THE THEORY OF TRANSFER PHENOMENA IN METALS IN STRONG MAGNETIC FIELDS

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Submitted to JETP editor June 24, 1964

J. Exptl. Theoret. Phys. (U.S.S.R.) **48**, 204-221 (January, 1965)

This paper is devoted to a theoretical investigation of different scattering mechanisms for electrons and phonons in transverse transfer phenomena in metals in strong magnetic fields; we elucidate the role played by phonons in heat transfer. If $\omega_H\tau_e \gg 1$ (τ_e is the electronic mean free flight time, ω_H the electronic Larmor frequency), one can write down transport equations for the electron and phonon distribution functions; an important feature of these equations is that they do not contain kinematic terms and that inhomogeneity gradients occur in the collision integrals. Using these equations we obtain general formulae for transport coefficients taking phonon-electron drag into account. We show that in sufficiently pure metals heat is basically transported by the phonons and not by the electrons. In the quantum mechanical case the phonon heat conductivity should show the same oscillations as the electron transport coefficients (the only difference consisting in a shift of the position of the maxima). We show that the relative amplitude of the quantum oscillations of all transport coefficients connected both with electron-impurity scattering and with electron-lattice vibration scattering may be of order unity. We show that the electron-phonon drag effect may appreciably affect the magnitude of the thermal emf.

1. INTRODUCTION

A number of papers[1,2] have dealt with a study of various aspects of galvanomagnetic effects in metals. Thermomagnetic effects have been studied less extensively than galvanomagnetic effects. In particular, the role played by phonons in heat transfer in metals has not been elucidated. Usually, only one scattering mechanism for electrons, the scattering by impurities, is taken into account in a study of galvano- and thermomagnetic effects. Electron-lattice vibration scattering plays an important role in sufficiently pure metals; this is well known to possibly lead to a drag of the phonons by the electrons.[1)] Since the phonon distribution is not an equilibrium one when there are electrical fields or gradients in the temperature or the chemical potential present, the conductivity and the thermal conductivity of pure metals must depend essentially upon the phonon scattering mechanism.

The present paper is devoted to a study of the effect of various mechanisms for the scattering of electrons and phonons on the transverse transfer phenomena in metals in strong magnetic fields, and we elucidate the roles played by electrons and phonons in heat transfer. We consider the case of closed Fermi surfaces and assume that the condition $\omega_H\tau_e \gg 1$, where ω_H is the electron Larmor frequency and τ_e the mean free flight time of an electron when there is no magnetic field, is fulfilled. If $\omega_H\tau_e \gg 1$ and if the electric field $\mathbf{E}$, and the gradients of the temperature T and of the chemical potential ζ are at right angles to the magnetic field, we can write down transport equations for the electron and phonon distribution functions; an important feature of these equations is that the electrical field and the inhomogeneities occur in the collision integrals. For weak electric fields and small inhomogeneities we can find solutions of the transport equations and this enables us to obtain general formulae for transfer coefficients. It then turns out that the heat current transferred by the phonons in sufficiently pure metals is appreciably larger than the heat current transferred by the electrons.

The situation where the thermal conductivity of metals is determined not by the electrons but by the phonons is similar to the situation occurring in a high-temperature plasma in a strong magnetic field. The thermal conductivity of the plasma is in that case also not determined by the electrons but by the photons emitted by the electrons in the mag-

[1)]L. Gurevich and Éfros [3] studied this effect for metals and the present authors [4] have studied it for a plasma.

netic field. If we consider only the electron-phonon interaction and neglect Umklapp processes, then the electrical current in the direction of the electrical field vanishes because of the total drag of the phonon gas by the electrons. In contradistinction to the electrical current, the heat current in the direction of the temperature gradient is non-vanishing, even when quasi-momentum is exactly conserved. We show that the phonon-electron drag can also appreciably affect the magnitude of the thermal emf.

It is well known that in the quantum mechanical case, when $\omega_H \gg T$, the transport coefficients display characteristic oscillations when the magnetic field is varied. We consider both oscillations connected with electron-impurity scattering and oscillations connected with the electron-phonon interaction, and we show that the relative amplitude of the oscillations may be of order unity and that the oscillations should be displayed not only in quantities connected with transfer by electrons, i.e., in the electrical conductivity and in the electronic thermal conductivity, but also in the phonon thermal conductivity.

2. PROBABILITIES FOR SCATTERING PROCESSES FOR ELECTRONS AND PHONONS

We describe the state of an electron in the crystalline lattice by three quantum numbers: the oscillation quantum number n, which takes on values 0, 1, 2, ..., the component p_3 of the electron quasi-momentum along the magnetic field **H**, and the continuous quantum number p_2 determining the coordinate ξ_1 of the center of the electron "Larmor orbit" along the x_1 axis [we assume the vector potential to be chosen in the form $\mathbf{A} = (0, Hx, 0)$]. One can show that $\xi_1 = p_2/eH + x_n(p_2, p_3)$, where x_n is a function of p_2 and p_3 with period aeH (a is the lattice constant) in p_2 and the reciprocal lattice period in p_3. In the case of a quadratic dispersion law $x_n = 0$; in the general case, however (but for closed Fermi surfaces), the quantity x_n is of the order of the electron Larmor radius. Assuming the Fermi surface to be closed, we shall neglect in the expression for ξ_1 the bounded quantity x_n in comparison with the quantity p_2/eH which can take on arbitrary values. We can assume the energy of an electron in the state (n, p_2, p_3) to be independent of p_2.

We shall determine the probabilities for scattering processes of electrons by impurities and by lattice vibrations. The electron-impurity scattering is elastic and the probability for it, i.e., the probability for the transition $(\kappa p_2) \to (\kappa' p_2')$, per unit time, can be written in the form

$$W_{ei}(\varkappa p_2, \varkappa' p_2') = \sum_q w_{ei}(\varkappa \mathbf{q}; \varkappa')\Delta(p_2 + q_2 - p_2'), \quad (1)$$

where

$$w_{ei}(\varkappa \mathbf{q}; \varkappa') = 2\pi |U_\mathbf{q}|^2 V n_i g_{nn'}(\mathbf{q})\Delta(p_3 + q_3 - p_3')\delta(\varepsilon_\varkappa - \varepsilon_{\varkappa'}),$$

ϵ_κ is the electron energy, $\kappa \equiv (n, p_3)$, $U_\mathbf{q}$ the Fourier component of the interaction energy between the electron and the impurity atom, V the volume of the solid, n_i the impurity concentration, and $g_{nn'}(\mathbf{q}) = |(\kappa p_2 | e^{iq_1 x_1} | \kappa' p_2')|^2$ ($\Delta(q) = 1$, if $q = 0$ and $\Delta(q) = 0$, if $q \neq 0$). In the case of a quadratic dispersion law, $\epsilon_\kappa = \omega_H(n + 1/2) + p_3^2/2m$ and

$$g_{nn'}(\mathbf{q}) = \left| \int_{-\infty}^{\infty} d\eta e^{i\alpha\eta} \varphi_n(\eta)\varphi_{n'}(\eta - \beta)\right|^2,$$

$$\varphi_n(\eta) = \frac{e^{-\eta^2/2}}{(2^n n! \sqrt{\pi})^{1/2}} H_n(\eta), \quad \alpha = \frac{q_1}{\sqrt{eH}}, \quad \beta = \frac{q_2}{\sqrt{eH}}, \quad (2)$$

where the $H_n(\eta)$ are the Hermite polynomials.[2)]

The main role in the electron-lattice vibration interaction is played by processes in which one phonon is absorbed or emitted by the electron. The probability for the transition per unit time of an electron from the state (κp_2) to a state $(\kappa' p_2')$ accompanied by the absorption of a phonon with wave vector **k** and frequency ω has the form

$$W_{ep}(\varkappa p_2, \mathbf{k}; \varkappa' p_2') = \sum_{K_2 K_3} \frac{g^2}{V}\omega g_{nn'}(\mathbf{k})\Delta(p_2 + k_2 - p_2' + K_2) \times \Delta(p_3 + k_3 - p_3' + K_3)\delta(\varepsilon_\varkappa + \omega - \varepsilon_{\varkappa'}), \quad (3)$$

where $g^2 = 2\pi^2 v_0 \rho / p_0^2$ is the electron-phonon interaction coupling constant, v_0 and p_0 the velocity and momentum on the Fermi surface, $\rho \approx 0.2$, and **K** a reciprocal lattice vector. We obtain the probability for the emission of a phonon from (3) by replacing **k** by $-\mathbf{k}$ and ω by $-\omega$ (in the δ-function). We shall not distinguish between phonons of different polarizations for the sake of simplicity.

If the Fermi surface lies entirely inside the Brillouin zone, the probability for Umklapp processes will be exponentially small ($\sim e^{-\epsilon_0/T}$, $\epsilon_0 \sim \zeta$). In that case we may neglect Umklapp processes in electron-phonon collisions. For more complex configurations the probability for Umklapp processes may be of the same order as for processes with exact quasi-momentum conservation. Unless we explicitly state the contrary,

2) We use units in which $\hbar = c = 1$.

we shall in the following neglect Umklapp processes in electron-phonon collisions (but we shall take them into account in phonon-phonon collisions, see below).

