INSTABILITY OF A POSITIVE COLUMN IN A MAGNETIC FIELD AT LOW TEMPERATURES

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We consider low-pressure positive column instability which is not directly related to the current or to collisions with charged particles. Such an instability arises in an inhomogeneous plasma if ionization and transfer of perturbations in an ambipolar field are taken into account; in particular, build-up of a purely azimuthal wave ($m = 1$, $k_z = 0$), whose phase velocity is close to the ion outflow rate in the ambipolar field, may take place. The condition for generation of this instability is similar to that observed experimentally[6]. The instability mechanism may also be operative in a strongly ionized plasma[7].

It was shown in several papers[1-3] that drift waves, connected with dissipative processes, can build up in an inhomogeneous plasma in the presence of a magnetic field. In particular, it follows from the paper of Timofeev[2], who used a quasi-classical approximation, that in a weak magnetic field (where the ion Larmor radius is larger than the characteristic dimension of the inhomogeneity) an oblique wave ($k_z \neq 0$) can build up, with a phase velocity on the order of the velocity of ion sound: $v_{ph} \sim \sqrt{T_e/M}$. When the mean free path and the Larmor radius of the ions are large ($l_i \gtrsim a$, $r_{iH} \gtrsim a$, where a = radius of the tube), it becomes essential to take into account transport processes in the ambipolar field; these processes cause the perturbations to run off towards the ends of the column, and play in the absence of collisions the role of a unique dissipation mechanism. These processes were not taken into account in papers devoted to drift instabilities in the plasma.

In the present paper we consider in detail the occurrence of instability in a low-pressure positive column in a longitudinal magnetic field, under the assumption that the longitudinal current does not influence the stability of the discharge and can be neglected in the stability analysis. It is assumed that the ions are accelerated in an ambipolar field from the points where they are created (the ions are produced with zero velocity) towards the wall (without collision with neutrals), the wall being under negative potential relative to the plasma column. The analysis is based on the kinetic equations for the electrons and ions. The problem of the equilibrium state of such a system (the Langmuir problem) is solved with allowance for the magnetic field. It is shown that when account is taken of the ionization and transport of perturbations in the ambipolar field, a mode with $m = 1$ can grow in the wave-number interval $0 \leq k_z \lesssim 1/a\sqrt{\omega_{eH}\tau_e}$. The phase velocity of these waves coincides in order of magnitude with the velocity of outflow of the ions in the ambipolar field:

$$2\sqrt{e\varphi_0/M} \lesssim v_{ph} \lesssim 2.2\sqrt{e\varphi_0/M},$$

where φ_0—value of the ambipolar potential on the edge of the column. The criterion for the excitation is in satisfactory agreement with experiment [6]. The instability mechanism considered, which is not connected directly with the current and with the collisions of the charged particles, may come into play in a strongly ionized plasma[7].

1. INITIAL EQUATIONS

The initial kinetic equations for the electrons and ions are of the form

$$\frac{\partial f_e}{\partial t} + \mathbf{v}\nabla f_e + \frac{e}{m_e}\frac{\partial f_e}{\partial \mathbf{v}}\nabla\varphi - [\mathbf{v}\boldsymbol{\omega}_{eH}]\frac{\partial f_e}{\partial \mathbf{v}} = Z\frac{n_e}{n_0}f_{0e} - \frac{f_e}{\tau_e} + \frac{n_e}{n_0}\frac{f_{0e}}{\tau_e}, \quad (1.1)$$

$$\frac{\partial f_i}{\partial t} + \mathrm{div}_r\mathbf{v}f_i - \frac{e}{M}\frac{\partial f_i}{\partial \mathbf{v}}\nabla\varphi = Zn_e\delta(\mathbf{v}). \quad (1.2)$$

Here $n_e = n_{e0} + n_e'$, where the zero subscript denotes the unperturbed quantity and the prime denotes its perturbation;

$$\omega_{eH} = e\mathbf{H}/m_e c,$$

Z—number of ionizations produced per unit time by a single electron, and τ_e—time between the collisions between the electrons and the neutrals. The ions are produced with zero velocity when neutral atoms are ionized by electrons, and this is taken into account by the δ-function.

