ω somewhat exceeding ω_{Le} is given by the formula

$$\frac{E_n}{E_0}\approx\frac{2n+1}{n(n+1)}\left(\frac{m\omega v_T}{eE_0}\right)^3\frac{N_ie^2e_i^2}{m^2v_T^3\omega}\ln\frac{eE_0}{m\omega v_T}\ln\frac{k_{max}}{k_{min}}. \tag{4.8}$$

It is necessary to note that the electrons will be heated under the action of a strong high-frequency field. In other words, as a result of collisions, the energy of the oscillatory motion of the electrons in the electric field will become random. During a time approximately equal to $t_0 = 1/\nu$ [see (3.10)] the energy of the thermal motion becomes equal to the energy of the oscillations after which the nonlinear effects considered by us cease to be clearly expressed. For a plasma of a density $\sim 10^{11}$ cm^{-3}, a temperature ~ 1 eV, and for $E_0 \sim 3$ kV/cm, $\omega = 2\times 10^{10}$ sec^{-1} such a time turns out to be $\sim 10^{-4}$ sec.

5. In this section we discuss the problem of the dependence of the appearance of new harmonics upon the polarization of the electric field. Let

$$\mathbf{E}(t) = \mathbf{E}_1\cos(\omega t+\delta_1) + \mathbf{E}_2\cos(\omega t+\delta_2). \tag{5.1}$$

Using Eq. (2.12) we can in this case easily write down the expression for the current density

$$\mathbf{j}^{(1)} = \sum_{l=0}^{\infty}\frac{4}{\pi}\frac{e^3N_ee_i^2N_i}{m\omega}\frac{1}{2l+1}\int_{-\infty}^{0} d\tau\cdot\tau\int d\mathbf{k}\frac{\mathbf{k}}{k^2}\exp\left\{-\frac{1}{2}k^2\tau^2v_T^2\right\}$$
$$\times J_{2l+1}\left(\frac{2ek}{m\omega^2}\left[w\left(\frac{\mathbf{k}}{k}\right)\right]^{1/2}\sin\frac{\omega\tau}{2}\right)$$

$$\times \cos\left\{ (2l+1)\left[\omega t + \frac{1}{2}\omega\tau + \varphi\left(\frac{\mathbf{k}}{k}\right)\right]\right\}, \quad (5.2)$$

where

$$w(\mathbf{n}) = (\mathbf{nE}_1)^2 + (\mathbf{nE}_2)^2 + 2(\mathbf{nE}_1)(\mathbf{nE}_2)\cos(\delta_1 - \delta_2), \quad (5.3)$$

$$e^{2i\varphi(n)} = (\mathbf{nE}_1 e^{i\delta_1} + \mathbf{nE}_2 e^{i\delta_2}) / (\mathbf{nE}_1 e^{-i\delta_1} + \mathbf{nE}_2 e^{-i\delta_2}). \quad (5.4)$$

Under the conditions when inequality (3.2) is applicable, we get from Eq. (5.2)

$$\mathbf{j}^{(1)} = \frac{e^2 N_e}{m\omega^2} \frac{2N_i e^2 e_i^2}{\sqrt{2\pi}\, m v_T^3} \ln \frac{k_{max}}{k_{min}} \sum_{l=0}^{\infty} \frac{1}{2l+1} \int d o_{\mathbf{n}} \mathbf{n}\, (w(\mathbf{n}))^{1/2}$$
$$\times \cos\{(2l+1)[\varphi(\mathbf{n}) + \omega t]\} \exp\left(-\frac{e^2 w(\mathbf{n})}{4m^2\omega^2 v_T^2}\right)$$
$$\times \left[I_l\left(\frac{e^2 w(\mathbf{n})}{4m^2\omega^2 v_T^2}\right) - I_{l+1}\left(\frac{e^2 w(\mathbf{n})}{4m^2\omega^2 v_T^2}\right)\right]. \quad (5.5)$$

Here $\mathbf{n}$ is a unit vector over the direction of which the integration is carried out.