Equations (1) and (3) determine the probabilities for electron transitions when there is no electrical field. If an electrical field **E** along the x_1 axis is applied to the solid, the energy of an electron in the state (κp_2) becomes equal to $\mathscr{E}_{\kappa p_2} = \epsilon_\kappa - e\xi_1 E$. Moreover, the form of the electron wave functions is changed. When studying transport equations we must know the probability for electron transitions when there is an electrical field $\mathbf{E} \perp \mathbf{H}$ present. These probabilities are given by Eqs. (1) and (3) if we replace in the δ-function ϵ_κ by $\mathscr{E}_{\kappa p_2}$. As far as the squares of the matrix elements $g_{nn'}$ are concerned, we may assume that they are the same as in the case where there is no electrical field.

We consider now phonon scattering processes. The probability for the absorption or emission of a phonon by an electron is determined by Eq. (3). Apart from these processes we shall take into account the scattering of phonons from the boundaries of the solid and phonon-phonon interaction processes. We may assume the scattering of phonons by the boundaries of the solid to be elastic. If the solid has the form of an infinite plate along **H** with a thickness L_1 in the x_1 direction, the probability for the scattering of a phonon from the boundaries of the plate (i.e., the probability for the transition $\mathbf{k} \rightarrow \mathbf{k}'$) can schematically be written in the form

$$W_{pw}(\mathbf{k}, \mathbf{k}') = (s/2L_1)\Delta(k_2 - k_2')\Delta(k_3 - k_3')\Delta(k_1 + k_1'), \qquad (4)$$

where s is the sound velocity.

The phonon-phonon interaction leads to different processes of scattering, fusion, and splitting of phonons. Phonon interaction processes in which quasi-momentum is not conserved play a particularly important role, since these processes may lead to the establishment of the equilibrium distribution of the phonons. We shall take the phonon-phonon interaction into account by introducing a phonon mean free flight time τ_{pp} relating to these processes. At high temperatures ($T > \Theta$, where Θ is the Debye temperature) this time is inversely proportional to T and at low temperatures $1/\tau_{pp} \sim e^{-b\Theta/T}$, $b \sim 1$ (the magnitude of $1/\tau_{pp}$ is appreciably larger than the probability for an Umklapp process in electron-phonon collisions for closed Fermi surfaces).

We note that phonon-impurity scattering does not play an important role since their mean free flight time relating to this process $\sim (n_e/n_i)(s/a')^3\omega^{-4}$ (n_e is the electron density, a' the dimensions of the impurity center) is appreciably larger than the flight time of a phonon with respect to processes where it is absorbed or emitted by electrons, provided only that $n_i/n_e < s/v_0$.

3. TRANSPORT EQUATIONS FOR ELECTRONS AND PHONONS

We denote by $f_\kappa(\xi, t)$ the electron distribution function, i.e., the number of electrons at time t in a state κ and with the center of their Larmor orbit near the point ξ. The electron density is connected with the distribution function through the relation

$$n_e(\xi, t) = \frac{eH}{(2\pi)^2}\int_{-\infty} dp_3 \sum_n f_\varkappa(\xi, t).$$

Assuming that $\omega_H \tau_e \gg 1$ and that the electron Larmor radius is appreciably smaller than the scale of the inhomogeneities (i.e., distances over which the distribution function changes appreciably) we can write down the following expression for the change in the electron distribution function per unit time caused by the electron-phonon interaction:

$$\dot{f}_\varkappa^{(ep)}(\xi, t) = \sum_{\varkappa'\mathbf{k}} w_{ep}(\varkappa\mathbf{k}, \varkappa') D_p(\varkappa\xi, \mathbf{k}; \varkappa'\xi_+) + \sum_{\varkappa'\mathbf{k}} w_{ep}(\varkappa'\mathbf{k}, \varkappa) D_p(\varkappa'\xi_-, \mathbf{k}; \varkappa\xi), \qquad (5)$$

where*

$$D_p(\varkappa\xi, \mathbf{k}; \varkappa'\xi_+) = f_{\varkappa'}(\xi_+, t)[1 - f_\varkappa(\xi, t)][1 + N_\mathbf{k}(\xi, t)] - f_\varkappa(\xi, t)[1 - f_{\varkappa'}(\xi_+, t)] N_\mathbf{k}(\xi, t),$$

$$w_{ep}(\varkappa\mathbf{k}, \varkappa') = \sum_{p_2} W_{ep}(\varkappa p_2, \mathbf{k}; \varkappa' p_2'), \quad \xi_\pm = \xi \pm \frac{1}{eH}[\mathbf{kn}],$$

$\mathbf{n} = \frac{\mathbf{H}}{H}$ and $N_\mathbf{k}(\xi, t)$

is the distribution function of phonons with wave vector **k** at the point ξ (one can introduce such a function provided the phonon wavelength is appreciably shorter than the scale of the inhomogeneities).

We draw attention to the fact that the particle distribution functions occur in the collision integral $\dot{f}_\kappa^{(ep)}$ at different points in space in contradistinction to the Boltzmann classical collision integral. This is connected with the fact that the position of the center of the electron Larmor "orbit" is

*$[\mathbf{k}\,\mathbf{n}] = \mathbf{k} \times \mathbf{n}$.

changed when a phonon is emitted or absorbed (the change ξ is equal to $\pm \mathbf{k} \times \mathbf{n}/eH$).

The change in the electron distribution function caused by electron-impurity scattering is of the form

$$\dot{f}_{\varkappa}^{(ei)}(\xi, t) = \sum_{\varkappa' \mathbf{q}} w_{ei}(\varkappa \mathbf{q}, \varkappa') D_i(\varkappa\xi, \mathbf{q}; \varkappa'\xi'), \tag{6}$$

where

$$D_i(\varkappa\xi, \mathbf{q}; \varkappa'\xi') = f_{\varkappa'}(\xi', t) - f_{\varkappa}(\xi, t), \quad \xi' = \xi - \frac{1}{eH}[\mathbf{qn}].$$

Equating the sum of $\dot{f}_K^{(ep)}$ and $\dot{f}_K^{(ei)}$ to zero, we get a transport equation to determine the electron distribution function in the stationary case:

$$\dot{f}_{\varkappa}^{(ep)} + \dot{f}_{\varkappa}^{(ei)} = 0. \tag{7}$$

We note that this equation does not contain kinematic terms connected with the presence of an electrical field or of temperature and chemical-potential gradients. The electrical field appears in the expression for the probabilities w_{ep} and w_{ei}, and the gradients implicitly in the distribution functions.

We now write down the phonon transport equation. The change in the phonon distribution function caused by their interaction with the electrons has the form

$$\dot{N}_{\mathbf{k}}^{(pe)}(\xi, t) = \sum_{\varkappa\varkappa' p_z'} w_{ep}(\varkappa \mathbf{k}, \varkappa') D_p(\varkappa\xi, \mathbf{k}; \varkappa'\xi_+). \tag{8}$$

The change in the phonon distribution function caused by their scattering from the boundaries of the solid (which is assumed to have the form of an infinitely long parallelepiped with transverse dimensions L_1 and L_2) can schematically be written in the form

$$\dot{N}_{\mathbf{k}}^{(pw)} = \frac{s}{2L_2}[N_{\mathbf{k}'}(\xi, t) - N_{\mathbf{k}}(\xi, t)] + \frac{s}{2L_1}[N_{\mathbf{k}''}(\xi, t) - N_{\mathbf{k}}(\xi, t)], \tag{9}$$

where $\mathbf{k}' = (k_1, -k_2, k_3)$, $\mathbf{k}'' = (-k_1, k_2, k_3)$.

Finally, the change in the phonon distribution function caused by the phonon-phonon interaction will be written in the form

$$\dot{N}_{\mathbf{k}}^{(pp)} = \frac{1}{\tau_{pp}}[N_{\mathbf{k}}^0(\xi) - N_{\mathbf{k}}(\xi, t)], \tag{10}$$

where $N_{\mathbf{k}}^0(\xi) = [e^{\omega/T(\xi)} - 1]^{-1}$ is the equilibrium phonon distribution at the local temperature $T(\xi)$.

