NEGATIVE ENERGY WAVES IN DISPERSIVE MEDIA

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It is shown that nonequilibrium transparent media can exhibit anomalous dispersion, in which case the energy of a monochromatic electromagnetic wave can be negative, i.e., the energy of the medium is lower in the presence of the wave than in the absence of the wave. Negative energies lead to a number of unusual phenomena in propagation and wave interaction.

THE energy of a monochromatic electromagnetic wave (frequency ω) that propagates in a dispersive isotropic transparent medium with magnetic permeability $\mu = 1$ is given by the expression[1] 1)

$$U = \frac{1}{8\pi}\left[\frac{d}{d\omega}(\varepsilon\omega)\langle E\rangle^2 + \langle H^2\rangle\right], \tag{1}$$

where ϵ is the dielectric constant. In the particular case of longitudinal waves ($H = 0$), with which we shall be concerned exclusively here, the wave energy is proportional to the derivative $d\epsilon/d\omega$ so that the sign of the energy depends on whether the medium exhibits normal dispersion $d\epsilon/d\omega > 0$ or anomalous dispersion $d\epsilon/d\omega < 0$.

It follows from the Kramers-Krönig relations that a region of transparency in a medium in thermodynamic equilibrium must also be a region of normal dispersion;[1] thus $U > 0$ in such a medium. On the other hand the Kramers-Krönig relations will not generally hold in their normal form in media which are not in thermodynamic equilibrium, since ϵ can have on the real ω axis a pole corresponding to an undamped oscillation even in the absence of an electric field. The statement concerning the positiveness of $d\epsilon/d\omega$ is also meaningless in such media. Media of this kind can exhibit negative dispersion in regions of transparency and the corresponding waves can exhibit negative energy.

An example of such a medium is a plasma in a magnetic field; the configuration can be anisotropic (with beams), or inhomogeneous in space. For example, the dielectric constant of a collisionless plasma with cold electrons and a highly anisotropic ion velocity distribution ($T_{\parallel}/T_{\perp} \to 0$) will be of the following form[2] (for the n-th harmonic of the ion cyclotron frequency $\Omega_i = eH/Mc$):

$$\varepsilon = 1 - \frac{\omega_0^2}{\omega^2}\frac{k_z^2}{k^2} - \frac{\Omega_0^2\zeta_n}{(\omega - n\Omega_i)^2}\frac{k_z^2}{k^2}, \tag{2}$$

where

$$\omega_0^2 = \frac{4\pi e^2 n_0}{m}, \qquad \Omega_0^2 = \frac{4\pi e^2 n_0}{M},$$

$$\zeta_n = I_n(x)e^{-x}, \qquad x = \frac{k_\perp^2 T_\perp}{M\Omega_i^2},$$

k is the wave number, k_z is the projection of the wave vector in the direction of the external magnetic field, I_n is the Bessel function of imaginary argument and order n. It is evident that $\partial\epsilon/\partial\omega < 0$ at a frequency ω somewhat below Ω_i, corresponding to a negative-energy wave.

Another example is an inhomogeneous plasma in a strong magnetic field with Maxwellian electron and ion velocity distributions. Close to the n-th harmonic of the ion cyclotron frequency the dielectric constant for transverse propagation ($k_z = 0$) is given by the relation[3]

$$\varepsilon = 1 + \frac{\omega_0^2 m}{Tk^2}\left(1 - \frac{\omega^*}{\omega}\right)\left(1 - \frac{\omega\zeta_n}{\omega - n\Omega_i}\right), \tag{3}$$

where $\omega^* = kv_0$ and v_0 is the electron drift velocity perpendicular to the density gradient $v_0 = T|\nabla n_0|/M\Omega_i n_0$. According to (3), the dispersion becomes negative near $\omega = n\Omega_i$ when $\omega < \omega^*$.

A negative-energy wave can exhibit a number of unusual features when it interacts with matter or with other waves. For example the introduction of ordinary dissipation, which corresponds to energy absorption [(in Eqs. (2) and (3) a dissipation mechanism of this kind might be Landau damping on the electrons or collisions of electrons with neutrals], does not damp the wave in time; instead, the wave grows. Indeed, the energy of the wave is absorbed by the medium if the conductivity $\sigma = (\omega/4\pi)\epsilon'' = (\pi/4\pi)\,\mathrm{Im}\,\epsilon$ is positive; it is evident from the

1) This expression also holds in the presence of spatial dispersion.

damping factor $\gamma = \epsilon''(\partial\epsilon'/\partial\omega)^{-1}$ that absorption implies negative damping in a medium with negative dispersion.