We introduce the notation

$$F_e = \int f_e dv_\perp; \quad \mathbf{K}_e = \int \mathbf{v}_\perp f_e dv_\perp; \quad F_i = \int f_i dv_\varphi dv_z.$$

Using (1.1), we obtain

$$\frac{\partial F_e}{\partial t} + v_z \frac{\partial F_e}{\partial z} + \operatorname{div}_\perp \mathbf{K}_e + \frac{e}{m_e}\frac{\partial \varphi}{\partial z}\frac{\partial F_e}{\partial v_z} = \frac{Zn_e}{n_0} F_{e0} - \frac{F_e}{\tau_e} + \frac{n_e}{n_0}\frac{F_{e0}}{\tau_e}, \qquad (1.3)$$

$$\frac{\partial \mathbf{K}_e}{\partial t} + v_z \frac{\partial \mathbf{K}_e}{\partial z} + \frac{e}{m_e}\frac{\partial \varphi}{\partial z}\frac{\partial \mathbf{K}_e}{\partial v_z} + \frac{T_e}{m_e}\nabla_\perp F_e - \frac{e}{m_e} F_e \nabla_\perp \varphi + [\mathbf{K}_{e0}\omega_{eH}] = -\frac{\mathbf{K}_e}{\tau_e}. \qquad (1.4)^*$$

2. EQUILIBRIUM

In equilibrium, the spatial distribution of the quantities is determined by the distance from the axis. Recognizing that

$$f_{i0} = f_{i0}(v_r)\delta(v_\varphi)\delta(v_z),$$

we obtain for the initial system of equilibrium equations:

$$\operatorname{div} \mathbf{K}_{e0} = ZF_{e0},$$

$$\frac{T_e}{m_e}\nabla_\perp F_{e0} - \frac{e}{m_e} F_{e0}\nabla_\perp \varphi = -\frac{1}{\tau_e}\mathbf{K}_{e0} - [\mathbf{K}_{e0}\omega_{eH}], \qquad (2.1)$$

$$\frac{v_r}{r}\frac{\partial r F_{i0}}{\partial r} - \frac{e}{M}\frac{d\varphi}{dr}\frac{\partial F_{i0}}{\partial v_r} = Zn_{e0}\delta(v_r). \qquad (2.2)$$

We assume that the plasma is quasi-neutral ($n_i = n_e = n$). Then we obtain from (2.1), after integrating with respect to dv_z,

$$\frac{1}{r}\frac{d}{dr} r\left(\frac{e}{m_e}\frac{d\varphi}{dr} n_0 - \frac{T_e}{m_e}\frac{dn_0}{dr}\right) = Z\frac{1 + (\omega_{eH}\tau_e)^2}{\tau_e} n_0. \qquad (2.3)$$

We shall assume henceforth that $(\omega_{eH}\tau_e)^2 \gg 1$, and leave off the zero subscript in the designation of the equilibrium quantities.

The equations for the characteristics of (2.2) are of the form

$$\frac{rdr}{v_r} = -r\, dv_r \Big/ \frac{e}{M}\frac{d\varphi}{dr} = \frac{drF_i}{Zn\delta(v_r)}, \qquad (2.4)$$

from which it follows that

$$\frac{Mv_r^2}{2} + e\varphi(r) = e\varphi(r'); \quad \frac{drF_i}{dr} = Z\frac{rn\delta(v_r)}{v_r}, \qquad (2.5)$$

where r' corresponds to the points of particle production,

$$rF_i = Z\int_{r'}^{r} r\frac{n\delta(v_r)}{v_r} dr = -\frac{M}{e} Z\frac{n(r')r'}{d\varphi/dr'}. \qquad (2.6)$$

*$[\mathbf{k}_e\omega_{eH}] = \mathbf{k}_e \times \omega_{eH}$.

For the ion density we get

$$n(r) = \int_0^{\sqrt{-2e\varphi(r)/M}} F_i dv_r = \frac{Z}{r}\int_0^r r' n(r')\left\{\frac{2e}{M}[\varphi(r') - \varphi(r)]\right\}^{-1/2} dr'. \qquad (2.7)$$

Equations (2.3) and (2.7) determine the spatial distribution of the density and potential under equilibrium.

