SOME COMMENTS ON THE SINGULARITIES OF COSMOLOGICAL SOLUTIONS OF THE EQUATIONS OF GRAVITATION

I. M. KHALATNIKOV

Institute of Physics Problems, Academy of Sciences, U.S.S.R.

Submitted to JETP editor July 2, 1964

J. Exptl. Theoret. Phys. (U.S.S.R.) 48, 261-271 (January, 1965)

A class of solutions of the gravitation equation is found which has a singularity in the energy density for $t \to 0$. In a certain sense these solutions are analogous to the case of collapse of dustlike matter, and they are characterized by the fact that in a synchronous reference system the world lines of the particles practically coincide with the lines of time. This situation occurs in the case in which the energy density does not depend on the coordinate along which the scale contraction takes place. In the second part of the article the solutions of the Maxwell equations for the electromagnetic fields under conditions of anisotropic collapse [1] are analyzed.

IN the present paper we discuss some properties of cosmological solutions of the equations of gravitation which have singularities with respect to the time. In the first part solutions are considered which in empty space have a fictitious singularity with respect to time, but in the presence of matter lead to a singularity in the energy density. Solutions of this sort are in a certain sense analogous to the general case of collapse of dustlike matter with the equation of state $p = 0$, which has been considered previously.[1,2] In the second part we discuss the behavior of electromagnetic fields under conditions of anisotropic collapse with a physical singularity with respect to time. This equation is of a certain interest both for cosmology and in connection with applications to the collapse of planets.

1. FICTITIOUS AND PHYSICAL SINGULARITIES IN THE COSMOLOGICAL SOLUTIONS OF THE EQUATIONS OF GRAVITATION

It has been shown previously [1,2] that in a synchronous reference system the general solution of the equations of gravitation has a fictitious singularity, which in general is not simultaneous. The first terms of the expansion of the metric near such a singularity can be put in the form

$$-ds^2 = -dt^2 + a_{ab}dx^a dx^b + 2(t-\varphi)a_{a3}dx^a dx^3 + (t-\varphi)^2 a_{33}(dx^3)^2 \tag{1.1}$$

(the quantities a_{ab}, a_{a3}, a_{33}, φ are functions of all three space coordinates).

The explanation of the appearance of such a fictitious singularity is that in a synchronous reference system geodesic lines, being coincident with lines of time, must inevitably intersect. It is also possible to construct a synchronous reference system such that in it a singularity of the type (1.1) will be reached simultaneously over all space. In a certain sense such a solution corresponds to a function φ which is zero, but the expansion of the metric near $\varphi = 0$ cannot be obtained by formally going to the limit $\varphi = 0$ in the solution (1.1).

The introduction of matter does not change the qualitative nature of the general solution. This can be seen easily from the fact that the matter moves (in the synchronous system) along world lines that do not coincide with lines of time and in general are not geodesic. The only exception is the case of dustlike matter (equation of state $p = 0$). Matter of this kind moves along geodesic lines. In this case the reference system can be chosen so that it is simultaneously synchronous and comoving—that is, it moves along with the matter. In a comoving system so chosen, however, the lines of time coincide with the world lines. Also the matter density goes into a caustic at infinity, since the world lines are simply focused onto it. It is obvious that such a singularity is in general not entirely a physical one, since, as has been proved earlier,[1,2] it can be removed by introducing an arbitrarily small deviation from zero pressure.

Strictly speaking, however, not every pressure is capable of removing this kind of singularity. We shall show that a situation analogous to that which we have described for dustlike matter will also occur when there is a pressure (that is, for an arbitrary equation of state), if the pressure is constant along the x^3 axis, along which the scale contraction occurs in the metric (1.1). The reason for this can be understood. In fact, if the pressure is constant along the x^3 axis, this means that there are no forces that can move the matter in the x^3t plane. But then it is obvious that a focusing of the world lines occurs in the x^3t plane, and a singularity appears in the energy density of the matter.

This kind of solution, however, is narrower than the general solution, which in the presence of matter contains eight arbitrary functions which depend on three coordinates.[1] An analysis which we shall carry out shows that these solutions contain only two arbitrary functions of the three coordinates, and some number of arbitrary functions of two coordinates.

