ON THE EQUATIONS OF QUANTUM FIELD THEORY

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A formulation of the fundamental axioms of quantum field theory is proposed which differs somewhat from the usual one. The condition that the equal time commutator of the current and the field operators have minimum singularity is taken as one of the axioms. Microcausality and the existence of a unitary S matrix are consequences of the basic axioms of the theory. From these axioms a closed set of equations can be derived for the S matrix elements which are off the mass shell in only one of the external four-momenta. In order to exclude undetermined subtraction terms, the equations are written in integro-differential form (in momentum space). This form permits one, in particular, to formulate the boundary conditions and to show that the number of independent constants ("charges") entering the equations apart from the particle masses is exactly equal to the number of matrix elements which do not vanish for any form of limit at infinity in the invariant variables. The iteration solution of the set of equations is identical with the renormalization expansion in perturbation theory.

1. INTRODUCTION

CAN the axiomatic method (A method), at present, compete successfully with the S matrix method[1)] (S method) in the theory of elementary particles? The present paper may, in some respect, be regarded as an attempt to give an affirmative answer to this question.

The appeal of the S method consists in its physical directness and its outward simplicity: the matrix elements are only considered on the mass shell and depend on the minimum number of variables, $3n-10$ ($n \geq 4$ is the number of external lines). But despite several important successes of this method, the general formulation of its equations is still far from complete: the analytic properties of the matrix elements become forbiddingly complicated as the number of external lines increases; the analysis of the five-line graph shows that one cannot, apparently, avoid going off into the complex plane of the invariant variables; and, finally, the basic principle of the S method, maximal analyticity consistent with unitarity, is not formulated in closed mathematical form.

The principal successes of the axiomatic method are mainly connected with investigations of analytic properties and the proof of various dispersion relations for the matrix elements.

However, the existing formulations of the basic equations in the A method suffer from excessive generality and complexity. We note that, apart from inessential differences, Lehmann, Symanzik, and Zimmermann,[1] Wightman,[2] and Nishijima[3] consider equations for vacuum expectation values of various products of the operators of the self-interacting fields in x space. These expectation values depend on the maximum number ($4n-10$, $n \geq 4$) of independent invariant variables. A formulation of these equations in p space and on the mass shell is impossible for the simple reason that these expectation values are singular for $p_i^2 \to m_i^2$. If one avoids these singularities by considering the S matrix elements on the mass shell, one can reduce the number of independent variables to $3n-8$, which are still two variables more than on the mass shell. It must be emphasized that the question of the number of independent variables is not an academic one. It is of the greatest importance as soon as one goes beyond the framework of perturbation theory. Furthermore, in the above-mentioned formulations of the axiomatic method it is not very well understood how the boundary conditions should be prescribed outside the frame of perturbation theory. The complexity of these formulations also shows up in the circumstance that it is impossible even

1) By S method one usual understands the attempt to formulate the theory in terms of so-called "observable" quantities – matrix elements of the S matrix for physical particles ($p_i^2 = m_i^2$), i.e., particles on the mass shell.

in perturbation theory to solve the integral equations, which are formally satisfied both by the renormalizable and the unrenormalizable versions of the theory.

In the formulation of the axiomatic method of Bogolyubov, Medvedev, and Polivanov [4] and Bogolyubov and Shirkov [5] (called the BMPSh method for brevity), there is great arbitrariness in the choice of the so-called quasilocal or subtraction terms. Even if this arbitrariness is reduced by additional assumptions, it was so far not clear by what considerations one can determine the form of these terms or how they can be eliminated from the final formulation.

The aim of the present paper is: 1) to formulate the fundamental axioms such that the arbitrariness in the definition of the matrix elements is narrowed down to the maximal extent and that possibly all nonrenormalizable theories are excluded; and 2) to derive from these axioms equations for the elements of the S matrix which are extrapolated off the mass shell in only one of the external four-momenta (the number of independent variables is here $3n-9$ instead of $3n-10$ on the mass shell).

In the lectures held on this topic, [6] these equations were found, but the fundamental assumptions were not formulated sufficiently clearly. In the present paper we give a formulation of the basic axioms which differs somewhat from the usual one (cf., e.g., [7]). Instead of the condition of local commutativity outside the light cone we introduce a more rigorous condition—we postulate the character of the singularity of the equal time commutators between the field operators and the current.[2)] Local commutativity, the existence of a unitary S matrix, and, apparently, the renormalizability of the theory are consequences of the other axioms. The number of arbitrary parameters ("charges") entering the solution of the system of equations, apart from the masses of the particles, is equal to the number of amplitudes which do not vanish for any form of limit at infinity in the invariant variables.

We find as a result that the axioms formulated in Sec. 2 are clearly sufficient for a derivation of the desired integro-differential equations. The question of the necessity and internal independence of these axioms remains still open. The axioms of the theory are briefly formulated in Sec. 2. In Sec. 3, we find the formal solution of the equal time commutation relations and prove the unitarity. In Sec. 4, a system of integro-differential equations for the matrix elements is derived. Section 5 is devoted to a discussion of these equations.