In the stationary case the sum of the collision integrals of the phonons equals $s(\partial N_{\mathbf{k}}^0/\partial\xi)\mathbf{k}/k$:

$$\dot{N}_{\mathbf{k}}^{(pe)} + \dot{N}_{\mathbf{k}}^{(pw)} + \dot{N}_{\mathbf{k}}^{(pp)} = s\frac{\mathbf{k}}{k}\frac{\partial N_{\mathbf{k}}^0}{\partial \xi}. \tag{11}$$

We now turn to solving the transport equations assuming the electrical field and the gradients of T and ζ to be sufficiently small. The transport equation for the electrons (7) is satisfied up to linear terms in $\mathbf{E}$, ∇T, and $\nabla\zeta$ by the functions

$$f_{\varkappa}(\xi) = \left[\exp\left\{\frac{\varepsilon_{\varkappa} - \zeta(\xi)}{T(\xi)}\right\} + 1\right]^{-1},$$

$$N_{\mathbf{k}}(\xi) = N_{\mathbf{k}}^0 + N_{\mathbf{k}}^0(1 + N_{\mathbf{k}}^0)U_{\mathbf{k}},$$

$$U_{\mathbf{k}} = uk_2 + vk_1. \tag{12}$$

Indeed, the quantities D_p and D_i are for these functions up to linear terms in $\mathbf{E}$, ∇T, and $\nabla\zeta$ equal to

$$D_p(\varkappa\xi, \mathbf{k}; \varkappa'\xi_+) = -N_{\mathbf{k}}^0 f_{\varkappa}^0(1 - f_{\varkappa'}^0)\Delta_{ep},$$

$$D_i(\varkappa\xi, \mathbf{q}; \varkappa'\xi') = f_{\varkappa}^0(1 - f_{\varkappa'}^0)\Delta_{ei},$$

$$\Delta_{ep} = U_{\mathbf{k}} + \frac{E}{HT}k_2 + \frac{1}{eH}[\mathbf{kn}]\frac{\partial}{\partial\xi}\frac{\varepsilon_{\varkappa'} - \zeta(\xi)}{T(\xi)},$$

$$\Delta_{ei} = \frac{E}{HT}q_2 + \frac{1}{eH}[\mathbf{qn}]\frac{\partial}{\partial\xi}\frac{\varepsilon_{\varkappa} - \zeta(\xi)}{T(\xi)}. \tag{12'}$$

Substituting these expressions into (5) and (6) and bearing in mind that $\xi - \xi_+ = -\mathbf{k} \times \mathbf{n}/eH$, $\xi - \xi' = \mathbf{q} \times \mathbf{n}/eH$, one verifies easily that the distributions (12) make $\dot{f}_K^{(ep)}$ and $\dot{f}_K^{(ei)}$ vanish.

We note that since the braces in (5) and (6) (sic!) are proportional to $\mathbf{E}$ and to ∇T and $\nabla\zeta$, we may assume with the accuracy considered here that the electrical field vanishes (as mentioned in the preceding section) in the expressions for the probabilities w_{ep} and w_{ei} occurring in front of the δ-functions.

Substituting (12′) into (11) we find u and v:

$$u = \frac{s^2}{\nu_2}\frac{\partial}{\partial x_2}\frac{1}{T} - \frac{\zeta}{eH}(\nu_{ep}^{(1)} - \nu_{ep}^{(0)})\frac{1}{\nu_2}\frac{\partial}{\partial x_1}\frac{1}{T} + \frac{1}{eHT}\frac{\nu_{ep}^{(0)}}{\nu_2}\frac{\partial\zeta}{\partial x_1} - \frac{1}{T}\frac{\nu_{ep}^{(0)}}{\nu_2}\frac{E}{H},$$

$$v = \frac{s^2}{\nu_1}\frac{\partial}{\partial x_1}\frac{1}{T} + \frac{\zeta}{eH}(\nu_{ep}^{(1)} - \nu_{ep}^{(0)})\frac{1}{\nu_1}\frac{\partial}{\partial x_2}\frac{1}{T} - \frac{1}{eHT}\frac{\nu_{ep}^{(0)}}{\nu_1}\frac{\partial\zeta}{\partial x_2}; \tag{13}$$

$$\nu_{ep}^{(r)} = \frac{eHL_1L_2}{2\pi}\sum_{\varkappa\varkappa'} w_{ep}(\varkappa, \mathbf{k}; \varkappa')\left(\frac{\varepsilon_{\varkappa'}}{\zeta}\right)^r(f_{\varkappa}^0 - f_{\varkappa'}^0),$$

$$\nu_\alpha = \nu_{ep}^{(0)} + \frac{1}{\tau_{pp}} + \frac{s}{L_\alpha}. \tag{14}$$

We note that $\nu_{ep}^{(0)}$ is the reciprocal of the lifetime of a phonon with wave vector $\mathbf{k}$ with respect to the electron-phonon interaction.

4. ELECTRICAL CURRENT AND HEAT CURRENT

We now determine the microscopic electrical current density $\mathbf{j}$ and the thermal current density $\mathbf{q}^{(e)}$ transferred by the electrons. The circular component j_+ of the current density is determined by the formula:[3)]

$$j_+ = j_1 + ij_2 = -ie\frac{E}{H}n_e + \frac{i}{H}(\nabla n_e\bar{\varepsilon}_\perp)_+$$
$$+\frac{ie}{2\pi L_3}\sum_{\varkappa\varkappa'\mathbf{k}} w_{ep}(\varkappa\mathbf{k};\varkappa')k_+D_p(\varkappa\xi,\mathbf{k};\varkappa'\xi_+)$$
$$+\frac{ie}{4\pi L_3}\sum_{\varkappa\varkappa'\mathbf{q}} w_{ei}(\varkappa\mathbf{q};\varkappa')D_i(\varkappa\xi,\mathbf{q};\varkappa'\xi')q_+;$$
$$n_e = \frac{1}{V}\sum_\varkappa f_\varkappa^0,\quad n_e\bar{\varepsilon}_\perp = \frac{1}{V}\sum_\varkappa \omega_H\left(n+\frac{1}{2}\right)f_\varkappa^0. \tag{15}$$

A similar formula is valid for the circular component $q_+^{(e)}$ of the heat current

$$q_+^{(e)} = q_1^{(e)} + iq_2^{(e)} = -i\frac{E}{H}n_e(\bar{\varepsilon}+\bar{\varepsilon}_\perp) + \frac{i}{eH}(\nabla n_e\overline{\varepsilon\varepsilon_\perp})_+$$
$$+\frac{i}{2\pi L_3}\sum_{\varkappa\varkappa'\mathbf{k}} w_{ep}(\varkappa\mathbf{k};\varkappa')\varepsilon_{\varkappa'}k_+D_p(\varkappa\xi,\mathbf{k};\varkappa'\xi_+)$$
$$+\frac{i}{4\pi L_3}\sum_{\varkappa\varkappa'\mathbf{q}} w_{ei}(\varkappa\mathbf{q};\varkappa')\varepsilon_{\varkappa'}q_+D_i(\varkappa\xi,\mathbf{q};\varkappa'\xi');$$
$$n_e\bar{\varepsilon} = \frac{1}{V}\sum_\varkappa \varepsilon_\varkappa f_\varkappa^0,\quad n_e\overline{\varepsilon\varepsilon_\perp} = \frac{1}{V}\sum_\varkappa \varepsilon_\varkappa\omega_H\left(n+\frac{1}{2}\right)f_\varkappa^0. \tag{16}$$

The first two terms in (15) and (16) are of a "collisionless" nature, and the next two terms are caused by electron scattering processes: the first one by scattering by lattice vibrations and the second one by impurity scattering. One can give a simple physical interpretation of the "collision" terms. They are the sum over different states of the product of the quantities

$$e(\xi-\xi_+) = -[\mathbf{kn}]/H,\quad e(\xi-\xi') = [\mathbf{qn}]/H,$$

corresponding to the transfer of charge from the point ξ to the points ξ_+ and ξ' (in the equation for j_+) and the quantities

$$\varepsilon_\varkappa\xi + \omega\xi - \varepsilon_{\varkappa'}\xi_+ = -\varepsilon_{\varkappa'}[\mathbf{kn}]/eH,$$
$$\varepsilon_\varkappa\xi - \varepsilon_{\varkappa'}\xi' = \varepsilon_\varkappa[\mathbf{qn}]/eH,$$

corresponding to the transfer of energy from the point ξ to the points ξ_+ and ξ' (in the equation for $q_+^{(e)}$) and the number of transitions per unit time.

[3)]Titeica [5] established this structure for the expression for j_+ for the case of a uniform distribution. The derivation of Eqs. (15) and (16) was for the general case of non-uniform distributions given in[6].

We must add to the thermal current $\mathbf{q}^{(e)}$ transferred by electrons the thermal current $\mathbf{q}^{(p)}$ transferred by phonons:

$$\mathbf{q}^{(p)} = \int N_\mathbf{k}(\xi)\,s\frac{\mathbf{k}}{k}\omega\frac{d\mathbf{k}}{(2\pi)^3}. \tag{17}$$

Substituting (12) into (15) and (16) we find the transport coefficients, that is the coefficients of $\mathbf{E}$, ∇T, and $\nabla\zeta$ in the expressions for the current $\mathbf{j}$ and the heat current $\mathbf{q} = \mathbf{q}^{(e)} + \mathbf{q}^{(p)}$:

$$\mathbf{j} = \hat{\sigma}\mathbf{E} - \hat{\sigma}'\nabla\zeta/e - \hat{\alpha}\nabla T,$$
$$\mathbf{q} - \zeta\mathbf{j}/e = \hat{\beta}\mathbf{E} - \hat{\beta}'\nabla\zeta/e - \hat{\gamma}\nabla T. \tag{18}$$

Each of these coefficients consists of two terms caused, respectively, by the electron-lattice vibrations and the electron-impurity scattering. We shall indicate these terms by superscripts (ep) and (ei) (for instance, $\hat{\sigma} = \hat{\sigma}^{(ep)} + \hat{\sigma}^{(ei)}$). The tensor $\hat{\gamma}^{(ep)}$ can be written as a sum $\hat{\gamma}^{(ep)} = \hat{\gamma}^{(ep)}(e) + \hat{\gamma}^{(ep)}(p)$, where $\hat{\gamma}^{(ep)}(e)$ and $\hat{\gamma}^{(ep)}(p)$ are the contributions to $\hat{\gamma}^{(ep)}$ from the electrons and the phonons.