The fictitious nature of the singularity in solutions of the type (1.1) can be especially simply seen with the example of a particular exact solution, namely

$$-ds^2 = -dt^2 + dx_1^2 + dx_2^2 + t^2dx_3^2. \qquad (1.2)$$

The metric (1.2) can be reduced to the Galilean metric by the simple transformation $t \sinh x_3 = \zeta$, $t \cosh x_3 = \tau$. Thus this particular solution in empty space has a fictitious (geometrical) singularity which is removed by a change to a different reference system.

Recently Kompaneets and Chernov[3] have obtained a particular solution of the gravitation equations in the presence of matter, which in first approximation is analogous to (1.2) and at the same time contains a singularity in the energy density. In the particular solution which they consider the pressure is altogether constant (independent of the coordinates); that is, the situation is that described earlier[1,2] for the case of dustlike matter ($p = 0$), and, as we have already said, it is exactly the same for the case of a constant pressure.

Now let us consider the case of simultaneous collapse [with 0 in the solution (1.1)], and examine the solution near $t = 0$. In the case in which we are interested the metric can be reduced to a form in which the interval is given by the formula

$$-ds^2 = -dt^2 + g_{ab}dx^adx^b + g_{a3}dx^adz + c^2dz^2, \qquad (1.3)$$

and in zeroth approximation

$$g_{ab} = a_{ab}, \qquad g_{a3} = a_{a3}t^2, \qquad c^2 = c_0{}^2t^2, \qquad (1.4)$$

where a_{ab}, a_{a3}, and c_0 are functions of the coordinates only. The reference system is still not completely prescribed by the metric (1.3). There is an arbitrariness which consists in the possibility of a transformation $z \to z(z', x, y)$. Accordingly it must be remembered that in the solution we shall obtain there is one arbitrary function associated with the reference system.

It is, however, extremely cumbersome to analyze the metric in the form (1.3). To simplify the task we first perform the analysis for the metric with $g_{a3} = 0$, and then include the effect of these terms by the method of small perturbations. This procedure is legitimate, since $g_{a3} < (g_{aa}g_{33})^{1/2}$, so that these terms are in fact small. In this situation the metric can be found in zeroth approximation with $g_{a3} = 0$, but in finding the next approximation it is already necessary to include the small corrections caused by the presence of g_{a3}. In the present case an analysis shows that the situation is still more favorable, and in the first two orders in powers of t none of the components of the curvature tensor except R_a^3 will involve terms in the g_{a3}. The particular tensor components R_a^3 will be used to find the first corrections to the zeroth-order values of the coefficients g_{a3}.

Accordingly we write the initial metric for the solution we are concerned with in the form

$$-ds^2 = -dt^2 + g_{ab}dx^adx^b + c^2dz^2 \quad (a, b = 1, 2). \qquad (1.5)$$

We introduce the notation

$$\varkappa_{ab} = g_{ab}, \qquad \lambda_{ab} = g_{ab}', \qquad \gamma = |g_{ab}|, \qquad (1.6)$$

where the dot means differentiation with respect to time, and the prime, with respect to the variable z.

The equations of gravitation, exact for the metric in the form (1.5), then take the following form:

$$R_0{}^0 = \frac{1}{2}(\ln\gamma)^{\cdot\cdot} + \frac{\ddot{c}}{c} + \frac{1}{4}\varkappa_{ab}\varkappa^{ab} = \frac{1}{3}\varepsilon(4u_0u^0 + 1), \qquad (1.7)$$

$$R_3{}^0 = \frac{1}{2}(\ln\gamma)^{\cdot\prime} + \frac{1}{4}\varkappa_{ab}\lambda^{ab} - \frac{\dot{c}}{2c}(\ln\gamma)' = \frac{4}{3}\varepsilon u_3u^0, \qquad (1.8)$$

$$R_a{}^0 = \frac{1}{2}(\ln\gamma)^{\cdot}{}_{;a} - \frac{1}{2}\varkappa_a{}^b{}_{;b} + \frac{\dot{c}_{,a}}{c} - \frac{c_{,b}}{2c}\varkappa_a{}^b = \frac{4}{3}\varepsilon u_au^0, \qquad (1.9)$$

$$R_3{}^3 = -\frac{1}{2c^2}(\ln\gamma)'' - \frac{1}{4c^2}\lambda_{ab}\lambda^{ab} + \frac{(\dot{c}\sqrt{\gamma})^{\cdot}}{c\sqrt{\gamma}} + \frac{1}{2c^2}\frac{\gamma'}{\gamma}\frac{c'}{c} - \frac{c_{;a}{}^{;a}}{c} = \frac{\varepsilon}{3}(4u_3u^3 + 1), \qquad (1.10)$$