Let us solve (2.3) and (2.7). To this end we reduce them to dimensionless form, introducing the quantities

$$e\varphi / T_e = -\Phi; \quad \alpha r = s; \quad \alpha = Z / \sqrt{2T_e/M}.$$

Equations (2.3) and (2.7) take in the new variables the form

$$\frac{1}{s}\frac{d}{ds} s\left(n\frac{d\Phi}{ds} + \frac{dn}{ds}\right) = -\beta n \qquad \beta = \frac{2m_e}{M}\frac{\tau_e\omega_{eH}^2}{Z};$$

$$n(s) = \frac{1}{s}\int_0^s \frac{s'n(s')ds'}{\sqrt{\Phi(s) - \Phi(s')}}. \qquad (2.8)$$

In the absence of a magnetic field ($\beta = 0$), this problem was solved by Langmuir and Tonks[4]. In this case $n = n_0 e^{-\Phi}$ and the system (2.8) reduces to a single equation. A solution of this nonlinear singular integral equation was obtained by Langmuir in the form of a power series

$$s = \sqrt{\Phi}(1 - 0.2\Phi - 0.026\Phi^2 - 0.0065\Phi^3 - 0.002\Phi^4 - \ldots).$$

Harrison and Tompson[5] obtained for the plane case in the absence of a magnetic field analytic solutions in terms of tabulated functions.

In the case of cylindrical symmetry and $\beta \neq 0$, it is difficult to obtain analytic solutions of the system (2.8), and we shall use the Langmuir procedure, seeking the solutions of (2.8) in the form of the following series:

$$s = a_1\rho + a_3\rho^3 + a_5\rho^5 + \ldots,$$

$$s' = a_1\rho_z + a_3\rho_z^3 + a_5\rho_z^5 + \ldots;$$

$$n(s) = b_0 + b_1\rho^2 + b_2\rho^4 + b_3\rho^6 + \ldots,$$

$$n(s') = b_0 + b_1\rho_z^2 + b_2\rho_z^4 + b_3\rho_z^6 + \ldots, \qquad (2.9)$$

where we introduce new variables

$$\rho = \sqrt{\Phi(s)}, \quad \rho_z = \rho\sin\theta = \sqrt{\Phi(s')}.$$

We rewrite the system (2.8), by changing over to the variable θ, in a form convenient for the integration by means of the series of (2.9):

$$n(s) = \frac{1}{s}\int_0^{\pi/2} n(\theta)s'(\theta)\frac{ds'}{d\rho_z} d\theta,$$

$$s\left(n\frac{d\Phi}{d\rho} + \frac{dn}{d\rho}\right) = -\beta\frac{ds}{d\rho}\rho\int_0^{\pi/2} \cos\theta\, n(\theta)s'(\theta)\frac{ds'}{d\rho_z} d\theta. \qquad (2.10)$$

Substituting the series (2.9) in (2.10), we easily obtain the coefficients a and b.

We present the first few coefficients:

$$a_1 = 1; \quad a_3 = -0.2(1+0.25\beta);$$

$$a_5 = -(1+0.25\beta)(0.026-0.0014\beta);$$

$$a_7 = -(1+0.25\beta)(0.0065-0.0024\beta-0.00075\beta^2);$$

$$a_9 = -(1+0.25\beta)(0.002-0.0015\beta-0.00064\beta^2 - 0.000049\beta^3);$$

$$b_1 = -b_0(1+0.25\beta); \quad b_2 = {}^1/_2 b_0(1+0.25\beta)(1+0.325\beta);$$

$$b_3 = -{}^1/_6 b_0(1+0.25\beta)(1+0.62\beta+0.1\beta^2);$$

$$b_4 = {}^1/_{24} b_0(1+0.25\beta)(1+0.96\beta+0.29\beta^2+0.03\beta^3).$$

When $\beta = 0$ all the coefficients above coincide with the Langmuir coefficients. We did not determine the higher-order coefficients, since we are interested in the relatively narrow region $0 < \beta \leq 6$. The point s_0, at which the derivative of the potential $d\Phi/ds$ becomes infinite, is identified[4] with the plasma limit beyond which the quasineutral analysis is not valid. In this sense, if $d\Phi/ds$ does not become infinite at any of the points s, then the plasma limit will be determined by the point at which the second derivative of the potential $d^2\Phi/ds^2$ becomes infinite. We encounter such a situation in the presence of a magnetic field, where, $d^2\Phi/ds^2$ becomes infinite starting with $\beta \sim 4$, and the corresponding values of s_0 and Φ_0 (the potential at the point s_0) provide a smooth continuation of the values of s_0 and Φ_0 corresponding to infinite $d\Phi/ds$. All the foregoing applies to the calculation of s_0 and Φ_0 by means of the coefficients a_1, a_3, a_5, a_7, and a_9.