According to (5.4), in the case of a plane-polarized field $\varphi(\mathbf{n}) = \delta$, and in that connection the sum over harmonics is retained. The opposite situation occurs for circular polarization, when E_1 and E_2 are equal in magnitude and at right angles to one another, and $\delta_2 - \delta_1 = \pi/2$. We find again, according to (5.4), that $\varphi(\mathbf{n})$ is now the azimuthal angle of the vector $\mathbf{n}$. According to (5.3), $w(\mathbf{n})$ does not depend on this angle. Therefore, in Eqs. (5.2) and (5.5) only the terms with $l = 0$ are non-vanishing. In particular, in the strong field limit we get

$$\mathbf{j}^{(1)} = \frac{e^2 N_e}{m\omega^2} \frac{4N_i e_i^2 m\omega^3}{eE_0^3} \ln\frac{k_{max}}{k_{min}} \{\mathbf{E}_1 \cos(\omega t + \delta_1)$$
$$- \mathbf{E}_2 \sin(\omega t + \delta_1)\}. \quad (5.6)$$

The difference between Eq. (5.6) and Eq. (4.4) consists not only in the absence of the higher harmonics, but also in the fact that in Eq. (5.6) there does not occur a double logarithmic expression. The cause of this difference can easily be understood if we note that the mean square of the electronic velocity for the field (5.1) is equal to

$$\langle v^2\rangle = 3v_T^2 + \frac{e^2}{m^2\omega^2}\{\mathbf{E}_1 \sin(\omega t + \delta_1) + \mathbf{E}_2 \sin(\omega t + \delta_2)\}^2. \quad (5.7)$$

For the case of circular polarization of the electric field, corresponding to Eq. (5.7) we get then at once

$$\langle v^2\rangle = 3v_T^2 + e^2 E_0^2 / m^2\omega^2. \quad (5.8)$$

In other words, only the direction of the electronic velocity is changed in a circularly polarized field, while its magnitude remains unchanged. The velocity therefore never vanishes and hence there does not appear the logarithm of the ratio of the velocity of the oscillations to the thermal velocity. On the other hand, the absence of the time-dependence of the electronic energy leads to a time-independent plasma conductivity. The cause of the appearance of harmonics disappears thus for a circularly polarized field.

6. We turn now to a consideration of a plasma in a strong magnetic field. The distribution function in zeroth approximation can, if we neglect the particle collisions, be written in the form

$$f_a^0(\mathbf{p}_a, t) = f_{a0}\left(\mathbf{p}_a - e_a \int_{-\infty}^{t} dt' \left\{\frac{1}{B^2}\mathbf{B}(\mathbf{E}(t')\mathbf{B})\right.\right.$$
$$+ \frac{1}{B}[\mathbf{E}(t')\mathbf{B}]\sin\Omega_a(t-t')$$
$$\left.\left. + \frac{1}{B^2}[\mathbf{B}[\mathbf{E}(t')\mathbf{B}]]\cos\Omega_a(t-t')\right\}\right), \quad (6.1)^*$$

where $\mathbf{B}$ is the magnetic field, and $\Omega_a = e_a B/m_a c$. Using the collision integral obtained earlier[2,3] which differs from (2.2) only in that $\mathbf{P}$ and $\mathbf{R}$ take into account a constant, uniform magnetic field, we can use (6.1) to obtain the following generalized Ohm law:

$$\frac{d\mathbf{j}_a}{dt} - \frac{e_a^2 N_a}{m_a}\mathbf{E}(t) + [\mathbf{\Omega}_a \mathbf{j}_a] = i\sum_b \frac{2}{\pi}\frac{N_a e_a^3 N_b e_b^2}{m_a}$$
$$\times \int_{-\infty}^{0} d\tau \int d\mathbf{k}\,\frac{\mathbf{k}}{k^4}\left\{\frac{1}{m_a}\left(\frac{(\mathbf{kB})^2}{B^2}\tau + \frac{\sin\Omega_a\tau}{\Omega_a}\frac{[\mathbf{kB}]^2}{B^2}\right)\right.$$
$$\left. + \frac{1}{m_b}\left(\frac{(\mathbf{kB})^2}{B^2}\tau + \frac{\sin\Omega_b\tau}{\Omega_b}\frac{[\mathbf{kB}]^2}{B^2}\right)\right\} F_a F_b^* \Phi_a \Phi_b^*. \quad (6.2)$$