$$R_a^3 = \frac{1}{2c^2}\left[-(\ln\gamma)'_{,a} + \lambda_a{}^b{}_{;b} + \frac{c_{,a}}{c}(\ln\gamma)' - \frac{c_{,b}}{c}\lambda_a{}^b\right]$$

$$= \frac{4}{3}\varepsilon u_a u^3, \tag{1.11}$$

$$R_a^b = \frac{1}{2}K\delta_a^b - \frac{1}{2c^2\sqrt{\gamma}}(\sqrt{\gamma}\lambda_a{}^b)' + \frac{1}{2c\sqrt{\gamma}}(c\sqrt{\gamma}\varkappa_a{}^b)^{\cdot}$$

$$+ \frac{c'}{2c^3}\lambda_a{}^b - \frac{c_{;a}{}^{;b}}{c} = \frac{\varepsilon}{3}(4u_a u^b + \delta_a^b). \tag{1.12}$$

In these equations all raising and lowering of indices is done with the two-dimensional tensor g_{ab}; the covariant differentiation is two-dimensional; and K is the scalar two-dimensional curvature. We take the equation of state in the ultrarelativistic form $\epsilon = 3p$.

We prescribe the metric in zeroth approximation in the form

$$g_{ab} = a_{ab}, \qquad c = c_0 t, \tag{1.13}$$

where a_{ab} and c_0 are functions which do not depend on the time. After ascertaining from (1.7)—(1.12) the degree of arbitrariness in the prescription of these functions, we shall find the subsequent terms in the expansion of the metric in a power series in t.

We make the assumption, which will be verified later, that the components of the tensor T_α^β are of higher order in the time than the main terms in the gravitational equations (1.10)—(1.12), i.e., that in the present case the situation is the same as in the anisotropic seven-function solution, where the terms caused by the matter came in only in the determination of corrections to the zeroth-order metric. Then from the requirement that the main terms in (1.10)—(1.12), which are of order $1/t^2$, must be equal to zero we have the conditions

$$\tfrac{1}{4}(\lambda_a{}^a)^2 - \tfrac{1}{4}\lambda_a{}^b\lambda_b{}^a = 0, \tag{1.10'}$$

$$(\lambda_a{}^b/c_0)_{;b} - (\lambda_b{}^b/c_0)_{;a} = 0, \tag{1.11'}$$

$$(\lambda_a{}^b\sqrt{\gamma}/c_0)' = 0. \tag{1.12'}$$

Integrating (1.12′), we get

$$\lambda_a{}^b = c_0 f_a{}^b(x_1, x_2)/\sqrt{\gamma}, \tag{1.14}$$

where f_a^b is an arbitrary function of the two variables (x^1, x^2). We now examine the equation for R_3^0. The main term in this equation is of the order $1/t$. Since, as we shall see, T_3^0 is of higher order $(\sim t^{1/3})$, it follows from (1.8) that

$$\lambda_a{}^a = 0. \tag{1.15}$$

We now show that when we use (1.14) it follows from (1.10′) and (1.15) that all of the components λ_a^b are zero. In fact, from (1.10′) we have

$$f_1{}^2 f_2{}^1 = f_1{}^1 f_2{}^2, \tag{1.16}$$

and from (1.15)

$$f_1{}^1 = -f_2{}^2. \tag{1.17}$$

Besides this there is one further connection which follows from the symmetry condition $f_{12} = f_{21}$:

$$f_1{}^1 a_{12} + f_1{}^2 a_{22} = f_2{}^1 a_{11} + f_2{}^2 a_{12}. \tag{1.18}$$

Eliminating the components f_2^2 and f_2^1 from the relations (1.16)—(1.18), we get the equation

$$a_{11}(f_1{}^1)^2 + 2f_1{}^1 f_1{}^2 a_{12} + (f_1{}^2)^2 a_{22} = 0. \tag{1.19}$$

Since the only physically admissible solutions of the equations of gravitation are those for which $\gamma = |g_{ab}| > 0$, the discriminant of the equation (1.19) is required to have a definite sign:

$$a_{12}{}^2 - a_{11}a_{22} < 0.$$

Then the equation (1.19) has only null solutions

$$f_1{}^1 = f_1{}^2 = 0, \tag{1.20}$$

and it follows from (1.17) and (1.18) that the other two components f_2^2 and f_2^1 are also identically equal to zero. The fact that f_a^b, and consequently also λ_a^b, must be zero means that in zeroth approximation the metric tensor g_{ab} does not depend on the coordinate z; that is, the functions a_{ab} depend only on the variables x^1 and x^2.