A similar amplification effect arises when the wave is reflected from the boundary of a medium in which the dispersion is of opposite sign. For example, an ordinary wave is amplified when reflected from a medium with negative dispersion when a negative energy wave is launched in the second medium. This effect would be easiest to observe in the reflection of a sound wave (which might be regarded as a longitudinal electromagnetic wave) from a tangential discontinuity, that is to say, a boundary between two regions of the same material moving with a velocity V. Transforming from the moving coordinate system to the laboratory coordinate system means that the wave frequency ω becomes $\omega + kV$; hence, in the case of supersonic flow ($V > c_s = \sqrt{T/M}$) the frequency of the wave propagating against the flow is reversed in sign and its energy becomes negative. (cf. [4].) It is easily shown that the reflection of a sound wave from a supersonic flow can lead to the growth of the wave. Furthermore, when $V > 2c_s$ the coefficient of reflection becomes infinite for any angle of incidence, that is to say, the incident wave need not be present; consequently it is possible to have pair production of waves with positive and negative energies. In this case two waves are launched and these propagate away from the boundary in opposite directions.

This production process can be expressed simply in the coordinate system in which one flow moves with velocity $v_0 = V/2$ along the x-axis and the other with velocity $-v_0$. In this coordinate system pair production corresponds to a wave frequency $\omega = 0$ and the ratio of the components, k_y (normal) and k_x, is $k_y/k_x = c_s/v_0$. In other words, there is a Cerenkov emission of sound waves from the perturbed separation surface. The power of this radiation is quadratic in the perturbation amplitude and can be arbitrary.

An analogous pair-production arises in an infinite plasma at rest. This will be the case when the dispersion equation $\epsilon(\omega) = 0$ exhibits multiple roots so that there are solutions with amplitudes growing linearly in time.

Since the sign of the energy (and correspondingly, the sign of the dispersion) depend on the coordinate system, in certain cases the negative-energy effect is nonessential and can be removed by conversion to another coordinate system. (In such a conversion the sign of the conductivity $\sigma = \omega\epsilon''/4\pi$ must also change since the real part of the frequency γ is invariant with respect to Galilean transformations.) However, if there are several flows which are separated in space or which interpenetrate each other the energy sign reversal due to the motion is a very real effect. In particular, the reversal in the sign of the conductivity can be a true physical feature.

An an example let us consider a weakly ionized plasma in a longitudinal electric field. If the directed (flow) velocity of the electrons V is appreciably smaller than the thermal velocity the electron conductivity in the coordinate system moving with the electrons $\sigma_0 = \omega'\epsilon''/4\pi$ is essentially independent of the velocity V and can be reregarded as constant when $\omega \ll \nu_e$ ($\omega' = \omega + kV$ is the frequency in the moving coordinate system and ν_e is the electron-neutral collision frequency). Thus, in the laboratory coordinate system we have

$$\sigma = \frac{\omega}{4\pi}\varepsilon'' = \frac{\omega}{\omega + kV}\sigma_0 \tag{4}$$

(we have taken account of the fact that the dielectric constant does not change in going from one coordinate system to the other).

The relation in (4) has been obtained by Gertsenshteĭn and Pustovoĭt[5] by direct calculation in the laboratory coordinate system. According to (4) the conductivity changes sign when the velocity V becomes greater than the phase velocity ω/k. The reversal of the sign of the conductivity can result in the excitation of waves with positive energy, but these waves themselves can only exist if there is a real difference between the mean velocities of the electrons and ions.

Negative energy effects can also arise in nonlinear interactions between waves. In this case the negative-energy wave can grow by virtue of the nonlinear transfer of energy to the ordinary wave; similarly the negative-energy wave can be damped by the acquisition of energy from the ordinary waves.

[1] L. D. Landau and E. M. Lifshitz, Élektrodinamika sploshnykh sred (Electrodynamics of Continuous Media), Gostekhizdat, 1957, p. 320, 338.

[2] A. B. Kitsenko and K. N. Stepanov, ZhTF **31**, 176 (1961), Soviet Phys. Tech. Phys. **6**, 127 (1961).

[3] A. B. Mikhaĭlovskiĭ and A. V. Timofeev, JETP **44**, 919 (1963), Soviet Phys. JETP **17**, 626 (1963).

[4] L. D. Landau and E. M. Lifshitz, Statisticheskaya fizika (Statistical Physics), Gostekhizdat, 1951, p. 226.

[5] M. E. Gertsenshteĭn and V. I. Pustovoĭt, JETP 43, 536 (1962), Soviet Phys. JETP **16**, 383 (1963).

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