Here

$$F_a = \exp\left\{ik\frac{e_a}{m_a}\int_t^{t+\tau} dt'\left(\mathbf{B}\frac{(\mathbf{E}(t')\mathbf{B})}{B^2}(t-t')\right.\right.$$
$$+ \frac{[\mathbf{E}(t')\mathbf{B}]}{B}\frac{1-\cos\Omega_a(t-t')}{\Omega_a}$$
$$\left. - \frac{[\mathbf{B}[\mathbf{BE}(t')]]}{B^2}\frac{\sin\Omega_a(t-t')}{\Omega_a}\right) + ik\frac{e_a}{m_a}\int_{-\infty}^{t+\tau} dt'\left(\mathbf{B}\frac{(\mathbf{E}(t')\mathbf{B})}{B^2}\tau\right.$$
$$+ \frac{[\mathbf{E}(t')\mathbf{B}]}{B}\frac{\cos\Omega_a(t-t') - \cos\Omega_a(t+\tau-t')}{\Omega_a}$$
$$\left.\left. + \frac{[\mathbf{B}[\mathbf{BE}(t')]]}{B^2}\frac{\sin\Omega_a(t-t') - \sin\Omega_a(t+\tau-t')}{\Omega_a}\right)\right\}, \quad (6.3)$$

$$\Phi_a = \int d\mathbf{p}_a f_{a0}(\mathbf{p})\exp\left\{i\mathbf{k}, \mathbf{B}\frac{(\mathbf{Bv}_a)}{B^2}\tau\right.$$
$$\left. - \frac{\sin\Omega_a\tau}{\Omega_a}\frac{[\mathbf{B}[\mathbf{Bv}_a]]}{B^2} + \frac{1-\cos\Omega_a\tau}{\Omega_a}\frac{[\mathbf{Bv}_a]}{B}\right\}. \quad (6.4)$$

In the case of a Maxwell distribution Eq. (6.4) becomes

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*$[\mathbf{E}(t')\mathbf{B}] = \mathbf{E}(t') \times \mathbf{B}$.

$$\Phi_a = \exp\left\{-\frac{1}{2}v_{Ta}^2\left[\frac{(\mathbf{kB})^2}{B^2}\tau^2 + 4\frac{[\mathbf{kB}]^2}{B^2\Omega_a^2}\sin^2\frac{\Omega_a\tau}{2}\right]\right\}, \tag{6.5}$$

where $v_T^2 = \kappa T_a/m_a$. Then

$$\langle v_a^2\rangle = 3v_{Ta}^2 + \frac{e_a^2}{m_a^2}\left(\int_{-\infty}^{t} dt'\left\{\frac{\mathbf{B}}{B^2}(\mathbf{E}(t')\mathbf{B}) + \sin\Omega_a(t-t')\right.\right.$$
$$\left.\left.\times\frac{[\mathbf{E}(t')\mathbf{B}]}{B} - \cos\Omega_a(t-t')\frac{[\mathbf{B}[\mathbf{BE}(t')]]}{B^2}\right\}\right)^2 \tag{6.6}$$

In the case of strong fields one must use just the second term on the right-hand side of this formula to determine the minimum impact parameter.

For an electrical field of the form (5.1) the last term in Eq. (6.6) becomes

$$\frac{e_a^2}{m_a^2}\left\{\frac{1}{B^2\omega^2}(\mathbf{B}, \mathbf{E}_1\sin(\omega t+\delta_1) + \mathbf{E}_2\sin(\omega t+\delta_2))^2\right.$$
$$+\frac{1}{(\omega^2-\Omega_a^2)^2}\left(\frac{\omega}{B^2}[\mathbf{B}[\mathbf{B}, \mathbf{E}_1\sin(\omega t+\delta_1)\right.$$
$$+\mathbf{E}_2\sin(\omega t+\delta_2)]] - \frac{\Omega_a}{B}[\mathbf{B}, \mathbf{E}_1\cos(\omega t+\delta_1)$$
$$\left.\left.+\mathbf{E}_2\cos(\omega t+\delta_2)]\right)^2\right\}. \tag{6.7}$$

Equation (6.7) as well as (6.2) are applicable only when $|\omega^2 - \Omega_a^2|$ is large compared with the square of the effective collision frequency. Since the collision frequency is relatively small, it is clear that for the case where the frequency of the variable field is close to the gyroscopic one, the field may turn out to be strong owing to cyclotron resonance.