2. FUNDAMENTAL AXIOMS[3)]

For simplicity, we consider the model of a self-interacting neutral field with mass m. We assume that there are no bound states.

Axiom 1. Definition of the state space: each state of the system corresponds to a vector in the Hilbert space H with a positive definite metric.

Axiom 2. Definition of the field $\varphi(x)$: for each point in x space there exists a finite bilinear form $\langle\Psi|\varphi(x)|\Phi\rangle$, where Ψ and Φ are arbitrary vectors of H. Furthermore we require $\langle\Psi|\varphi(x)|\Phi\rangle^* = \langle\Phi|\varphi(x)|\Psi\rangle$, i.e., the operator $\varphi(x)$ is hermitian.

Axiom 3. Lorentz invariance: to each inhomogeneous Lorentz transformation (a, Λ)[4)] there corresponds a unitary operator $U(a,\Lambda)$ in H. These operators form a representation of the inhomogeneous Lorentz group. Covariance of the field means

$$U(a,\Lambda)\varphi(x)U^{-1}(a,\Lambda) = \varphi(\Lambda x + a). \tag{2.1}$$

Axiom 4. Spectral property: the operator of energy-momentum $\hat{P}_\mu$ is an operator for infinitesimally small translations $U(a)$:

$$U(a) = \exp(i\hat{P}a). \tag{2.2}$$

The eigenvalues of $\hat{P}_\mu$ lie in the upper light cone. There is a unique vacuum state $|0\rangle$ in H,

$$\hat{P}_\mu|0\rangle = 0. \tag{2.2a}$$

We also assume that $\langle 0|\varphi(x)|0\rangle = 0$.

Axiom 5. Minimal singularity: we introduce the current operator

$$j(x) \equiv (\Box - m^2)\varphi(x) \equiv K_x\varphi(x), \tag{2.3}$$

where m is the mass of the single-particle state, and require that the commutator $[\varphi(x), j(y)]_-$ for equal times have the weakest admissible (by the transformation properties of the fields) singularity, i.e., for scalar operators

$$[\varphi(x), j(y)]_-|_{x_0=y_0} = 0. \tag{2.4}$$

For spinor operators the analogous relation has $\delta(\mathbf{x}-\mathbf{y})$ on the right-hand side.

Axiom 6. Completeness: we construct the in field operator

2) This postulate may be called the principle of minimal singularity.

3) Cf. [7]

4) Here a is a translation, and Λ a rotation in four-dimensional space.

$$\varphi_{in}(x) \equiv \varphi(x) + \int \Delta^R(x - x', m) j(x') d^4x', \qquad (2.5)$$

where $\Delta^R(x, m)$ is the retarded Green's function of the Klein-Gordon equation. We assume that

$$[\varphi_{in}(x),\ \varphi_{in}(y)]_- = -i\Delta(x - y,\ m) \qquad (2.6)$$

and that the space of in states exhausts H:

$$H^{in} = H. \qquad (2.7)$$

The operator $\varphi_{in}(x)$ may be expanded in terms of creation and annihilation operators:

$$\varphi_{in}(x) = (2\pi)^{-3/2} \int \frac{d\mathbf{p}}{2E(\mathbf{p})} \{e^{ipx} a_{in}^{(+)}(\mathbf{p}) + e^{-ipx} a_{in}^{(-)}(\mathbf{p})\} \qquad (2.8)$$

with the commutation relation

$$[a_{in}^{(-)}(\mathbf{p}),\ a_{in}^{(+)}(\mathbf{p}')]_- = 2p_0\delta(\mathbf{p} - \mathbf{p}'),$$

$$p_0 = +\sqrt{\mathbf{p}^2 + m^2} \equiv E(\mathbf{p}). \qquad (2.9)$$

The main difference with respect to the usual formulations lies in Axiom 5. In the usual axiomatic method one introduces the axiom of microcausality to define the local properties of the theory. It is easy to see that microcausality in the sense of local commutativity is a consequence of Axioms 3, 5, and 6. Indeed, it follows from Lorentz invariance and (2.4) that

$$[\varphi(x),\ j(y)]_- = 0 \qquad (x - y)^2 < 0, \qquad (2.10)$$

i.e., local commutativity of φ and j, and hence

$$[j(x), j(y)]_- = 0, \qquad (x - y)^2 < 0. \qquad (2.11)$$

It was shown earlier,[8] that (2.5), (2.6), and (2.10) lead to

$$[\varphi(x), \varphi(y)]_- = 0, \qquad (x - y)^2 < 0. \qquad (2.12)$$

Thus our theory is microcausal.[5)]

Let us now find a criterion for the existence of a unitary S matrix operator in our theory. Besides the in field, let us also introduce the out field:

$$\varphi_{out}(x) \equiv \varphi(x) + \int \Delta^A(x - x', m) j(x') d^4x', \qquad (2.13)$$

where $\Delta^A(x, m)$ is the advanced Green's function for the Klein-Gordon equation. We note that by translation invariance

$$\partial\varphi_{in,\,out}(x) / \partial x^\mu = i[\hat{P}_\mu,\ \varphi_{in,\,out}(x)]_- \qquad (2.14)$$

regardless of the commutation relations satisfied by φ_{in} and φ_{out}; by definition of the vacuum state, (2.2a), we have

$$|0\rangle = |0\rangle_{in} = |0\rangle_{out}, \qquad (2.15)$$

5)A detailed comparison of the various formulations of microcausality may be found in [8].