Before proceeding now to find the subsequent terms in the expansion of the metric in powers of t, let us determine the nature of the variation with time of the energy ϵ and the velocity components u_α. To do this we use the law of conservation of entropy, which, since $\sigma \sim \epsilon^{3/4}$, can be written for the equation of state $\epsilon = 3p$ in the form

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}(\varepsilon^{3/4}u^i\sqrt{-g}) = 0 \tag{1.21}$$

and the equations of hydrodynamics, which are contained in the gravitational equations,

$$T_i{}^k{}_{;k} = 0, \qquad T_i{}^k = \tfrac{4}{3}\varepsilon u_i u^k + \delta_i{}^k. \tag{1.22}$$

Again making the assumption, to be confirmed by the results, that in Eqs. (1.21) and (1.22) the main terms are those with differentiations with respect to time, we get two conditions [4]:

$$tu_0\varepsilon^{3/4} = \text{const}, \qquad u_\alpha\varepsilon^{1/4} = \text{const}. \tag{1.23}$$

We then use the identity $u_i u^i = -1$. There are two possibilities: either all of the covariant components are of the same order $(u_0^2 \approx u_\alpha u^\alpha)$, or else $u_0^2 \cong 1$. In the former case we find from (1.23) $u_0 \sim 1/t$, $\epsilon \sim$ const. By calculating $T_0^0 \sim \epsilon u_0^2 \sim t^{-2}$ we convince ourselves that this result is in contradiction with (1.7), since R_0^0 is known to be of higher order in t. There remains the second possibility. Then from (1.23) we get

$$\varepsilon = \varepsilon_0 / t^{4/3}, \qquad u_\alpha = u_\alpha^{(0)} t^{1/3}. \tag{1.24}$$

We note that the relation (1.24) does not exclude the possibility that a particular component u_α may have a more rapid decrease with t.

Since we now know the law of variation with time of the components T_0^0 and T_a^b, we easily discover that the expansion of the components of the metric tensor goes in powers of $t^{2/3}$. We therefore write it in the form

$$g_{ab} = a_{ab} + b_{ab} t^{2/3} + \ldots, \tag{1.25}$$

$$c = c_0 t + c_1 t^{5/3}. \tag{1.26}$$

In addition it follows from (1.11) that the component u_3 must decrease with the time at least according to a $t^{5/3}$ law.

In calculating subsequent terms of the expansion of the metric we allow for the fact that in the general case in which we are interested there are nonvanishing components of the metric tensor g_{a3}. As was already pointed out, they can be taken into account as a small perturbation in the equations (1.7)–(1.12). Owing to the fact that in zeroth approximation $\lambda_a^b = 0$, in the first approximation (after the zeroth) corrections coming from the g_{a3} appear only in the equations (1.11); in the left member there is an additional term in R_a^3

$$\frac{1}{c\sqrt{\gamma}} (\varkappa_a{}^3 \sqrt{\gamma} c)^{\cdot}. \tag{1.27}$$

Writing the components g_{a3} in the form

$$g_{a3} = t^2 (a_{a3} + b_{a3} t^{2/3} + \ldots), \tag{1.28}$$

we find that in first approximation the additional term (1.27) in R_a^3 is equal to

$$\frac{1}{3c_0^2 t^{1/3}} \left\{ a_{a3} \left(b_c^c - \frac{2c_1}{c_0} \right) + \frac{8}{3} b_{a3} \right\}. \tag{1.29}$$

This term is of the same order as the first nonvanishing terms in (1.11).