The transport coefficients defined by the electrical current are of the form

$$\sigma_{11}^{(ep)} = \sigma_{11}^{(ep)'} = \frac{1}{H^2T}\frac{1}{V}\sum_\mathbf{k} k_2^2N_\mathbf{k}^0(1+N_\mathbf{k}^0)\nu_{ep}^{(0)}\left(1-\frac{\nu_{ep}^{(0)}}{\nu_2}\right),$$
$$\sigma_{11}^{(ei)} = \sigma_{11}^{(ei)'} = \frac{1}{2H^2T}\frac{1}{V}\sum_\mathbf{k} q_2^2\nu_{ei}^{(0)},$$
$$\sigma_{12} = -\sigma_{21} = \frac{en_e}{H},\qquad \sigma_{12}' = -\sigma_{21}' = \frac{e}{H}\frac{\partial}{\partial\zeta}n_e\bar{\varepsilon}_\perp,$$
$$\alpha_{11}^{(ep)} = \frac{\zeta}{eH^2T^2}\frac{1}{V}\sum_\mathbf{k} k_2^2N_\mathbf{k}^0(1+N_\mathbf{k}^0)(\nu_{ep}^{(1)}-\nu_{ep}^{(0)})\left(1-\frac{\nu_{ep}^{(0)}}{\nu_2}\right),$$
$$\alpha_{11}^{(ei)} = \frac{\zeta}{2eH^2T^2}\frac{1}{V}\sum_\mathbf{q} q_2^2(\nu_{ei}^{(1)}-\nu_{ei}^{(0)}),$$
$$\alpha_{12} = \frac{1}{H}\frac{\partial}{\partial T}n_e\bar{\varepsilon}_\perp + \frac{s^2}{HT^2}\frac{1}{V}\sum_\mathbf{k} k_2^2N_\mathbf{k}(1+N_\mathbf{k}^0)\frac{\nu_{ep}^{(0)}}{\nu_2}, \tag{19}$$

where $\nu_{ep}^{(r)}$ is defined by Eq. (14) and

$$\nu_{ei}^{(r)} = \frac{eHL_1L_2}{2\pi}\sum_{\varkappa\varkappa'} w_{ei}(\varkappa\mathbf{q};\varkappa')\left(\frac{\varepsilon_\varkappa}{\zeta}\right)^r f_\varkappa^0(1-f_\varkappa^0). \tag{20}$$

(The expressions for σ_{22}, α_{22}, $-\alpha_{21}$ are obtained from σ_{11}, α_{11}, α_{12} by replacing k_2 and q_2 by k_1 and q_1 and ν_2 by ν_1.)

The transport coefficients defined by the heat current have the form

$$\beta_{11}^{(ep)} = \beta_{11}^{(ep)'} = \frac{1}{eH^2T}\frac{1}{V}$$
$$\times\sum_\mathbf{k} k_2^2N_\mathbf{k}^0(1+N_\mathbf{k}^0)(\nu_{ep}^{(1)}-\nu_{ep}^{(0)})\left(1-\frac{\nu_{ep}^{(0)}}{\nu_2}\right),$$

$$\beta_{11}^{(ei)} = \beta_{11}^{(ei)'} = \frac{\zeta}{2eH^2T}\frac{1}{V}\sum_{\mathbf{q}} q_2^2(\nu_{ei}^{(1)} - \nu_{ei}^{(0)}),$$

$$\beta_{12} = -\beta_{21} = \frac{n_e}{H}(\bar{\varepsilon}_\perp + \bar{\varepsilon} - \zeta),$$

$$\beta_{12}' = \frac{T}{H}\frac{\partial}{\partial T} n_e\bar{\varepsilon}_\perp + \frac{s^2}{HT}\frac{1}{V}\sum_{\mathbf{k}} k_1^2 N_{\mathbf{k}}^0(1 + N_{\mathbf{k}}^0)\frac{\nu_{ep}^{(0)}}{\nu_1},$$

$$\gamma_{11}^{(ep)}(e) = \left(\frac{\zeta}{eHT}\right)^2\frac{1}{V}\sum_{\mathbf{k}} k_2^2 N_{\mathbf{k}}^0(1 + N_{\mathbf{k}}^0)$$

$$\times\left(\nu_{ep}^{(2)} - 2\nu_{ep}^{(1)} + \nu_{ep}^{(0)} - \frac{1}{\nu_2}(\nu_{ep}^{(1)} - \nu_{ep}^{(0)})^2\right),$$

$$\gamma_{11}^{(ep)}(p) = \frac{s^4}{T^2}\frac{1}{V}\sum_{\mathbf{k}} k_1^2 N_{\mathbf{k}}^0(1 + N_{\mathbf{k}}^0)\frac{1}{\nu_1},$$

$$\gamma_{11}^{(ei)} = \frac{1}{2}\left(\frac{\zeta}{eHT}\right)^2\frac{1}{V}\sum_{\mathbf{q}} q_2^2(\nu_{ei}^{(2)} - 2\nu_{ei}^{(1)} + \nu_{ei}^{(0)}),$$

$$\gamma_{12} = -\gamma_{21} = \frac{1}{eH}\frac{\partial}{\partial T} n_e(\overline{\varepsilon\varepsilon_\perp} - \zeta\bar{\varepsilon}_\perp)$$

$$+ \frac{\zeta s^2}{eHT^2}\frac{1}{V}\sum_{\mathbf{k}}\left(\frac{k_1^2}{\nu_1} + \frac{k_2^2}{\nu_2}\right)(\nu_{ep}^{(1)} - \nu_{ep}^{(0)})N_{\mathbf{k}}^0(1 + N_{\mathbf{k}}^0). \quad (21)$$

(The expressions for β_{22}, γ_{22}, $-\beta'_{21}$ are obtained from β_{11}, γ_{11}, and β'_{12} by replacing k_2 and q_2 by k_1 and q_1 and ν_1 by ν_2.)

The tensors $\hat{\alpha}$, $\hat{\beta}'$, $\hat{\sigma}$, and $\hat{\gamma}$ satisfy the symmetry relations for transport coefficients:

$$\beta_{ik}'(\mathbf{H}) = T\alpha_{ki}(-\mathbf{H}),\ \sigma_{ik}(\mathbf{H}) = \sigma_{ki}(-\mathbf{H}),$$

$$\gamma_{ik}(\mathbf{H}) = \gamma_{ki}(-\mathbf{H}).$$

However, $\sigma_{11} \neq \sigma_{22}$, $\gamma_{11} \neq \gamma_{22}$. This is connected with the fact that the phonon mean free path in the general relations (19) (21) is assumed to be equal to the dimensions of the solid. Only when $L_1 = L_2$ will the diagonal components of the tensors $\hat{\sigma}$ and $\hat{\gamma}$ become the same.

We note that the tensor $\hat{\sigma}'$ is not the same as the tensor $\hat{\sigma}$ and that the tensor $\hat{\beta}'$ is not the same as the tensor $\hat{\beta}$. In other words, the Einstein relations connecting the electrical conductivity and the diffusion coefficients are not satisfied for the microscopic current. However, if we introduce the macroscopic current occurring in the Maxwell equations, $\mathbf{J} = \mathbf{j} - \operatorname{curl}\mathbf{M}$ ($\mathbf{M}$ is the magnetization current) the Einstein relations will be satisfied for $\mathbf{J}$ since in $\mathbf{J}$ the quantities $\nabla\zeta$ and $\mathbf{E}$ do not appear separately, but in the combination $e\mathbf{E} - \nabla\zeta$.[7,8]