where $|0\rangle_{in}$ and $|0\rangle_{out}$ are defined by

$$a_{in,\,out}^{(-)}(\mathbf{p})\,|0\rangle_{in,\,out} = 0.$$

Furthermore, if $\varphi_{out}(x)$ satisfies the free field commutation relations,

$$[\varphi_{out}(x),\ \varphi_{out}(y)]_- = -i\Delta(x - y, m), \qquad (2.16)$$

it follows from (2.6) and (2.15) that

$$\varphi_{out}(x) = S^+\varphi_{in}(x)S, \qquad (2.17)$$

where the unitary operator S is called, by definition, the S matrix. It is easy to show from (2.5), (2.6), and (2.13) that $\varphi_{out}(x)$ satisfies (2.16) if and only if

$$\int \Delta(x - x',\ m)\Delta(y - y',\ m) \left\{\frac{\delta j(x')}{\delta\varphi_{in}(y')} - \frac{\delta j(y')}{\delta\varphi_{in}(x')} + i[j(x'), j(y')]_-\right\} d^4x' d^4y' = 0. \qquad (2.18)$$

This is now the criterion for the existence of a unitary S matrix expressed through the properties of the current operators. The S matrix is related to the current operator through

$$\int \Delta(x - x', m) \left\{ j(x') - iS^+ \frac{\delta S}{\delta\varphi_{in}(x')} \right\} d^4x' = 0. \qquad (2.19)$$

We shall show in the following (Sec. 3) that (2.18) follows from Axioms 1 to 6.

From the point of view of the Lagrangian formalism condition (2.4) means that the current j(x) contains no odd derivatives of $\varphi(x)$ with respect to x_0. Thus, Axiom 5 excludes a wide class of nonrenormalizable interactions with derivatives.

We thus arrive at the result that the equal time commutator condition (2.4) contains more information than the condition of local commutativity (2.12). The equations derived from Axioms 1 to 6 will therefore have less "degrees of freedom" than in the usual axiomatic formulation.

3. FORMAL SOLUTION OF THE COMMUTATION RELATIONS

In this section we find a formal solution for the matrix element of the current which follows from the equal time commutation relations (2.4) and the other axioms. We call this solution formal because, although finite, it is expressed as a difference of two indeterminate and generally divergent terms. We note that in deriving the desired integro-differential equations (Sec. 4) one can in general avoid this intermediate step of a formal solution. Nevertheless, we have chosen this procedure for two reasons: first, for the sake of a simple proof of unitarity, and second, to exhibit most clearly the similarities and differences with respect to the methods of LSZ and BMPSh (cf. Sec. 5).

Let us rewrite (2.4) in momentum space for an arbitrary matrix element between in states:

$$\int e^{-i\mathbf{px}} \langle m | [\varphi(\mathbf{x}), j(0)]_- | l\rangle \, d\mathbf{x} = 0. \tag{3.1}$$

Here, for example, $|l\rangle \equiv |\mathbf{q}_1, \ldots, \mathbf{q}_l\rangle$ is the vector of the in state of l particles with the momenta $\mathbf{q}_j$, $j = 1, 2, \ldots, l$ and the energies $E_j = +(q_j^2 + m^2)^{1/2}$.

We note that we need not consider the vacuum matrix element $\langle 0 | [\varphi(x), j(0)]_- | 0\rangle$, since it automatically satisfies (3.1).[6]

Let us introduce the notation

$$r_\pm(m|p|l) = \pm\langle m| [a_{in}^{(\mp)}(\pm\mathbf{p}), j(0)]_- |l\rangle. \tag{3.2}$$

Using (2.5), (2.8), (2.9), and the completeness of the in states, we rewrite (3.1) in the form

$$r_+(m|p|l) - r_-(m|p|l) = I(m|p|l); \tag{3.3}$$

$$I(m|p|l) = 2(2\pi)^{3/2}E(\mathbf{p}) \sum_n \langle m|j(0)|n\rangle\langle n|j(0)|l\rangle$$
$$\times\left\{ -\frac{\delta(\mathbf{p}_n - \mathbf{p}_l + \mathbf{p})}{m^2 - (p_n - p_l)^2} + \frac{\delta(\mathbf{p}_n - \mathbf{p}_m - \mathbf{p})}{m^2 - (p_n - p_m)^2} \right\}, \tag{3.4a}$$

$$p_l^0 \to p_l^0 + i\varepsilon, \qquad p_m^0 \to p_m^0 - i\varepsilon, \tag{3.4b}$$

where p_m, p_n, p_l are the total four-momenta of the $|m\rangle$, $|n\rangle$, and $|l\rangle$ states; $p_n = (p_n^0, \mathbf{p}_n)$, etc.