The results of the calculations are listed in the table, from which we see that Φ_0 and s_0 decrease with increasing β, or, what is the same, with increasing magnetic field, and the ratio $s_0/\sqrt{\Phi_0}$ remains approximately constant, oscillating within narrow limits. This can be easily understood, recognizing that with increasing magnetic field a decrease takes place in the flux of charged particles to the wall. In the stationary state, the number of particles produced upon ionization should be compensated by the flow-off to the wall, and therefore

$$s_0 / \sqrt{\Phi_0} \approx \text{const.}$$

	β							
	0	0,5	1	2	3	4	5	6
Φ_0	1,15	1,11	1.06	1	0.95	0,89	0.80	0,65
s_0	0.78	0.74	0.72	0,67	0,64	0.61	0.58	0.56
$s_0/\sqrt{\Phi_0}$	0.73	0,71	0.7	0,67	0.66	0,65	0.65	0,68

Simple qualitative estimates enable us to determine the asymptotic behavior of the potential and the density for large values of β. To this end we write down (2.7) in the form

$$n(r) = \frac{Z}{r\sqrt{2T_e/M}} \int_0^r \frac{r' n(r')\,dr'}{\sqrt{\Phi(r) - \Phi(r')}}. \tag{2.11}$$

As can be seen from the table, s_0 decreases with increasing magnetic field and for a very strong field ($\beta \gg 1$) we should have

$$Za / \sqrt{2T_e/M} \sim s_0 \ll 1.$$

In the main part of the tube, the potential Φ ($\Phi \sim s_0^2$) should therefore be small. Consequently, in the first equation of (2.8) we can neglect the term with the potential, and the density distribution in the main part of the tube will be determined by a Bessel function:

$$n = n_0 J_0(\sqrt{\beta} s).$$

In a layer of thickness x adjacent to the wall (x is measured from the wall), where the electron mobility cannot be neglected, we have

$$\Phi \sim 1, \quad n/n_0 \sim x/a \sim s_0.$$

Thus, for large values of β the main change in the potential occurs in a layer next to the wall, of thickness $x \sim s_0 a$ ($s_0 \ll 1$).

3. STABILITY

In the analysis of the stability, we start by linearizing (1.1) and (1.2) with respect to the perturbations. In the case when $\omega_{eH}\tau_e \gg 1$, we can use for the electrons the drift approximation

$$\mathbf{v}_\perp = c[\mathbf{EH}]/H^2.$$

The perturbation of each quantity A is defined as

$$A' = A'(r)\exp\{-i\omega t + im\varphi + ik_z z\}.$$

In this approximation we obtain for the perturbation of the electron density

$$n_e' = \frac{i\gamma}{1-\gamma\Phi^*/\tau_e}\left\{\frac{mc}{Hr}\frac{dn}{dr}\Phi^* + \frac{k_z e}{T_e} n(r)\Phi_1\right\}\varphi'(r), \tag{3.1}$$

where

$$\Phi^* = \frac{\pi}{k_z} W(\xi); \quad \Phi_1 = -\frac{i}{k_z}\frac{1}{\gamma}[1 + i\sqrt{\pi}\,\xi W(\xi)],$$

$$W(\xi) = e^{-\xi^2}\left(1 + \frac{2i}{\sqrt{\pi}}\int_0^\xi e^{t^2} dt\right);$$

$$\xi = \gamma\frac{\sqrt{\pi}}{k_z}\left(\omega + \frac{i}{\tau_e} - \frac{mc}{Hr}\frac{d\varphi}{dr} - iZ\right), \quad \gamma = \left(\frac{m_e}{2\pi T_e}\right)^{1/2}.$$