For an electric field of the form

$$\mathbf{E} = \sum_i \mathbf{E}_i\cos(\omega t+\delta_i) \tag{6.8}$$

Equation (6.3) takes the form

$$F_a = \prod_r \exp\left\{i\frac{e_a}{m_a}\frac{(\mathbf{k}[\mathbf{E}_r\mathbf{B}])}{\omega B(\omega^2-\Omega_a^2)}[\sin(\omega t+\delta_r)\right.$$
$$-\sin(\omega[t+\tau]+\delta_r)\Big] + i\frac{e_a}{m_a}\left(\frac{(\mathbf{kB})(\mathbf{BE}_r)}{B^2\omega^2}\right.$$
$$\left.+\frac{B^2(\mathbf{kE}_r)-(\mathbf{kB})(\mathbf{BE}_r)}{B^2(\omega^2-\Omega_a^2)}\right)$$
$$\left.\times[\cos(\omega t+\delta_r) - \cos(\omega[t+\tau]+\delta_r)]\right\} \tag{6.9}$$

The formulae obtained here enable us to obtain for a given polarization of the field the necessary expressions for the nonlinear conductivity. As an example of an application we shall consider the case of perpendicular magnetic and electric fields which is of practical interest. The electric field has the form (5.1) with mutually perpendicular components of equal magnitude ($E_1 = E_2 = E_0$) and $\delta_2 - \delta_1 = \pi/2$; we can then write Eq. (6.9) in the form

$$F_a = \exp\left\{i\frac{2e_aE_0k\sin\theta}{m_a\omega(\omega-\Omega_a)}\sin\frac{\omega\tau}{2}\right.$$
$$\left.\times\sin\left(\omega t+\delta_1+\frac{1}{2}\omega\tau+\psi\right)\right\}, \tag{6.10}$$

where θ and ψ are the polar and azimuthal angles of the vector $\mathbf{k}$ in a coordinate system with the polar axis along the magnetic field.

For frequencies of the variable field larger than the ionic gyroscopic frequency and for magnetic fields in which the ionic Larmor radius is large compared with characteristic impact parameters of collisions, we can neglect the ionic motion. Using Eqs. (6.5) and (6.10) we can then write the generalized Ohm's law in the following form:

$$\frac{d\mathbf{j}}{dt} + [\Omega_e\mathbf{j}] = \frac{e^2N_e}{m}[\mathbf{E}_1\cos(\omega t+\delta_1) - \mathbf{E}_2\sin(\omega t+\delta_1)]$$
$$-\mathbf{E}_1[s_1\sin(\omega t+\delta_1) + s_2\cos(\omega t+\delta_1)]$$
$$-\mathbf{E}_2[s_1\cos(\omega t+\delta_1) - s_2\sin(\omega t+\delta_1)], \tag{6.11}$$

where

$$s_1 = \frac{e^2N_e}{m}\frac{8ee_i^2N_i}{mE_0}\int_{-\infty}^{0} d\tau\cos\frac{\omega\tau}{2}\int_{k_{min}}^{k_{max}} dk\cdot k\int_0^{\pi/2} d\theta\sin^2\theta\left\{\tau\cos^2\theta\right.$$
$$\left.+\frac{1}{\Omega_e}\sin\Omega_e\tau\sin^2\theta\right\}\exp\left\{-\frac{1}{2}v_T^2k^2\left[\tau^2\cos^2\theta\right.\right.$$
$$\left.\left.+\frac{4}{\Omega_e^2}\sin^2\theta\sin^2\frac{\Omega_e\tau}{2}\right]\right\}J_1\left(\frac{2eE_0k\sin\theta}{m\omega(\omega-\Omega_e)}\sin\frac{\omega\tau}{2}\right), \tag{6.12}$$

while s_2 differs only in that $\cos\omega\tau/2$ is replaced by the sine of the same argument.