When deriving Eqs. (19) and (21) we neglected Umklapp processes in electron-phonon collisions. If we take those into account, the transport coefficients will be determined by the equations

$$\sigma_{11}^{(ep)} = \frac{e^2}{T}\frac{1}{V}\sum_{\mathbf{k}} N_{\mathbf{k}}^0(1 + N_{\mathbf{k}}^0)\sum_{\substack{\varkappa\varkappa'\\p_2p_2'}} W_{ep}(\varkappa p_2,\ \mathbf{k};\ \varkappa' p_2')$$

$$\times(f_\varkappa^0 - f_{\varkappa'}^0)(\xi - \xi')\left(\frac{k_2}{eH}\frac{\nu_{ep}^{(0)}}{\nu_2} + \xi - \xi'\right),$$

$$\alpha_{11}^{(ep)} = \frac{1}{T}\beta_{11}^{(ep)} = \frac{e}{T^2}\frac{1}{V}\sum_{\mathbf{k}} N_{\mathbf{k}}^0(1 + N_{\mathbf{k}}^0)$$

$$\times\sum_{\substack{\varkappa\varkappa'\\p_2p_2'}} W_{ep}(\varkappa p_2,\ \mathbf{k};\ \varkappa' p_2')(f_\varkappa^0 - f_{\varkappa'}^0)(\varepsilon_{\varkappa'} - \zeta)(\xi - \xi')$$

$$\times\left(\frac{k_2}{eH}\frac{\nu_{ep}^{(0)}}{\nu_2} + \xi - \xi'\right),$$

$$\gamma_{11}^{(ep)}(e) = \frac{1}{T^2}\frac{1}{V}\sum_{\mathbf{k}} N_{\mathbf{k}}^0(1 + N_{\mathbf{k}}^0)$$

$$\times\left\{\sum_{\substack{\varkappa\varkappa'\\p_2p_2'}} W_{ep}(\varkappa p_2,\ \mathbf{k};\ \varkappa' p_2')(f_\varkappa^0 - f_{\varkappa'}^0)\right.$$

$$\left.\times(\xi - \xi')^2,\ (\varepsilon_{\varkappa'} - \zeta)^2 + \zeta^2\left(\frac{k_2}{eH}\right)^2\frac{(\nu_{ep}^{(0)} - \nu_{ep}^{(1)})^2}{\nu_2}\right\};$$

$$\nu_{ep}^{(r)} = \sum_{\substack{\varkappa\varkappa'\\p_2p_2'}} W_{ep}(\varkappa p_2,\ \mathbf{k};\ \varkappa' p_2')(f_\varkappa^0 - f_{\varkappa'}^0)\left(\frac{\varepsilon_{\varkappa'}}{\zeta}\right)^r. \quad (22)$$

If in an infinite and perfectly pure metal the Umklapp processes are neglected both in phonon-phonon and in electron-phonon collisions, then $\nu_{ep}^{(0)} \equiv \nu_2$ and $k_2 + eH(\xi - \xi') = 0$. Therefore $\sigma_{11}^{(ep)}$ and $\alpha_{11}^{(ep)}$ vanish. This is connected with the effect of the complete drag of the phonons by the electrons which occurs when the momentum conservation law is strictly observed. In contradistinction to $\sigma_{11}^{(ep)}$ and $\alpha_{11}^{(ep)}$ the quantity $\gamma_{11}^{(ep)}$ does not vanish when this law is strictly observed.

5. CONNECTION BETWEEN IRREVERSIBLE CURRENTS AND ENTROPY CHANGE

We show that the irreversible parts of the electrical and the thermal currents determined by the diagonal parts of the transport tensors are connected through simple relations with the rate of change of the entropy of the solid. The entropy is clearly the sum of the electron entropy $S^{(e)}$ and the phonon entropy $S^{(p)}$:

$$S^{(e)} = -\frac{eH}{2\pi}\sum_{\varkappa}\int d\xi\,\{f_\varkappa\ln f_\varkappa + (1 - f_\varkappa)\ln(1 - f_\varkappa)\},$$

$$S^{(p)} = -\frac{1}{L_1L_2}\sum_{\mathbf{k}}\int d\xi\,\{N_{\mathbf{k}}\ln N_{\mathbf{k}} - (1 + N_{\mathbf{k}})\ln(1 + N_{\mathbf{k}})\}.$$

Using the expressions for the change in the distribution functions (5) and (6) we show easily that

$$\dot{S} = \dot{S}^{(e)} + \dot{S}^{(p)} = -\frac{eH}{2\pi}\int d\xi \sum_{\varkappa\varkappa'\mathbf{k}} w_{ep}(\varkappa\mathbf{k};\ \varkappa')\{f_{\varkappa'}(\xi_+)$$

$$\times [1-f_\varkappa(\xi)][1+N_\mathbf{k}(\xi)] - f_\varkappa(\xi)[1-f_{\varkappa'}(\xi_+)]N_\mathbf{k}(\xi)\}$$

$$\times \ln\frac{f_\varkappa(\xi)[1-f_{\varkappa'}(\xi_+)]N_\mathbf{k}(\xi)}{[1-f_\varkappa(\xi)]f_{\varkappa'}(\xi_+)[1+N_\mathbf{k}(\xi)]} - \frac{eH}{4\pi}$$

$$\times \int d\xi \sum_{\varkappa\varkappa'\mathbf{q}} w_{ei}(\varkappa\mathbf{q};\ \varkappa')\{f_{\varkappa'}(\xi') - f_\varkappa(\xi)\}$$

$$\times \ln\frac{f_\varkappa(\xi)[1-f_{\varkappa'}(\xi')]}{[1-f_\varkappa(\xi)]f_{\varkappa'}(\xi')} + \frac{1}{2L_1L_2}\int d\xi \sum_\mathbf{k}\left\{\frac{s}{2L_2}(N_\mathbf{k}-N_{\mathbf{k}'})\right.$$

$$\left.\times \ln\frac{N_\mathbf{k}(1+N_{\mathbf{k}'})}{(1+N_\mathbf{k})N_{\mathbf{k}'}} + \frac{s}{2L_1}(N_\mathbf{k}-N_{\mathbf{k}''})\ln\frac{N_\mathbf{k}(1+N_{\mathbf{k}''})}{N_{\mathbf{k}''}(1+N_\mathbf{k})}\right\}. \quad (23)$$

(Here we have omitted for the sake of simplicity the well-known expression for the change in the phonon entropy caused by the phonon-phonon interaction.[9])

Substituting (12) into (23) and neglecting terms containing the field and the gradients in powers higher than the second, we get

$$\dot{S} = \frac{eH}{2\pi}\int d\xi \sum_{\varkappa\varkappa'\mathbf{k}} w_{ep}(\varkappa\mathbf{k};\ \varkappa')f_\varkappa{}^0(1-f_{\varkappa'}{}^0)N_\mathbf{k}{}^0\Delta_{ep}^2$$

$$+ \frac{eH}{4\pi}\int d\xi \sum_{\varkappa\varkappa'\mathbf{q}} w_{ei}(\varkappa\mathbf{q};\ \varkappa')f_\varkappa{}^0(1-f_{\varkappa'}{}^0)\Delta_{ei}{}^2$$

$$+ \frac{1}{4L_1L_2}\int d\xi \sum_\mathbf{k} N_\mathbf{k}{}^0(1+N_\mathbf{k}{}^0)$$

$$\times \left\{\frac{s}{L_2}(U_\mathbf{k}-U_{\mathbf{k}'})^2 + \frac{s}{L_1}(U_\mathbf{k}-U_{\mathbf{k}''})^2\right\}. \quad (24)$$

To find the irreversible parts of the currents, which we shall denote by $\mathbf{j}'$ and $\mathbf{q}'$ (they correspond to the diagonal components of the transport tensors), we introduce generalized "forces":

$$\mathbf{X} = \mathbf{E} - \nabla\frac{\zeta}{e}, \quad \mathbf{Y} = \nabla\frac{1}{T}.$$

Then

$$\mathbf{j}' = \frac{1}{2V}\frac{\partial\dot{S}}{\partial\mathbf{X}}, \quad \mathbf{q}' = \frac{1}{2V}\frac{\partial\dot{S}}{\partial\mathbf{Y}}.$$

Using (23) we get for $\mathbf{j}'$ and $\mathbf{q}'$ expressions which are exactly the same as the expressions for the irreversible parts of the current and the thermal current in Eqs. (19) and (21).

6. TRANSPORT COEFFICIENTS IN THE CLASSICAL CASE

We determine first the transport coefficients in the classical case when $\omega_H \ll T$. Since Eqs. (19) and (21) are valid for $\omega_H\tau_e \gg 1$, we must assume that the temperature is sufficiently low, $T \ll \Theta$ (in the opposite case, $\tau_e \approx 1/T$ and the condition for classical behavior will be incompatible with the condition $\omega_H\tau_e \gg 1$).

If $\omega_H \ll T$, we may change in Eqs. (14) and (20), which determine the "collision frequencies" $\nu_{ep}^{(r)}$ and $\nu_{ei}^{(r)}$, from a summation over n to an integration over the transverse components $p_\perp$ of the electron. Moreover, we can put in the scattering probabilities w_{ep} and w_{ei}: $\mathbf{H} = 0$. As a result the quantities $\nu_{ep}^{(r)}$ and $\nu_{ei}^{(r)}$ become

$$\nu_{ep}^{(r)}(\mathbf{k}) = g^2\omega\frac{1}{(2\pi)^3}\int\frac{(\varepsilon_\mathbf{p}+\omega)^r}{\zeta^r}[f^0(\varepsilon_\mathbf{p}) - f^0(\varepsilon_\mathbf{p}+\omega)]$$

$$\times\ \delta(\varepsilon_\mathbf{p}+\omega-\varepsilon_{\mathbf{p}+\mathbf{k}})\,d\mathbf{p},$$

$$\nu_{ei}^{(r)}(\mathbf{q}) = |U_\mathbf{q}|^2 n_i\frac{1}{(2\pi)^2}\int\left(\frac{\varepsilon_\mathbf{p}}{\zeta}\right)^r f^0(\varepsilon_\mathbf{p})[1-f^0(\varepsilon_\mathbf{p})]$$

$$\times\ \delta(\varepsilon_\mathbf{p}-\varepsilon_{\mathbf{p}+\mathbf{q}})\,d\mathbf{p}.$$