Substituting (1.25) and (1.26) in the equations (1.7)–(1.12), we find the conditions which determine the quantities $u_a^{(0)}$ and b_{a3}. In this way we find

$$R_0^0 = -\frac{1}{9} b_a^a + \frac{10}{9} \frac{c_1}{c_0} = -\varepsilon_0, \tag{1.7″}$$

$$R_3^0 = -\frac{1}{6} b_a^{a\prime} = 0, \tag{1.8″}$$

$$R_a^0 = (\ln c_0)_{,a} = \frac{4}{3} \varepsilon_0 u_a^{(0)}, \tag{1.9″}$$

$$R_3^3 = \frac{10}{9} \frac{c_1}{c_0} + \frac{1}{3} b_a^a + \frac{1}{2c_0} \left(b_a^{a\prime} \frac{1}{c_0} \right)' = \frac{\varepsilon_0}{3}, \tag{1.10″}$$

$$R_a^3 = \left(\frac{b_a^{b\prime}}{c_0} \right)_{;b} - \left(\frac{b_b^{b\prime}}{c_0} \right)_{;a} + \frac{1}{3c_0^2} \left\{ a_{a3} \left(b_c^c - \frac{2c_1}{c_0} \right) + \frac{8}{3} b_{a3} \right\} = 0, \tag{1.11″}$$

$$R_a^b = \frac{2}{9} b_a^b - \frac{1}{2c_0\sqrt{\gamma}} \left(\frac{b_a^{b\prime}\sqrt{\gamma}}{c_0} \right)' = \frac{1}{3} \varepsilon_0 \delta_a^b. \tag{1.12″}$$

Using (1.8″), we find from (1.7″) and (1.10″)

$$b_a^a = \varepsilon_0 (x^1, x^2), \tag{1.30}$$

$$c_1 / c_0 = -{}^3/_5 \varepsilon_0 (x^1, x^2). \tag{1.31}$$

The quantity $u_a^{(0)}$ can be determined from (1.9″)

$$u_a^{(0)} = {}^3/_4 (\ln c_0)_{,a} / \varepsilon_0. \tag{1.32}$$

From (1.11″) we find the quantity b_{a3}

$${}^8/_3 b_{a3} = -c_0^2 (b_a^{b\prime} / c_0)_{;b} - {}^1/_3 a_{a3} (b_c^c - 2c_1 / c_0). \tag{1.33}$$

The components of the tensor b_a^b are determined by (1.12″), from which it can be seen that these quantities are not arbitrary functions of all three coordinates. In particular, Eqs. (1.11″) and (1.12″) are automatically satisfied in the case in which $b_a^{b\prime} = 0$. We then have

$$b_a^b = {}^3/_2 \varepsilon_0 \delta_a^b. \tag{1.34}$$

If we regard Eq. (1.12″) as a differential equation in the variable z, we can still find functions of two variables on which the quantity b_a^b will depend.

In this way we can convince ourselves that the metric in question contains only three arbitrary functions of the three coordinates: c_0 and a_{a3}. Furthermore one function depends on the choice of reference system. Consequently there remain only two physical functions which fix the solution in the presence of matter. In the general solution, on the other hand, there must be eight such functions.[1]

According to Eq. (1.24), in the solution we have obtained there is an essential singularity in the energy density, although in the null metric the singularity is obviously of a geometrical nature. An important point for the understanding of this situation is that the energy density and the pressure do not depend on the coordinate z. Therefore no force along the z axis acts on a particle of the matter. The velocity u_3 of the particle along this axis goes to zero more rapidly than the components in other directions. Thus the particles are at rest, as it were, on lines t = const in the tz plane, in the absence of forces that could give them a velocity along z. In other words, in this case the world lines coincide with the geodesic lines. As a result of the change of the metric with time the particles of matter will finally be focused for $t \to 0$, since all distances along z are rapidly contracting. Thus in the tz plane we have precisely the sort of situation that would occur for dustlike matter ($p = 0$) in a synchronous reference system. In this case also the dustlike matter could be focused, as has been shown earlier,[2] and lead to the appearance of a physical singularity.

In conclusion we note that the particular solution we have described will obviously be unstable, and the presence of a pressure gradient along the z axis inevitably leads to defocusing and the destruction of the singularity (cf.[2]).