The reason for the absence of higher harmonics in Eq. (6.11) is the same as for the case of circular polarization when there is no magnetic field and is connected with the fact that the electronic velocity is constant. The effective intensification of the electric field in the vicinity of cyclotron resonance is shown clearly in Eq. (6.12). Under conditions when the electronic gyroscopic frequency is not larger than the Langmuir frequency, we get from (6.12)

$$s_1 = \frac{e^2N_e}{m}\frac{4e_i^2N_i(\omega-\Omega_e)}{E_0^2\omega}\int_{-\infty}^{0} d\tau\cdot\tau\cot\frac{\tau}{2}\int_{k_{min}}^{k_{max}} dk$$
$$\times\exp(-v_T^2k^2\tau^2/2\omega^2)\left\{-\cos\left[\frac{2eE_0k}{m\omega(\omega-\Omega_e)}\sin\frac{\tau}{2}\right]\right.$$
$$\left.+\frac{m\omega(\omega-\Omega_e)}{2eE_0k\sin\tau/2}\sin\left[\frac{2eE_0k}{m\omega(\omega-\Omega_e)}\sin\frac{\tau}{2}\right]\right\}, \tag{6.13}$$

and the corresponding expression for s_2 is obtained by replacing the cotangent by unity.

When inequality (3.2) is satisfied, Eq. (6.12) simplifies and becomes

$$s_1 = \frac{e^2 N_e}{m} \frac{8 N_i e_i^2 (\omega - \Omega_e)}{v_T E_0^2} \ln \frac{k_{max}}{k_{min}} \left\{ -\sqrt{\frac{\pi}{2}} \right.$$
$$\times \exp\left[-\frac{e^2 E_0^2}{2 m^2 v_T^2 (\omega - \Omega_e)^2} \right] + \left| \frac{m v_T (\omega - \Omega_e)}{e E_0} \right|$$
$$\left. \times \int_0^\infty \frac{dx}{x} \sin x \exp\left[-\frac{x^2}{2} \frac{m^2 v_T^2 (\omega - \Omega_e)^2}{e^2 E_0^2} \right] \right\}. \quad (6.14)$$

For strong fields we then get

$$s_1 = \frac{e^2 N_e}{m} \frac{4\pi N_i e_i^2 m (\omega - \Omega_e)^2}{|e E_0^3|} \operatorname{sgn}(\omega - \Omega_e) \ln \frac{k_{max}}{k_{min}}$$
$$\text{when} \quad \frac{e^2 E_0^2}{m^2 v_T^2 (\omega - \Omega_e)^2} \gg 1. \quad (6.15)$$

The function s_2 is under those conditions relatively small.

In experimental conditions it is often convenient to work in a region where the electronic gyroscopic frequency, the Langmuir frequency, and the frequency ω of the varying field are not very different, one from the other. Under such conditions Eq. (6.14) is a fair approximation.[4)] A more accurate (numerical) result can be obtained using Eqs. (6.13) and (6.12).

4)We considered in an earlier paper[6] the influence of the electrical field upon the interaction of colliding particles in a strongly magnetized plasma, leading to the departure of particles from the interaction region because of drift in crossed fields; in that paper we showed that the double logarithmic expression which occurs as usual becomes dependent on the drift velocity and thereby on the electrical field strength. We note that it then turns out to be more convenient to introduce a maximum collision time $\tau_{max} \approx p^{3/2} \sqrt{(m/e^2)}$, where p is the impact parameter, rather than a cut-off at small impact parameters in the integration. We must use the same procedure also to obtain an asymptotic expression when inequality (3.2) is violated, when an electron may rotate several times in the collision region.

[1] Veksler, Gekker, Gol'ts, Delone, Kononov, Kudrevatova, Luk'yanchikov, Rabinovich, Savchenko, Sarksyan, Sergeĭchev, Silin, Tsopp, Levin, and Muratov, Radiative acceleration of a plasma, Contribution to the International Accelerator Conference, Dubna, August 1963.

[2] V. P. Silin, JETP **38**, 1771 (1960), Soviet Phys. JETP **11**, 1277 (1960).

[3] V. P. Silin, JETP **41**, 861 (1961), Soviet Phys. JETP **14**, 617 (1962).

[4] L. D. Landau, JETP **7**, 203 (1937), Phys. Zs. Sowjetun. **10**, 154 (1936), Collected Papers (Gordon and Breach and Pergamon Press, 1965) p. 163.

[5] R. Balescu, Statistical Mechanics of Charged Particles, Interscience Publishers, New York, 1963.

[6] V. P. Silin, O provodimosti plazmy v sil'nykh elektricheskom i magnitnom polyakh, Plasma Conductivity in Strong Electric and Magnetic Fields) Preprint Lebedev Institute No. 52 (1964); DAN SSSR (in press).

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