We put

$$Y = i\int \left(\frac{m}{r} v_\varphi + k_z v_z\right) f_i' dv_\varphi dv_z.$$

From (1.2) we easily obtain a system of two equations relative to F_i' and Y:

$$\frac{v_r}{r}\frac{\partial r F_i'}{\partial r} - \frac{e}{M}\frac{d\varphi}{dr}\frac{\partial F_i'}{\partial v_r} = i\omega F_i' - Y + \frac{e}{M}\frac{\partial \varphi'}{\partial r}\frac{\partial F_i}{\partial v_r} + Z n_e'\delta(v_r);$$

$$\frac{v_r}{r}\frac{\partial r Y}{\partial r} - \frac{e}{M}\frac{d\varphi}{dr}\frac{\partial Y}{\partial v_r} = i\omega Y + \frac{e}{M} F_i\left(\frac{m^2}{r^2} + k_z^2\right)\varphi'(r). \quad (3.2)$$

The equations for the characteristics of the system (3.2) are of the form

$$\frac{r dr}{v_r} = -r dv_r \Big/ \frac{e}{M}\frac{d\varphi}{dr} = dr F_i' \times \left[i\omega F_i' - Y + \frac{e}{M}\frac{\partial \varphi'}{\partial r}\frac{\partial F_i}{\partial v_r} + Z n_e'\delta(v_r)\right]^{-1};$$

$$\frac{r dr}{v_r} = -r dv_r \Big/ \frac{e}{M}\frac{d\varphi}{dr} = dr Y \times \left[i\omega Y + \frac{e}{M} F_i\left(\frac{m^2}{r^2} + k_z^2\right)\varphi'(r)\right]^{-1}. \quad (3.3)$$

Each of these systems has as one of its characteristics the integral of motion

$$M v_r^2 / 2 = e\varphi(r') - e\varphi(r),$$

where r′—points of particle production. With the aid of this integral and the second characteristic we can integrate each equation of (3.2). Let us integrate the first equation of the system (3.2). We write down the characteristics of this equation:

$$v_r(r, r')\frac{dr F_i'}{dr} = r\left[i\omega F_i' - Y + \frac{e}{M}\frac{\partial \varphi'}{\partial r}\frac{\partial F_i}{\partial v_r} + Z n_e'\delta(v_r)\right],$$
$$M v_r^2 / 2 = e\varphi(r') - e\varphi(r). \quad (3.4)$$

We further obtain

$$r F_i'(r, r') = \int_{r'}^{r} \rho\left[i\omega F_i'(\rho, r') - Y(\rho, r') + \frac{e}{M}\frac{\partial \varphi'}{\partial \rho}\frac{\partial F_i}{\partial v_r(\rho, r')}(\rho, r') + Z n_e'\delta(v_r)\right]\frac{d\rho}{v_r(\rho, r')}, \quad (3.5)$$

$$n_i'(r) = \int_0^{\sqrt{-2e\varphi(r)/M}} F_i' dv_r = -\frac{1}{r}\frac{e}{M}\int_0^r \frac{d\varphi}{dr'}\frac{1}{v_r(r, r')} dr' \int_{r'}^{r} \frac{\rho}{v_r(\rho, r')} \times \left[i\omega F_i'(\rho, r') - Y(\rho, r') + \frac{e}{M}\frac{\partial \varphi'}{\partial \rho}\frac{\partial F_i}{\partial v_r(\rho, r')} + Z n_e'\delta(v_r)\right] d\rho. \quad (3.6)$$

By reversing the order of integration in (3.6), we break up this expression into two parts:

$$n_i'(r) = -\frac{1}{r}\frac{e}{M}\int_0^r \rho d\rho \int_0^\rho \frac{d\varphi}{dr'}\frac{i\omega F_i'(\rho, r') - Y(\rho, r')}{v_r(r, r') v_r(\rho, r')} dr' - \frac{1}{r}\frac{e}{M}\int_0^r \rho d\rho \int_0^\rho \frac{d\varphi}{dr'}\left[\frac{e}{M}\frac{\partial \varphi'}{\partial \rho}\frac{\partial F_i(\rho, r')}{\partial v_r(\rho, r')} + Z n_e'\delta(v_r)\right] \times \frac{dr'}{v_r(r, r') v_r(\rho, r')}. \quad (3.7)$$

The first of these will be integrated with respect to dr′ approximately, and the second exactly. In the first integral, recognizing that, of the two velocities $v_r(r, r')$ and $v_r(\rho, r')$, the main contribution to the value of the integral with respect to dr′ is made by $v_r(\rho, v')$, we take the velocity $v_r(r, r')$ outside the sign of integration with respect to dr′ at the point $r' = \rho$. The second equation of (3.2) is integrated in analogy with the first.