The right-hand side of (3.4a) can be written as a difference

$$I(m|p|l) = R_+(m|p|l) - R_-(m|p|l), \tag{3.5}$$

$$R_\pm(m|p|l)$$
$$= i(2\pi)^{-3/2}\int e^{ip_\pm x}\langle m|\theta(-x_0)[j(x), j(0)]_-|l\rangle\, d^4x$$
$$= (2\pi)^{3/2}\sum_n \langle m|j(0)|n\rangle\langle n|j(0)|l\rangle$$
$$\times\left\{ \frac{\delta(\mathbf{p}_n - \mathbf{p}_m - \mathbf{p})}{E_m - E_n \pm E(\mathbf{p}) - i\varepsilon} - \frac{\delta(\mathbf{p}_n - \mathbf{p}_l + \mathbf{p})}{E_n - E_l \pm E(\mathbf{p}) - i\varepsilon} \right\},$$
$$p_\pm^0 = \pm E(\mathbf{p}). \tag{3.6}$$

This separation is a purely formal operation, since the $R_\pm$ are in general divergent expressions (although the difference $R_+ - R_-$ must always be finite!).

Let us now show that the (formal) solution of (3.3) to (3.6) is

$$r_\pm(m|p|l) = R_\pm(m|p|l) + K(m|l), \tag{3.7}$$

where $K(m|l)$ is an arbitrary function independent of p.

[6] This is easily proved by setting $p_m^\mu = 0$, $p_l^\mu = 0$ in (3.4) and using invariance against space reflection.

Proof. As shown earlier,[8] we have owing to local commutativity (2.12)

$$r_\pm(m|p|l) = i(2\pi)^{-3/2}\int e^{ip_\pm x}d^4x\,\langle m|\theta(-x_0)[j(x), j(0)]_-$$
$$+ \Lambda(x)|l\rangle, \tag{3.8}$$

where $\Lambda(x)$ is an arbitrary quasilocal operator.[7] Substituting (3.8) in (3.3) and comparing with (3.5), we find that $\Lambda(x)$ can not contain derivatives of $\delta(x)$, i.e.,

$$\Lambda(x) = \delta(x)\lambda(0), \tag{3.9}$$

where λ is an arbitrary operator. From this we immediately obtain (3.7) with

$$K(m|l) = i(2\pi)^{-3/2}\langle m|\lambda(0)|l\rangle.$$

Let us prove the unitarity of the theory. Condition (2.18) is a simple consequence of (3.8) and (3.9). Formulas (2.6) to (2.9) allows us to write $r_\pm(m|p|l)$ in the form

$$r_\pm(m|p|l) = i(2\pi)^{-3/2}\int e^{ip_\pm x}\left\langle m\left|\frac{\delta j(0)}{\delta\varphi_{in}(x)}\right|l\right\rangle d^4x.$$

Comparing this expression with (3.8) and (3.9) and omitting terms which give no contribution on the mass shell, we find

$$\delta j(0)/\delta\varphi_{in}(x) = -i\theta(-x_0)[j(0), j(x)]_- + \delta(x)\lambda,$$

which immediately leads to (2.18).

Important remark. Equation (3.3) was not covariant, neither formally nor implicitly. Solution (3.7) consists of a sum of two invariant terms: $R_\pm(m|p|l)$ and $K(m|l)$. Indeed, the Lorentz invariance of $R_\pm$ follows from the local commutativity (2.11) of the operators $j(x)$ and Axiom 3, and the invariance of $K(m|l)$ from the invariance properties of $r_\pm$ and $R_\pm$.

The next step uses in an essential way the invariance properties of each term of (3.7).

4. INTEGRO-DIFFERENTIAL EQUATIONS

The solution (3.7) cannot satisfy us: it contains the unknown, generally divergent function $K(m|l)$. Using the invariance properties of the amplitudes entering in (3.7), we may go over to differential equations and eliminate $K(m|l)$. We do this first for the graphs with three, four, and five external lines,[8] and then generalize the results. These am-

[7] I.e., the operator $\Lambda(x)$ may contain terms $\sim\delta(x)$ and any finite derivatives of $\delta(x)$.

[8] As already noted in Sec. 3, the two-line graph, or the single-particle Green's function, does not enter in the system of equations for the r functions. In the S method, where all four-momenta are on the mass shell, the equations do not contain the three-line graphs, either.

plitudes have been selected because of the dimensionality four of pseudo-Euclidian space: the number of independent invariants of three-, four-, and five-line graphs coincides with the number of scalar products of independent four-momenta.[9)] The invariance of the $r_\pm$ functions does not forbid them to depend on the sign of the zeroth components of the four-vectors whose squares were chosen as independent variables. Since, in this section, we consider only such variations of the invariants in which the sign of their zeroth components is unaltered, we may disregard this dependence.