Bearing in mind that $T \ll \zeta$, we find

$$\nu_{ep}{}^{(0)} = \frac{g^2s}{(2\pi)^2}\left(\frac{p_0}{v_0}\right)^2\omega,$$

$$\nu_{ep}^{(r)} - \nu_{ep}^{(0)} = g^2s\left(\frac{p_0}{v_0}\right)^2\frac{T^2}{\zeta}r\left\{G_1\left(\frac{\omega}{T}\right)\right.$$

$$\left.+\frac{r-1}{2}\frac{T}{\zeta}G_2\left(\frac{\omega}{T}\right)\right\}, \quad (25)$$

$$\nu_{ei}^{(r)} = |U_\mathbf{q}|^2 n_i m^2\frac{1}{2\pi q}\int_{q^2/8m}^\infty d\varepsilon f^0(\varepsilon)[1-f^0(\varepsilon)]\left(\frac{\varepsilon}{\zeta}\right)^r, \quad (26)$$

where

$$G_l(x) = \frac{1}{(2\pi)^2}\int_{-\infty}^\infty\frac{y^l\,dy}{(1+e^{-y})(1+e^{y-x})}.$$

Substitution of (25) into (19) leads to the following expression for σ_{11}:

$$\sigma_{11} = \frac{n_e e^2}{m\omega_H{}^2}\left(\frac{1}{\tau_{ep}{}^{(2)}} + \frac{1}{\tau_{ei}}\right), \quad (27)$$

where

$$\frac{1}{\tau_{ep}{}^{(\alpha)}} = T\left(\frac{T}{\Theta}\right)^4\frac{\rho}{2(2\pi)^3}\int_0^\infty\frac{x^5dx}{(e^x-1)(1-e^{-x})(1+A_\alpha x)}$$

$$(\alpha = 1, 2)$$

$$\tau_{ei} \approx \frac{n_e}{n_i}\frac{a}{v_0}, \quad A_\alpha = \frac{1}{2}\rho\frac{s}{v_0}T\left(\frac{1}{\tau_{pp}} + \frac{s}{L_\alpha}\right)^{-1},$$

$$\Theta = p_0 s.$$

For a bulk solid ($A_\alpha \gg 1$)[4)]

4) L. Gurevich and Éfros [3] found the quantity $\tau_{ep}^{(2)}$ for $A_\alpha \gg 1$.

$$\frac{1}{\tau_{ep^{(2)}}} \approx \frac{\pi}{30}\left(\frac{T}{\Theta}\right)^4 \frac{v_0}{s}\left(\frac{1}{\tau_{pp}} + \frac{s}{L_\alpha}\right).$$

In the case of a film ($A_\alpha \ll 1$)

$$\frac{1}{\tau_{ep^{(2)}}} \approx \frac{15\zeta(5)}{2\pi^3}\rho T\left(\frac{T}{\Theta}\right)^4.$$

Since at low temperatures $1/\tau_{pp} \approx e^{-b\Theta/T}$ when $T \ll \Theta$ the quantity $\sigma_{11}^{(ep)}$ will for a bulk solid be exponentially small. This conclusion arrived at while neglecting Umklapp processes in electron-phonon collisions retains its validity also when the latter are taken into account, provided only that the Fermi surface lies completely inside the Brillouin zone. For more complicated configurations when the probability for electron-phonon collisions which do not conserve quasi-momentum may be of the same order as the probability for collisions with strict quasi-momentum conservation, the quantity $\sigma_{11}^{(ep)}$ will for $T \ll \Theta$ vary not exponentially with temperature but as a power of the temperature (proportional to T^n, where according to (22) $n \geqq 3$).

We note that in the classical case the tensors $\hat{\sigma}$ and $\hat{\sigma}'$ are the same.

We now give expressions for the components of the tensor $\hat{\alpha}$:

$$\alpha_{11}^{(ep)} = \rho \frac{n_e}{H}\frac{T}{\omega_H}\left(\frac{T}{\Theta}\right)^4 2\pi^2 \int_0^\infty \frac{x^4 G_1(x)}{(e^x-1)(1+xA_2)}dx,$$

$$\alpha_{11}^{(ei)} = \frac{2\pi^2}{3}\frac{n_e}{H}\frac{T}{\zeta}\frac{1}{\omega_H\tau_{ei}},$$

$$\alpha_{12} = \frac{\pi^2}{2}\frac{T}{\zeta}\frac{n_e}{H} + \frac{n_e}{H}\left(\frac{T}{\Theta}\right)^3 \times \int_0^\infty \frac{A_2 x^5 dx}{(e^x-1)(1-e^{-x})(1+xA_2)}. \quad (28)$$

If $A_2 \gg 1$,

$$\alpha_{11}^{(ep)} = (2\pi)^2 \frac{n_e}{H}\left(\frac{T}{\Theta}\right)^4 \frac{v_0}{\omega_H s}\left(\frac{1}{\tau_{pp}} + \frac{s}{L_2}\right),$$

$$\alpha_{12} = -\alpha_{21} = \frac{\pi^2}{2}\frac{n_e}{H}\left\{\frac{T}{\zeta} - \frac{8\pi^2}{15}\left(\frac{T}{\Theta}\right)^3\right\}. \quad (29)$$

If $A_2 \ll 1$,

$$\alpha_{11}^{(ep)} = 2\pi^2\rho\frac{n_e}{H}\frac{T^2}{\omega_H\Theta}\int_0^\infty \frac{x^4 G_1(x)}{e^x-1}dx,$$

$$\alpha_{12} = \frac{\pi^2}{2}\frac{n_e}{H}\frac{T}{\zeta}. \quad (30)$$

We note that the second term in the expression for α_{12} in (29) is caused by the phonon-electron drag effect. This term will be larger than the first one when $T > \frac{1}{2}\Theta\sqrt{(\Theta/\zeta)}$.

If the Fermi surface is such that the probability for electron-phonon conditions with non-conservation of quasi-momentum has the same order of magnitude as the probability for such collisions with quasi-momentum conservation, there is practically no drag of the phonons and the behavior of the tensor $\hat{\alpha}$ will be different: there will not be a second term in the component α_{12}. The phonon drag may thus be manifest in the off-diagonal components of the tensor $\hat{\alpha}$ determining the thermal emf.

Let us finally consider the tensors defined by the heat current. Since $\beta_{ik}(H) = \beta'_{ik}(H) = T\alpha_{ki}(-H)$ it is sufficient to give only the components of $\hat{\gamma}$. The diagonal components of $\hat{\gamma}^{(ei)}$ are connected with $\sigma_{11}^{(ei)}$ by the Wiedemann-Franz relations

$$\gamma_{11}^{(ei)} = \gamma_{22}^{(ei)} = \frac{\pi^2}{3}\frac{T}{e^2}\sigma_{11}^{(ei)}, \quad (31)$$

and the diagonal elements of $\hat{\gamma}^{(ep)}(e)$ and $\hat{\gamma}^{(ep)}(p)$ are determined by the formulae

$$\gamma_{11}^{(ep)}(e) = 2\pi^2\rho\left(\frac{T}{\Theta}\right)^4\left(\frac{T}{\omega_H}\right)^2\frac{n_e}{m} \times \int_0^\infty dx\frac{x^4}{e^x-1}\left\{G_2(x) - \frac{G_1^2(x)}{G_0(x)}\frac{xA_2}{1+xA_2}\right\},$$

$$\gamma_{11}^{(ep)}(p) = \frac{2}{\rho}\left(\frac{T}{\Theta}\right)^2\frac{n_e}{m} \times \int_0^\infty \frac{A_1 x^4}{(e^x-1)(1-e^{-x})(1+xA_1)}dx. \quad (32)$$

We note that these quantities do not vanish as $L \to \infty$, $\tau_{pp}^{-1} = 0$. The off-diagonal components of $\hat{\gamma}$ are of the form

$$\gamma_{12} = -\gamma_{21} = \frac{1}{24\pi}\frac{n_e}{m}\frac{T}{\omega_H} - (2\pi)^2\frac{n_e}{m}\frac{T}{\omega_H}\left(\frac{T}{\Theta}\right)^3 \times \int_0^\infty\left(\frac{A_1}{1+xA_1} + \frac{A_2}{1+xA_2}\right)\frac{x^4 G_1(x)}{e^x-1}dx. \quad (33)$$

In the limiting case $A_\alpha \gg 1$ Eqs. (32) and (33) give

$$\gamma_{11}^{(ep)}(e) = 2\pi^2\rho\frac{n_e}{m}\left(\frac{T}{\omega_H}\right)^2\left(\frac{T}{\Theta}\right)^4 \times \int_0^\infty dx\frac{x^4}{e^x-1}\left[G_2(x) - \frac{G_1^2(x)}{G_0(x)}\right],$$

$$\gamma_{11}^{(ep)}(p) = \frac{12\zeta(3)}{\rho}\frac{n_e}{m}\left(\frac{T}{\Theta}\right)^2,$$

$$\gamma_{12}=-\gamma_{21}=\frac{1}{24\pi}\frac{n_e}{m}\frac{T}{\omega_H}-\frac{n_e}{m}\frac{T}{\omega_H}\left(\frac{T}{\Theta}\right)^3 2(2\pi)^2 \times\int_0^\infty\frac{x^4G_1(x)}{e^x-1}dx. \quad (34)$$