As a result we obtain approximately

$$n_i'(r) = \frac{1}{r}\int_0^r \rho \frac{i\omega n_i'(\rho) - Y^*(\rho) + Z n_e'(\rho)}{v_r(r, \rho)} d\rho - \frac{1}{r}\frac{e}{M}\int_0^r \rho \frac{d\varphi'}{d\rho} d\rho \int_0^\rho \frac{\partial F_i(\rho, r')}{\partial r'}\frac{dr'}{v_r(r, r')},$$

$$Y^*(r) = \int_0^{\sqrt{-2e\varphi(r)/M}} Y(r, v_r) dv_r. \quad (3.8)$$

Analogously, the integration of the second equation of (3.2) leads to the following equation for the function Y* (r):

$$Y^*(r) = \frac{1}{r}\int_0^r \rho\left[i\omega Y^*(\rho) + \frac{e}{M} n(\rho)\left(\frac{m^2}{\rho^2} + k_z^2\right)\varphi'(\rho)\right] \times \frac{d\rho}{v_r(r, \rho)}. \quad (3.9)$$

Equations (3.1), (3.8), and (3.9) are the starting points for the derivation of the dispersion equation.

For simplicity let us consider the case of a metallic wall, on which the perturbed potential vanishes. Inasmuch as n_e' is proportional to φ', the perturbation of the electron density also vanishes on the wall in this case. From the condition of quasi-neutrality on the wall ($n_e' = n_i' = 0$), we obtain the dispersion equation

$$\int_0 \rho \frac{(i\omega + Z) n'(\rho) - Y^*(\rho)}{v_r(a, \rho)} d\rho - \frac{e}{M}\int_0^a \rho \frac{d\varphi'}{d\rho} d\rho \int_0^\rho \frac{\partial F_i(\rho, r')}{\partial r'}\frac{dr'}{v_r(a, r')} = 0. \quad (3.10)$$

The distributions of the equilibrium densities and the potential are chosen in the form

$$\varphi = -Ar^2, \quad n = n_0(1 - Br^2).$$

Let us determine, for the chosen potential and velocity profiles, the derivative $\partial F_i(r, r')/\partial r'$, which is essential for the analysis that follows. The aggregate of velocities corresponding to the distribution function of the ions F_i at the point r, is determined by the set of points $r' \to 0 < r' < r$, inasmuch as the ions are accelerated towards the wall. In the (r, r') representation, on the basis of (2.6), the distribution function is

$$F_i(r, r') = \frac{M}{2eA}\frac{Z}{r}n_0(1 - Br'^2), \quad 0 < r' < r;$$
$$F_i(r, r') = 0, \qquad r' > r. \tag{3.11}$$

From this we get

$$\frac{\partial F_i}{\partial r'} = \frac{Zn_0M}{2eA}\frac{1}{r}[-2Br' - (1 - Br'^2)\delta(r' - r) + \delta(r')]. \tag{3.12}$$

Inasmuch as we have confined ourselves to examination of the mode $m = 1$, the perturbed electric radial field on the axis $(r = 0)$ should differ from zero and the radial dependence of the perturbed potential, taking into account the symmetry of the problem, is chosen in the form

$$\varphi'(r) = \varphi_1 r(1 - r^2/a^2).$$

With the aid of (3.12) we obtain from (3.10) by simple calculations

$$\frac{i(i\omega + Z)}{1 - \gamma\Phi^*/\tau_e}\frac{\pi}{16}a^2\gamma\left[\frac{k_z e}{T_e}\left(1 - \frac{Ba^2}{2}\right)\Phi_1 - \frac{2Bc}{H}\Phi^*\right]$$
$$-\frac{3\pi}{16}a^4B_1 - \frac{\pi}{4}a^2C_1 - \frac{\pi}{2}D_1 - \frac{Z\pi}{8A}\left(1 - \frac{3}{4}Ba^2\right) = 0, \tag{3.13}$$

$$B_1 = \frac{3\pi}{16}\frac{e}{M}\frac{B}{a^2}\frac{q}{1 - 3\pi i\omega q/16},$$

$$C_1 = -\frac{\pi}{4}\frac{e}{M}\left(B + \frac{1}{a^2}\right)\frac{q}{1 - \pi i\omega q/4},$$

$$D_1 = \frac{\pi}{2}\frac{e}{M}\frac{q}{1 - \pi i\omega q/2}, \quad q = \sqrt{\frac{M}{2eA}}, \tag{3.14}$$

where by virtue of the small longitudinal mobility of the ions (the ions are not magnetized) we have neglected the longitudinal perturbation of the potential in the ion equation (3.9).