Unless noted otherwise, we shall in the following only talk about the r_+ functions [according to (3.6), $r_\pm$ differ by the formal replacement $E(p) \to -E(p)$]. It is therefore convenient to introduce a simpler notation:

$$r(1, 2, \ldots k \,|\, k+1, \ldots m) \equiv \langle \mathbf{p}_1, \ldots \mathbf{p}_k \,|\, j(0) \,|\, \mathbf{p}_{k+1}, \ldots \mathbf{p}_m \rangle, \quad (4.1)$$

$$R(1, 2, \ldots \tilde{i}, \ldots k \,|\, k+1, \ldots m) \equiv i(2\pi)^{-3/2} \int e^{ip_ix} \langle \mathbf{p}_1, \ldots 0_i, \ldots \mathbf{p}_k \,|\, \theta(-x_0)[j(x), j(0)]_- \,|\, \mathbf{p}_{k+1}, \ldots \mathbf{p}_m \rangle d^4x, \quad (4.2)$$

where 0_i denotes the absence of a particle with momentum $\mathbf{p}_i$.

It is clear that, if we expand this expression in a complete set of <u>in</u> states and go over everywhere to multiple commutators of the type (4.4), the R function is expressed as a sum of bilinear combinations of r functions.

In order to exclude from the discussion the so-called unconnected graphs, we shall assume that

$$\mathbf{p}_i \neq \mathbf{p}_j, \qquad i = 1, \ldots k; \quad j = k+1, \ldots m. \quad (4.3)$$

Then the r function is given by the multiple commutator:[10)]

$$r(1, \ldots k \,|\, k+1, \ldots m) = (-1)^{m-k} \langle 0 \,|\, [a_{in}^{(-)}(\mathbf{p}_1)\,[\cdots[a_{in}^{(-)}(\mathbf{p}_k)\,[a_{in}^{(+)}(\mathbf{p}_{k+1})\,[\ldots[a_{in}^{(+)}(\mathbf{p}_m), j(0)]\ldots]]\ldots] \,|\, 0\rangle. \quad (4.4)$$

9)If n is the number of external lines, then the number of independent four-momenta is equal to m = n − 1; the number of scalar products is equal to (n − 1)(n − 2)/2, and the number of independent invariants is 3n − 9, n > 4. For n = 4,5 we have (n − 1)(n − 2)/2 = 3n − 9. For the three-line graph this relation is obvious.

10)It is easy to show that, by changing the matrix elements of the current into multiple commutators of the type (4.4) in the equations for the r functions, we can lift the restriction (4.3) in the final expression and consider the momenta $\mathbf{p}_i$ and $\mathbf{p}_j$ as arbitrary.

<u>A. Three-line graph.</u> We have in general three different r functions:

$$r(1|2), \quad r(12|0), \quad r(0|12).$$

It is seen from the definition (4.4) and the solution (3.7) that

$$r(0|12) = r(12|0)^*$$

and that the equation for $r(1|2)$ is obtained from the equation for $r(0|12)$ by changing the sign of the four-momentum p_1. Hence, we have essentially one equation for the different regions of values of the invariant $(p_1+p_2)^2 \equiv s$.[11)]

The solution (3.7) for $r(0|12)$ can be expressed through two different R functions:

$$r(0\,|\,12) \equiv \langle 0 \,|\, j(0) \,|\, \mathbf{p}_1, \mathbf{p}_2\rangle = \begin{cases} R(0\,|\,\tilde{1}2) + K(0\,|\,2) & (4.5a) \\ R(0\,|\,1\tilde{2}) + K(0\,|\,1) & (4.5b) \end{cases}.$$

The function $r(0|12)$ depends only on the single invariant

$$s = (p_1 + p_2)^2 \equiv (E_1 + E_2)^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2;$$

$K(0|2)$ and $K(1|0)$ are constants, since $p_1^2 = p_2^2 = m^2$.

Differentiating (4.5a) and (4.5b) with respect to s, we find the desired equation for the three-line graph

$$\partial r(0\,|\,12)/\partial s = \partial R(0\,|\,\tilde{1}2)/\partial s \quad (4.6a)$$

and the supplementary condition

$$\partial R(0\,|\,\tilde{1}2)/\partial s = \partial R(0\,|\,1\tilde{2})/\partial s. \quad (4.6b)$$

<u>B. Four-line graph.</u> In this case, for example, the r function

$$r(12|3) \equiv \langle \mathbf{p}_1, \mathbf{p}_2 | j(0) | \mathbf{p}_3 \rangle$$

depends on three independent invariants, which may be chosen as

$$s = (p_1 + p_2)^2, \quad u = (p_2 - p_3)^2, \quad t = (p_1 - p_3)^2. \quad (4.7)$$

If the momentum $p_4 = p_1 + p_2 - p_3$ lies on the mass shell, i.e., $p_4^2 = m^2$, then s, u, and t are related, as usual, by $s + u + t = 4m^2$.

The solution (3.7) can be written in terms of three different R functions:

$$r(12\,|\,3) = R(\tilde{1}2\,|\,3) + K(2\,|\,3), \quad (4.8a)$$

$$r(12\,|\,3) = R(1\tilde{2}\,|\,3) + K(1\,|\,3), \quad (4.8b)$$

$$r(12\,|\,3) = R(12\,|\,\tilde{3}) + K(12\,|\,0). \quad (4.8c)$$

11)In this paper we do not touch upon the problem of the analytic properties of the r functions when all momenta except one are on the mass shell. These properties must follow from the equations.