If $A_\alpha \ll 1$, we have

$$\gamma_{11}^{(ep)}(e)=\gamma_{22}^{(ep)}(e)=2\pi^2\rho\frac{n_e}{m}\left(\frac{T}{\omega_H}\right)^2\left(\frac{T}{\Theta}\right)^4\int_0^\infty\frac{x^4G_2(x)}{e^x-1}dx,$$

$$\gamma_{11}^{(ep)}(p)=\frac{\pi^2}{15}\frac{n_e}{m}\left(\frac{T}{\omega_H}\right)^2\frac{L\omega_H}{v_0},$$

$$\gamma_{12}=-\gamma_{21}=\frac{1}{24\pi}\frac{n_e}{m}\frac{T}{\omega_H}. \quad (35)$$

Let us compare the contributions to the heat current from the electrons and the phonons for the case of a bulk sample:

$$\frac{\gamma_{11}^{(ep)}(e)}{\gamma_{11}^{(ep)}(p)}\sim\rho^2\left(\frac{T}{\Theta}\right)^2\left(\frac{T}{\omega_H}\right)^2, \quad L\gg\frac{v_0}{\rho T}.$$

This quantity is appreciably smaller than unity when $T \sim \omega_H$, provided $\omega_H^2 \ll \Theta^2/\rho^2$. The quantity $\gamma_{11}^{(ei)}$ connected with the electron-impurity scattering can clearly be neglected if the impurity concentration is sufficiently small

$$\frac{n_i}{n_e}\ll\frac{1}{\rho}\frac{T}{\zeta}\left(\frac{\omega_H}{\Theta}\right)^2.$$

For $T \sim 10^2$ °K, $H \sim 10^4$ G, this gives $n_i/n_e < 10^{-6}$ to 10^{-5}. For sufficiently pure metals heat can thus basically be transferred by the phonons rather than by the electrons.

7. TRANSPORT COEFFICIENTS IN THE QUANTUM-MECHANICAL CASE

We turn to a consideration of the transport coefficients in the quantum mechanical case when $\omega_H \gg T$ (both cases $T < \Theta$ and $T > \Theta$ may then be realized). First of all we consider those parts of the transport coefficients which are connected with the electron-impurity scattering. Assuming a quadratic dispersion law we can according to (1) write the quantity $\nu_{ei}^{(r)}$ in the form

$$\nu_{ei}^{(r)}(\mathbf{q})=\frac{n_i meH}{2\pi|q_3|}|U_q|^2\sum_{nn'}q_{nn'}(q_\perp)\left(\frac{\varepsilon_\varkappa}{\zeta}\right)^r f^0(\varepsilon_\varkappa)[1-f^0(\varepsilon_\varkappa)],$$

$$p_3=\frac{m}{q_3}\left\{\omega_H(n-n')-\frac{q_3^2}{2m}\right\}. \quad (36)$$

It follows from the definition (2) that the function $q_{nn'}$ is appreciably non-vanishing when $q_\perp \lesssim \omega_H/v_0$. If the range R of the forces between the electron and the impurity atom is sufficiently small, so that $R \ll v_0/\omega_H$, we can thus assume that $U_q \approx U_0$.

Substituting (36) into (19) and (21) leads to the following equations for the transport coefficients caused by electron-impurity scattering:[5)]

$$\sigma_{11}^{(ei)}=\frac{e^2m^2\omega_H|U_0|^2n_i}{4\pi^3T}\int_0^\infty dEf(E)(1-f(E)) \times\sum_{n=0}^N\frac{2n+1}{\sqrt{E-\varepsilon}}\sum_{n'=0}^N\frac{1}{\sqrt{E-\varepsilon_\perp'}},$$

$$\alpha_{11}^{(ei)}=\frac{e\zeta m^2\omega_H|U_0|^2n_i}{4\pi^3T^2}\int_0^\infty dEf(E)(1-f(E))\left(\frac{E}{\zeta}-1\right) \times\sum_{n=0}^N\frac{2n+1}{\sqrt{E-\varepsilon_\perp}}\sum_{n'=0}^N\frac{1}{\sqrt{E-\varepsilon_\perp'}},$$

$$\gamma_{11}^{(ei)}=\frac{\zeta^2m^2\omega_H|U_0|^2n_i}{4\pi^3T^2}\int_0^\infty dEf(E)(1-f(E))\left(\frac{E}{\zeta}-1\right)^2 \times\sum_{n=0}^N\frac{2n+1}{\sqrt{E-\varepsilon_\perp}}\sum_{n'=0}^N\frac{1}{\sqrt{E-\varepsilon_\perp'}}, \quad (37)$$

where $\epsilon_\perp = \omega_H(n + 1/2)$ and the integer N is connected with E by the relation $E = \omega_H(N + 1/2) + \omega_H\delta$, $0 \leqq \delta < 1$.

These expressions diverge for $E = \omega_H(n + 1/2)$; this is connected with the fact that the first Born approximation is inapplicable when $p_3 = 0$. However, as the divergence is a logarithmic one, Eqs. (37) lead to the correct results when the temperature is not too low ($T \gtrsim \omega_H$) and also far from resonances when $\zeta = \omega_H(N + 1/2) + \omega_H\eta$ ($0 \leq \eta < 1$) is not close to $\omega_H(N + 1/2)$ (the temperature can then be arbitrary).

We give the expressions for the transport coefficients referring to these cases

$$\sigma_{11}^{(ei)}=\frac{n_ee^2}{m\omega_H^2\tau_{ei}}\left\{1+\frac{5}{2}\left(\frac{\omega_H}{\zeta}\right)^{1/2} \times\sum_{r=1}^\infty\frac{(-1)^r}{\sqrt{2r}}\Psi(\alpha_r)\cos\left(\frac{2\pi r\zeta}{\omega_H}-\frac{\pi}{4}\right)\right\},$$

$$\alpha_{11}^{(ei)}=\frac{2\pi^2}{3}\frac{T}{e\zeta}\frac{n_ee^2}{m\omega_H^2}\frac{1}{\tau_{ei}}\left\{1-\frac{5}{2}\frac{\zeta}{T}\sqrt{\frac{\omega_H}{T}} \times\sum_{r=1}^\infty\frac{(-1)^r}{\sqrt{2r}}\Psi'(\alpha_r)\cos\left(\frac{2\pi r\zeta}{\omega_H}+\frac{\pi}{4}\right)\right\},$$

5)Several authors [1] have obtained the expression for $\sigma_{11}^{(ei)}$.

$$\gamma_{11}^{(ei)}=\frac{\pi^2}{3}\frac{n_eT}{m\omega_H^2\tau_{ei}}\left\{1-\frac{5}{2}\left(\frac{\omega_H}{\zeta}\right)^{1/2}\right.$$
$$\left.\times\sum_{r=1}^{\infty}\frac{(-1)^r}{\sqrt{2r}}\Psi''(\alpha_r)\cos\left(\frac{2\pi r\zeta}{\omega_H}-\frac{\pi}{4}\right)\right\}. \quad (38)$$

Here $\Psi(x)=x/\sinh x$, $\alpha_r=2\pi^2 rT/\omega_H$. We see that the relative amplitude of the oscillations of $\sigma_{11}^{(ei)}$ and $\gamma_{11}^{(ei)}$ is of the order $(\omega_H/\zeta)^{1/2}$.

If $T \ll \omega_H$ and the distance from a resonance sufficiently large ($1 \gg \eta \gg \omega_H/\zeta$) we have

$$\sigma_{11}^{(ei)}\approx\frac{n_ee^2}{m\omega_H\tau_{ei}}\left(1+\frac{5}{4}\sqrt{\frac{\omega_H}{\zeta\eta}}\right),$$
$$\gamma_{11}^{(ei)}\approx\frac{\pi^2}{3}\frac{n_eT}{m\omega_H\tau_{ei}}\left(1+\frac{5}{4}\sqrt{\frac{\omega_H}{\zeta\eta}}\right). \quad (39)$$

The second term in the brackets determines the amplitude of the oscillations. When $\eta \sim \omega_H/\zeta$ the relative amplitude of the oscillations becomes thus of the order unity. The same situation also occurs when $\eta < \omega_H/\zeta$.