Let us consider the case $k_z = 0$, when the perturbation $m = 1$ reduces to a shift of the pinch as a whole towards the wall of the tube. When $k_z \to 0$, $\xi \to \infty$ and

$$W(\xi) = \frac{i}{\sqrt{\pi}}\frac{1}{\xi}\left(1 + \frac{1}{2\xi^2} + \ldots\right),$$

$$\Phi^* \sim i\frac{\sqrt{\pi}}{k_z}\frac{1}{\xi}; \quad \Phi_1 \sim \frac{i}{2\gamma k_z\xi^2} \sim 0.$$

The dispersion equation (3.13) reduces to

$$\frac{\pi cBa^2(i\omega + Z)}{8H(\omega + 2Ac/H - iZ)} - \frac{e}{M}q\left\{\left(\frac{3\pi}{16}\right)^2\frac{Ba^2}{1 - 3\pi i\omega q/16}\right.$$
$$\left. - \left(\frac{\pi}{4}\right)^2\frac{1 + Ba^2}{1 - \pi i\omega q/4} + \left(\frac{\pi}{2}\right)^2\frac{1}{1 - \pi i\omega q/2}\right\}$$
$$-\frac{Z\pi}{8A}\left(1 - \frac{3}{4}Ba^2\right) = 0. \tag{3.15}$$

On the stability boundary $\text{Im}\,\omega = 0$, and, equating the real and imaginary parts of (3.15) to zero (with allowance for the fact that $2Ac/H \gg \omega$, and Z, a condition satisfied when the ion Larmor radius is much larger than the tube radius), we obtain the system of two equations

$$Ba^2 - \frac{0.53Ba^2}{1 + 0.35x} + \frac{1.24(1 + Ba^2)}{1 + 0.62x} - \frac{9.84}{1 + 2.46x} = 0,$$

$$\frac{\pi}{4}Zq\left(\frac{5}{4}Ba^2 - 1\right) - \frac{0.35Ba^2}{1 + 0.35x} + \frac{0.62(1 + Ba^2)}{1 + 0.62x}$$
$$-\frac{2.46}{1 + 2.46x} = 0, \tag{3.16}$$

where $x = (\omega q)^2$.

Recognizing that $Zq \sim s_0/\sqrt{\Phi_0} \sim 0.7$, we can easily show from (3.16) and (3.15) that when $Ba^2 > 0.9$, oscillations begin to build up with frequency

$$\omega \approx 2\sqrt{eA/M} = 2a^{-1}\sqrt{e\varphi_0/M},$$

and these oscillations can be called "ambipolar" sound, since they are generated only if transport in the ambipolar field is taken into account. Since $e\varphi_0 \sim T_e$, the frequency of these oscillations coincides in order of magnitude with the frequency of ion sound: $\omega \sim a^{-1}\sqrt{T_e/M}$.

If we neglect transport in the ambipolar field and let v_r in (3.9) and (3.10) approach zero, then we obtain Timofeev's dispersion equation[2]:

$$(i\omega + Z)n' - \frac{ie}{M\omega}n\frac{m^2}{r^2}\varphi' = 0, \tag{3.17}$$

where n' is defined by (3.1). An investigation of (3.17) shows that if account of electron diffusion, i.e., terms with $k_z \neq 0$ ($|\xi| \gg 1$), is taken in (3.1), the ion-sound oscillations with $\omega \approx -m\sqrt{T_e/M}/r$ can build up.

If $k_z = 0$, the role of diffusion is assumed by transport in the ambipolar field. In addition, ionization must be taken into account for the generation of the instability in question.