Here K(2|3), K(1|3), and K(12|0) depend only on u, t, and s, respectively. Therefore, by differentiating (4.8a), (4.8b), and (4.8c), we obtain three equations

$$\partial r(12|3)/\partial s = \partial R(\widetilde{1}2|3)/\partial s, \qquad (4.9a)$$

$$\partial r(12|3)/\partial u = \partial R(1\widetilde{2}|3)/\partial u, \qquad (4.9b)$$

$$\partial r(12|3)/\partial t = \partial R(12|\widetilde{3})/\partial t \qquad (4.9c)$$

and three supplementary conditions

$$\partial R(\widetilde{1}2|3)/\partial s = \partial R(\widetilde{1}2|3)/\partial s, \qquad (4.10a)$$

$$\partial R(1\widetilde{2}|3)/\partial u = \partial R(12|\widetilde{3})/\partial u, \qquad (4.10b)$$

$$\partial R(\widetilde{1}2|3)/\partial t = \partial R(12|\widetilde{3})/\partial t. \qquad (4.10c)$$

These equations are valid for $s \geq 4m^2$, $u \leq 0$, and $t \leq 0$. In order to obtain equations for the other regions of values of the invariants, we must write equations for r(123|0), r(1|23), etc.

C. Five-line graph. The function

$$r(12|34) \equiv \langle \mathbf{p}_1, \mathbf{p}_2 | j(0) | \mathbf{p}_3, \mathbf{p}_4 \rangle \qquad (4.11)$$

depends on six independent invariants

$$s_{12} = (p_1 + p_2)^2, \quad s_{34} = (p_3 + p_4)^2,$$
$$s_{ij} = (p_i - p_j)^2, \quad i = 1, 2; \quad j = 3, 4.$$

Using (3.7), the function r(12|34) can be written in terms of four different R functions:

$$r(12|34) = R(\widetilde{1}2|34) + K(2|34), \qquad (4.12a)$$

$$r(12|34) = R(1\widetilde{2}|34) + K(1|34), \qquad (4.12b)$$

$$r(12|34) = R(12|\widetilde{3}4) + K(12|4), \qquad (4.12c)$$

$$r(12|34) = R(12|3\widetilde{4}) + K(12|3). \qquad (4.12d)$$

By formal differentiation of each of these equations with respect to the invariants on which the corresponding K term does not depend, we obtain six equations

$$\partial r(12|34)/\partial s_{ij} = \partial R_i/\partial s_{ij},$$

$$i = 1, \quad j = 2, 3, 4; \quad i = 2, \quad j = 3, 4; \quad i = 3, \quad j = 4 \qquad (4.13)$$

and six supplementary conditions

$$\partial R_i/\partial s_{ij} = \partial R_j/\partial s_{ij}, \quad i, j = 1, 2, 3, 4, \quad i \neq j, \qquad (4.14)$$

where, for example, $R_1 \equiv R(\widetilde{1}2|34)$, etc.

D. Six-line graph. Now the complications begin. Let us write the system (3.7) for the six-line graph:

$$r(12|345) = R_i + K_i, \quad i = 1, 2, 3, 4, 5, \qquad (4.15)$$

where $R_1 \equiv (\widetilde{1}2|345)$, etc., $K_1 \equiv K(2|345)$, etc.

The number of invariant scalar products is equal to ten. As independent variables we may choose any nine; for example, all except $s_{45} = (p_4 + p_5)^2$. The first three K terms of (4.15) depend on this invariant, the last two K terms do not. The same situation holds (because of symmetry) for any other choice of nine independent invariants.

In order to eliminate the K terms from (4.15), we proceed in the following way.[12] Let us differentiate the first three equations (4.15) with respect to s_{ij}, where i = 1, 2, 3; the values of j depend on the values of i and are written down below. We have

$$\frac{\partial r}{\partial s_{ij}} = \frac{\partial R_i}{\partial s_{ij}} + \frac{\partial K_i}{\partial s_{45}} \frac{\partial s_{45}}{\partial s_{ij}}, \qquad i = 1, \quad j = 2, 3, 4, 5;$$

$$i = 2, \quad j = 1, 3, 4, 5; \quad i = 3, \quad j = 1, 2, 4, 5. \quad (4.16a)$$

Six more equations are obtained from the last two equations (i = 4, 5) in (4.15):

$$\frac{\partial r}{\partial s_{ij}} = \frac{\partial R_i}{\partial s_{ij}}, \qquad j = 1, 2, 3. \qquad (4.16b)$$