Let us now consider those parts of the transport coefficients which are connected with the electron-phonon interaction. Substituting (3) into (14) and summing over p_3 and p_3' we write $\nu_{ep}^{(r)}$ in the form

$$\nu_{ep}^{(r)}=\frac{1}{2}\frac{eH\rho}{p_0}\frac{\omega}{|k_3|}\sum_{nn'}g_{nn'}(k_\perp)\left(\frac{\varepsilon_\varkappa+\omega}{\zeta}\right)^r$$
$$\times[f(\varepsilon_\varkappa)-f(\varepsilon_\varkappa+\omega)],$$
$$p_3=\frac{m}{k_3}\left\{\omega_H(n-n')+\omega-\frac{k_3^2}{2m}\right\}. \quad (40)$$

The simplest is the high-temperature case, $T \gtrsim \Theta$. Replacing in that case $f(\epsilon)-f(\epsilon+\omega)$ by $\omega f(\epsilon)(1-f(\epsilon))/T$ and neglecting in the expression for p_3 the quantity $\omega\sim\Theta$ as compared to $k_3^2/2m\sim\zeta$, we get

$$\nu_{ep}^{(r)}(\mathbf{k})\approx\frac{1}{2}\frac{eH\rho}{p_0}\frac{\omega^2}{T|k_3|}\sum_{nn'}g_{nn'}(k_\perp)\left(\frac{\varepsilon_\varkappa}{\zeta}\right)^r f(1-f),$$
$$p_3=\frac{m}{k_3}\left\{\omega_H(n-n')-\frac{k_3^2}{2m}\right\}. \quad (41)$$

Since the mean free flight time of the phonons for $T \gtrsim \Theta$ is very small we may assume their distribution to be an equilibrium one and put in Eqs. (19) and (21) $\nu_1=\nu_2=\infty$. Using (41) we obtain for $\sigma^{(ep)}$, $\alpha^{(ep)}$, and $\gamma^{(ep)}$ expressions which differ from those for $\sigma^{(ei)}$, $\alpha^{(ei)}$, and $\gamma^{(ei)}$ in that τ_{ei} is replaced by $4\pi^3/\rho T$. At high temperatures the electron-phonon scattering behaves thus like electron-impurity scattering.

We consider, finally, the case of very low temperatures $T \ll s\omega_H/v_0$. When $T \ll \Theta$ phonons with wave vectors $k\sim T/s$ play the main role. On the other hand, we can for large n and $k \ll \omega_H/v_0$ replace the function $g_{nn'}$ in Eq. (40) by $\delta_{nn'}$. If $T \ll s\omega_H/v_0$, we have thus $g_{nn'}\approx\delta_{nn'}$ and

$$\nu_{ep}^{(r)}(\mathbf{k})\approx\frac{eH\rho}{2p_0}\frac{\omega}{|k_3|}\sum_n\left(\frac{\varepsilon_\varkappa+\omega}{\zeta}\right)^r\{f(\varepsilon_\varkappa)-f(\varepsilon_\varkappa+\omega)\},$$
$$p_3=\frac{m}{k_3}\left(\omega-\frac{k_3^2}{2m}\right). \quad (42)$$

The main part in that sum is played by only one term corresponding to the minimum value of $|\epsilon_K-\zeta|$. It is reached for a value of $n=n_0$ determined by the condition

$$\zeta-p_3^2/2m=\omega_H(n_0+1)+T\varphi,$$
$$-\omega_H/2\leqslant T\varphi<\omega_H/2.$$

The quantity φ is clearly a periodic function of $\zeta-p_3^2/2m$ with period ω_H and can be written as a Fourier series

$$\varphi(z)=\frac{\omega_H}{2T}\left\{\frac{2}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\sin\frac{2\pi rz}{\omega_H}-1\right\},$$
$$z=\zeta-\frac{p_3^2}{2m}=\zeta+\frac{1}{2}\xi T\left(1-\frac{T}{8\varepsilon_0}\xi x^2\right)-\frac{\varepsilon_0}{x^2},\quad \xi=\frac{\omega}{T},$$
$$x=\frac{k_3}{|k_3|},\quad \varepsilon_0=\frac{ms^2}{2}. \quad (43)$$

Retaining in $\nu_{ep}^{(r)}$ only the one term with $n=n_0$ we get

$$\nu_{ep}^{(r)}=\frac{\omega_H s\rho}{2v_0|x|}(N^0+1)^{-1}(e^\varphi+1)^{-1}(1+e^{-\varphi-\xi})^{-1}.$$

Using this expression we can, in principle, determine the quantities $\sigma_{11}^{(ep)}$, $\alpha_{11}^{(ep)}$, and $\gamma_{11}^{(ep)}(e)$. We restrict ourselves here to the evaluation of the quantity $\gamma_{11}^{(ep)}(p)$ determined by the phonon thermal conductivity. To do this, we expand the function $1/\nu_{ep}^{(0)}$ which is a periodic function of z (as is the function $\varphi(z)$) in a Fourier series:

$$\frac{1}{\nu_{ep}^{(0)}}=\frac{2v_0|x|}{s\omega_H\rho}\left\{1+\sum_{r=-\infty}^{\infty}\frac{\exp(2\pi irz/\omega_H)}{1+2\pi irT/\omega_H}\right\}.$$

Substituting this expression into (21) we get

$$\gamma_{11}^{(ep)}(p)=\frac{2v_0T^3}{s^2\omega_H\rho}\frac{1}{(2\pi)^2}\int_0^\infty\frac{\xi^4d\xi}{(e^\xi-1)(1-e^{-\xi})}\int_0^1dx\,x(1-x^2)$$
$$\times\left\{1+\sum_{r=-\infty}^{\infty}\frac{\exp(2\pi irz/\omega_H)}{1+2\pi irT/\omega_H}\right\}. \quad (44)$$

Assuming that $\epsilon_0 \ll \omega_H$, $\epsilon_0 \ll T$ we can in (44) integrate over x. Using also the definition (43) of the function φ, we find

$$\gamma_{11}^{(ep)}(p)=\frac{v_0T}{\omega_Hs^2\rho}\left\{\frac{\pi^2}{15}+\frac{\pi T}{6\alpha\omega_H}\varphi(\zeta)-\frac{T}{\pi\alpha\omega_H}\int_0^\infty\frac{\xi\varphi(\zeta-\alpha\omega_H\xi^2/2\pi)}{e^\xi-1}d\xi\right\}\qquad(44')$$

where $\zeta = N\omega_H + \eta\omega_H$, $0 \leqq \eta < 1$, N a positive integer and $\alpha = \pi T_2/8\epsilon_0\omega_H$. When $\alpha \ll 1$ the region of ξ for which $0 \leqq z \leqq \omega_H$ gives the main contribution to the integral in (44′). In that integral $\varphi(z) = -z/T$ and

$$\int_0^\infty\frac{\xi\varphi(\zeta-\alpha\omega_H\xi^2/2\pi)}{e^\xi-1}d\xi=\frac{\omega_H}{2T}\left\{-\frac{\pi}{3}\left(\eta-\frac{1}{2}\right)+\frac{\pi^2\alpha}{15}-\frac{\pi^2}{6}-\int_{2\pi\eta/\alpha}^\infty\frac{dz}{e^{\sqrt{z}}-1}\right\}$$

Substituting this expression into (44′) we get finally

$$\gamma_{11}^{(ep)}(p)=\frac{v_0T}{\omega_Hs^2\rho}\left\{\frac{\pi^2}{30}+\frac{1}{2\pi\alpha}\int_{2\pi\eta/\alpha}^\infty\frac{dz}{e^{\sqrt{z}}-1}\right\}.\qquad(45)$$

We note that $\gamma_{11}^{(ep)}(p)$ is a periodic function of the chemical potential with period ω_H. The maximum value of $\gamma_{11}^{(ep)}(p)$ is reached for $\eta = 1$, i.e., for $\zeta = N\omega_H$:

$$[\gamma_{11}^{(ep)}(p)]_{max}\approx 2p_0/3\rho T.\qquad(46)$$

We recall that the maxima of $\sigma_{11}^{(ei)}$, $\alpha_{11}^{(ei)}$, and $\gamma_{11}^{(ei)}$ are reached in the points $\zeta = \omega_H(N + 1/2)$. The shift of the maxima of $\gamma_{11}^{(ep)}(p)$ relative to to the maxima of $\gamma_{11}^{(ei)}$ is connected with the fact that the quantity $\gamma_{11}^{(ep)}(p)$ is determined by the function $1/\nu_{ep}^{(0)}$ while the quantities $\sigma_{11}^{(ei)}$, $\alpha_{11}^{(ei)}$, and $\gamma_{11}^{(ei)}$ are determined by $\nu_{ei}^{(0)}$.

When $\eta \sim 1$, the second term in (45) is exponentially small so that

$$\gamma_{11}^{(ep)}(p)\approx\frac{\pi^2}{30}\frac{v_0T}{\omega_Hs^2\rho}.\qquad(47)$$

Since that quantity is appreciably less than $[\gamma_{11}^{(ep)}(p)]_{max}$ the amplitude of the oscillations in $\gamma_{11}^{(ep)}(p)$ is also determined by Eq. (46). The interval in which $\gamma_{11}^{(ep)}(p)$ as function of the chemical potential is close to its maximum value is of the order $\alpha\omega_H/2\pi$. A comparison of $\gamma_{11}^{(ep)}(p)$ with $\gamma_{11}^{(ei)}$ gives

$$\gamma_{11}^{(ei)}/\gamma_{11}^{(ep)}(p)\sim\rho(\zeta/\omega_H)^2n_i/n_e.$$

We see that, if $n_i/n_e < \rho^{-1}(\omega_H/\zeta)^2$ the phonons and not the electrons will play the main role in the heat transfer in the quantum case.

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Translated by D. ter Haar
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