If $k_z \neq 0$ ($|\xi| \gg 1$) we have

$$\Phi^* \sim i\frac{\sqrt{\pi}}{k_z}\frac{1}{\xi}\left(1 + \frac{1}{2\xi^2}\ldots\right), \quad \Phi_1 \sim \frac{i}{2\gamma k_z\xi^2}.$$

and only the first term in the dispersion equation (3.15) will change and take the form ($2Ac/H \gg \omega$, Z)

$$\frac{\pi}{16}\frac{Ba^2}{A}(i\omega + Z)\left[1 + \frac{1}{2\xi^2} - \frac{H}{4Bc}\frac{e}{T_e}\left(1 - \frac{Ba^2}{2}\right)\right.$$
$$\left.\times\left(\frac{i}{\tau_e} + 2A\frac{c}{H}\right)\frac{1}{\xi^2}\right] \Big/ \left[1 - \frac{iH}{4Ac\tau_e\xi^2}\right].$$

When $A \approx T_e/ea^2$ and $(1 - Ba^2/2)/Ba^2 \approx 1$ (in fact, as follows from the solution of the Langmuir problem, the latter expression lies in the interval 0.5—1), the expression in the square brackets ≈ 1 and the stability criterion depends little on k_z. If we make no assumptions whatever concerning the relative value of the quantity

$$\frac{1}{Ba^2}\left(1 - \frac{Ba^2}{2}\right),$$

then in the wave-number interval

$$0 < k_z \lesssim 1 / a\sqrt{\omega_{eH}\tau_e}$$

the conditions for the excitation of oblique waves are somewhat less stringent than the condition for the excitation of a purely azimuthal wave. The critical value of Ba^2 at the instant of excitation, corresponding to $k_z \approx 1/a\sqrt{\omega_{eH}\tau_e}$, is equal to approximately 0.8, and the frequency at the instant of excitation is $\omega \approx 2.2\sqrt{eA/M}$.

Using the results of the solution of the Langmuir equation in a magnetic field, let us determine the critical value of the magnetic field and the ratio $Z(H = A_{cr})/Z(H = 0)$, using the critical values of Ba^2, obtained from the solution of the problem of the stability of the column. We calculate Ba^2 using the first three coefficients of the expansion of the density (2.9), b_1, b_2, and b_3. It can be thought that such a calculation would lead to a value of Ba^2 which is 10% or more too large (the accuracy of the calculation decreases with increasing β). The value of Ba^2 is exaggerated by 10% when $H = 0$ ($\beta = 0$). (In this case the density is determined from the analytic expression $n|_{r=a} = n_0 e^{-\Phi_0}$ [4] and the value of the overestimate can be determined.)

Under the assumption that when $\beta \neq 0$ the use of the coefficients b_1, b_2, and b_3 yields values of Ba^2 overestimated by 10—20%, we obtain approximately $Ba^2 = 0.8$ when $\beta = 1.5$—2 and $Ba^2 = 0.9$ when $\beta = 2$—3. With the aid of the table we get

$$Z(\beta = 2) / Z(\beta = 0) = 0.86;$$

$$Z(\beta = 3) / Z(\beta = 0) = 0.81.$$

The experiments of Nedospasov et al [6] show that at the instant of instability

$$Z(H = H_{cr}) / Z(H = 0) \approx 0.7.$$

Setting β_{cr} equal to 2 or 3, let us calculate the critical value of the magnetic field for a plasma in the installation of Nedospasov et al [6], where $T_e = 4$ eV, $M = 200$ atomic units (mercury), $l_e = 25$ cm, and $a = 1.5$ cm. The instability condition $\beta > \beta_{cr}$ can be written in the form

$$\omega_{eH}^2 \gtrsim (s_0\beta)_{cr}\frac{M^{1/2}T_e}{m_e^{3/2}l_e a}. \tag{3.18}$$

From this we obtain under the above conditions that $H_{cr} = 25$ Oe when $\beta = 2$ and $H_{cr} = 28$ Oe when $\beta = 3$. In the experiment [6] noticeable symptoms of plasma instability were observed at $H \approx 30$ Oe. The instability was accompanied by noise and by a clearly pronounced peak at 80 kcs. The calculated value of the frequency at the instant of the instability is $\omega \approx a^{-1}\sqrt{T_e/M} \approx 80$ kcs.

The instability mechanism considered can be manifest in a strongly ionized plasma, inasmuch as it is not connected directly with charged-particle collisions. Thus, in a strongly ionized collisionless plasma, in the installation of Buchel'nikova et al [7], under conditions when $r_{iH} \gtrsim a$, excitation of peaks with frequency $\omega \approx 70$ kcs was observed. The calculated value of the frequency is close to that observed in the experiment.

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