Excluding from these equations $\partial K_i/\partial s_{45}$ (i = 1, 2, 3), we find finally nine equations:

$$\frac{\partial r}{\partial s_{ij}} = \frac{\partial R_i}{\partial s_{ij}} + \left(\frac{\partial R_5}{\partial s_{i5}} - \frac{\partial R_i}{\partial s_{i5}}\right) \frac{\partial s_{45}}{\partial s_{ij}} \left(\frac{\partial s_{45}}{\partial s_{i5}}\right)^{-1},$$

$$i = 1, \quad j = 2, 3, 4, 5; \quad i = 2, \quad j = 3, 4, 5; \quad i = 3, \quad j = 4, 5; \qquad (4.17)$$

and six supplementary conditions

$$\left(\frac{\partial R_5}{\partial s_{i5}} - \frac{\partial R_i}{\partial s_{i5}}\right) \frac{\partial s_{45}}{\partial s_{i5}} = \left(\frac{\partial R_4}{\partial s_{i4}} - \frac{\partial R_i}{\partial s_{i4}}\right) \frac{\partial s_{45}}{\partial s_{i4}}, \qquad i = 1, 2, 3;$$

$$\frac{\partial R_i}{\partial s_{ij}} + \left(\frac{\partial R_5}{\partial s_{i5}} - \frac{\partial R_i}{\partial s_{i5}}\right) \frac{\partial s_{45}}{\partial s_{ij}} \left(\frac{\partial s_{45}}{\partial s_{i5}}\right)^{-1}$$

$$= \frac{\partial R_i}{\partial s_{ij}} + \left(\frac{\partial R_5}{\partial s_{j5}} - \frac{\partial R_j}{\partial s_{j5}}\right) \frac{\partial s_{45}}{\partial s_{ij}} \left(\frac{\partial s_{45}}{\partial s_{j5}}\right)^{-1},$$

$$i = 1, \quad j = 2, 3; \quad i = 2, \quad j = 3. \qquad (4.18)$$

E. n-line graph. The generalization to an arbitrary r function is now easy. The number of independent invariants is N = 3n − 9, where n = m + 1 is the number of external lines. The easiest way to select these invariants is the following. Let us denote the vectors not connected through the conservation law by p_i (i = 1, ..., n − 1). We take any three of these, say p_1, p_2, and p_3, and span with $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ a three-dimensional basis. We construct the invariants (p_1p_2), (p_1p_3), and (p_2p_3) which depend only on p_1, p_2, and p_3. Now it is clear that with each new vector three more invariants (p_1p_i), (p_2p_i), and (p_3p_i) are added. It is easy to check

[12]The author thanks R. É. Kallosh for proposing this method of eliminating the K terms.

that the number of these invariants is precisely equal to $N = 3 + 3(n-4) = 3n - 9$ and that they are independent. After this one must apply the procedure described on the example of the six-line graph. As a result we obtain N equations and six supplementary conditions.

The general form of the equations is

$$\partial r(1, 2, \ldots, k | k+1, \ldots, m) / \partial s_j = M_j(s_1, \ldots, s_N), \tag{4.19}$$

where $M_j(s_1, \ldots, s_N)$ are certain linear combinations of the first derivatives of the corresponding R functions, and s_j, $j = 1, 2, \ldots, N$ are independent invariants.

5. DISCUSSION

A. Evidently, the supplementary conditions are a consequence of the exact equations for the r functions [just like a number of other conditions, for example, the unitarity condition (2.18)]. The supplementary conditions are automatically fulfilled in the solution of the equations by perturbation theory (cf. [6], Appendix B). The importance of these conditions shows up in attempts to find approximate equations outside the framework of perturbation theory. In particular, the relativistically invariant cut-off of the number of particles in the intermediate state presents no problem. But this involves in general a violation of the supplementary conditions. This in turn leads to a violation of microcausality and the corresponding analytic properties of the r functions. We obtain approximate, relativistically invariant but, in a certain sense, "nonlocal" equations.

B. It is clear that, if the system of equations (4.19) has a solution, then it determines each r function up to an arbitrary constant λ_n. The number of independent constants λ_n entering in the solution is exactly equal to the number of r functions which do not vanish for any choice of limit at infinity in the independent invariants $|s_j|$.

In our model of neutral scalar (or pseudoscalar) particles a finite renormalized solution may be obtained by perturbation theory if we assume that all r functions, except the ones corresponding to three- and four-line graphs, vanish at infinity. The expansion of the solution with respect to the corresponding constants λ_3 and λ_4 gives the renormalized perturbation series corresponding to the interaction Lagrangian $L(x) \sim \lambda_3\varphi^3 + \frac{1}{4}\lambda_4\varphi^4$ with the essential difference that here the Feynman graphs are defined only for all external four-momenta except one on the mass shell.

If at least one amplitude with $n > 4$ is different from zero at infinity, then the iteration solution leads to the appearance of divergent expressions which are nonrenormalizable from the Lagrangian point of view. Whether there exists a finite solution outside perturbation theory for this case is not yet clear. In any case, it will evidently be a nonanalytic function of the corresponding constant λ_n for $\lambda_n \to 0$.

If the solution is such that at least one r function with $n \geq 4$ is different from zero at infinity, then the corresponding renormalization constant,

$$Z^{-1} = \int_{m^2}^{\infty} \rho(\varkappa^2)\, d\varkappa^2,$$

[where $\rho(\kappa^2)$ is the Källén-Lehmann spectral function [9]] is divergent.

C. As is known, [1-5] the S matrix elements are in the axiomatic method expressed in terms of the vacuum expectation values of T products of the field operators $\varphi(x)$ or the current $j(x)$. In order to obtain equations for such matrix elements or τ functions, one must from the very beginning consider mixed matrix elements $\langle m^{\underline{out}} | j(0) | l^{\underline{in}} \rangle$ instead of (3.2). The equations for the τ functions have the same appearance as those for the r functions. The difference consists in the fact that the R functions on the right-hand side of (3.7) are everywhere replaced by the T functions [cf. (4.2)]:

$$T(\tilde{1}, 2, \ldots, k \,|\, k+1, \ldots, m) = (-i)(2\pi)^{-3/2} \int e^{ip_1x_1} d^4x_1$$
$$\times \langle \mathbf{p}_2, \ldots, \mathbf{p}_k | Tj(x_1) j(0) | \mathbf{p}_{k+1}, \ldots, \mathbf{p}_m \rangle.$$

It is clear that, by expanding in a complete set (of in or out states), the T function will be expressed in terms of τ and r functions. Hence the set of equations for the τ functions is not closed. At the first stage of the investigation it is therefore simpler to deal with the equations for the r functions. Moreover, the analytic properties of the r functions have been studied more deeply. In momentum space, the r functions in which the number of ingoing and outgoing lines is not higher than two agree on the mass shell with the corresponding τ functions.

D. The transition to the integro-differential form of the basic equations (4.19) allows us, first, to exclude from the equations the indeterminate (and in general, divergent) subtraction K terms [cf. (3.7)]; second, to formulate the boundary conditions outside the framework of perturbation theory and to determine the number of independent constants entering in the theory; and finally, to perform the integration along an arbitrary path in the space of invariants, which is of particular importance for those amplitudes where the boundary conditions include the point at infinity. We emphasize that the transition from (3.7) to the integro-differential equations was in an essential manner based on the invariance properties of the amplitudes.

E. In the LSZ method[2] the matrix element r_+ [(3.2)] is transformed to the form

$$r_+(m|p|l)=i(2\pi)^{-3/2}\int e^{ip_+x}d^4xK_x\langle m|\theta(-x_0)\times[\varphi(x),\ j(0)]_-|l\rangle. \quad (5.1)$$

In order to compare this expression with (3.7) we must separate in (5.1) the quasi-local terms which have a polynomial dependence on p_+. Performing the differentiation, we find

$$r_+(m|p|l)=i(2\pi)^{-3/2}\int e^{ip_+x}\langle m|\theta(-x_0)[j(x),\ j(0)]_- - \delta(x_0-y_0)[\dot{\varphi}(x),\ j(0)]_- + iE(\mathbf{p})\delta(x_0-y_0)[\varphi(x),\ j(0)]_-|l\rangle\, d^4x. \quad (5.2)$$

If, in addition to the basic axioms of the LSZ method, we set

$$[\varphi(\mathbf{x}),\ j(0)]_-=0,$$

then (5.2) agrees with (3.7) with the condition

$$K(m|l)=-i(2\pi)^{-3/2}\int e^{ip_+x}d^4x\langle m|\delta(x_0-y_0)[\dot{\varphi}(x),j(0)]_-|l\rangle.$$

By covariance considerations, this term is independent of p_+. To eliminate it, we must therefore proceed in the same fashion as described in Sec. 4.

In the case of scalar (or pseudoscalar and spinor) particles we can thus obtain the desired equations in the LSZ method only if we postulate the additional condition (2.4). It is therefore natural to adopt this condition as one of the fundamental axioms, the more so since, as shown in the present paper, we then do not have to include microcausality and unitarity among the basic axioms.

It is true that, if we start from the assumption that the fundamental field is a spinor field and all Bose particles are composites formed by these fundamental spinor particles, then we do not have to introduce additional assumptions besides the usual axioms of the LSZ method in order to obtain the desired equations. However, these equations have no finite solutions within perturbation theory. The problem of the existence of finite solutions outside perturbation theory remains open.

F. The causality condition is in the BMPSh method[4,5]

$$\delta j(x)\,/\,\delta\varphi_{in}(y)=-i\theta(x_0)\,[j(x),j(y)] + \text{arbitrary quasi-local terms}. \quad (5.3)$$

Hence we obtain solution (3.7) in the BMPSh method if we make the additional assumption that the quasi-local terms in (5.3) contain no derivatives of δ functions, i.e., if we restrict the strength of the singularity of the matrix elements. The subsequent procedure of eliminating the indeterminate quasi-local terms is the same as the one described in Sec. 4.

G. Thus the main difference with respect to other work on the axiomatic method consists, first, in the imposition of additional restrictions on the strength of the singularity of the quasi-local terms and second, in the consistent elimination of these terms from the equations. For an invariant formulation of the equations one must choose the coordinate system most adapted to each particular case. A number of examples in perturbation theory has been considered in the lectures of the author.[6] There the reader can also find a list of the problems which are being investigated at present.

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329