An analysis analogous to this can be made for the case of the equation of state $p = 0$ (dustlike matter). As has already been pointed out, in this case the physical singularity coincides with the coordinate singularity, since when there are no forces ($p = 0$) the world lines of the particles coincide with the geodesic lines. The expansions of all functions near the singularity go in integer powers of t, and the energy has a singularity of the form

$$\varepsilon = \varepsilon_0 / t. \qquad (1.35)$$

2. ELECTROMAGNETIC FIELDS AND ANISOTROPIC COLLAPSE

In a paper by E. Lifshitz and the writer[4] a broad class of solutions of the gravitational equations has been found, in which there is a physical singularity with respect to time (anisotropic collapse). This type of solution is described in a synchronous reference system ($g_{00} = -1$, $g_{0\alpha} = 0$) by a metric tensor

$$g_{\alpha\beta} = \sum t^{2p_i} l_\alpha^{(i)} l_\beta^{(i)} \qquad (\alpha, \beta = 1, 2, 3). \qquad (2.1)$$

The three three-dimensional vectors $l^{(i)}$, and also the exponents p_i, are functions of the three space coordinates. The exponents p_i satisfy two relations:

$$\sum p_i = \sum p_i^2 = 1. \qquad (2.2)$$

They run through values in the ranges

$$-1/3 < p_1 < 0, \quad 0 < p_2 < 2/3, \quad 2/3 < p_3 < 1.$$

When the conditions (2.2) are satisfied all of the components of the gravitational equations, except $R_\alpha^0 = 0$, are automatically satisfied by the metric (2.1). The equations $R_\alpha^0 = 0$ give three conditions connecting the vectors $l^{(i)}$. In addition, in the general case with arbitrary exponents p_i there is one further condition on the function $l^{(1)}$:

$$\mathbf{l}^{(1)} \operatorname{rot} \mathbf{l}^{(1)} = 0. \qquad (2.3)^*$$

A solution of the form (2.1) exists both in empty space and in the presence of matter. An interesting question is that of the influence of an electromagnetic field on the nature of this solution. Here there are essentially two questions. One is as to how the solutions of the Maxwell equations behave for a given collapsing metric. In the application to planetary collapse this type of problem was formulated by Ginzburg and Ozernoĭ,[5] who considered the case of the centrally symmetrical collapse of a star with the equation of state $p = 0$ and having a magnetic dipole moment. The other question is that of the possibility of an inverse influence of electromagnetic fields on the nature of the collapse.

We shall answer the second question first, and show that the contribution of the electromagnetic field to the asymptotic behavior for $t \to 0$ is never larger than that of matter with the ultrarelativistic equation of state $\epsilon = 3p$. The electromagnetic field tensor satisfies the equations[6]

$$\frac{1}{\sqrt{-g}} \frac{\partial (F^{ik}\sqrt{-g})}{\partial x^k} = \frac{4\pi}{c} j^i,$$

$$\frac{\partial F_{ik}}{\partial x^l} + \frac{\partial F_{li}}{\partial x^k} + \frac{\partial F_{kl}}{\partial x^i} = 0, \qquad (2.4)$$

where F_{ik} can be expressed in terms of the potential A_i of the field

$$F_{ik} = A_{k;\,i} - A_{i;\,k} = \partial A_k / \partial x^i - \partial A_i / \partial x^k.$$

The current j^i satisfies the equation of continuity

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} (j^i \sqrt{-g}) = 0. \qquad (2.5)$$

We now call attention to the fact that, no matter what may be the mechanism of conductivity or the origin of the electromagnetic field, there is a maximum value of the current which cannot be exceeded. The maximum value of the current is the value that it would have if all of the charges moved with the velocity of the medium. It is obvious that the relation

$$j^\alpha = u^\alpha j^0 / u^0 \qquad (\alpha = 1, 2, 3).$$

holds for the components of such a current. We recall that in the anisotropic solution[1,4]

$$u^\alpha \sim u^0 t^{-p_3}, \qquad \varepsilon \sim t^{-2(1-p_3)}, \qquad u_0 \sim t^{(1-3p_3)/2}. \qquad (2.6)$$

Using (2.6) and the fact that $p_3 < 1$, we note that the main term in the equation of continuity (2.5) is the term with the time derivative. It immediately follows that

*rot = curl.

$$j^0\sqrt{-g} = \text{const}, \qquad (2.7)$$

$$j^0 \sim t^{-1}, \qquad j^\alpha \sim t^{-1-p_3}. \qquad (2.8)$$

Equation (2.8) determines the law of increase with time of the components of the maximum current.

Next, using (2.8), we integrate the Maxwell equations (2.4) and find the laws of variation with time of the components of the electromagnetic field tensor [again the main terms in the equations (2.4) are those containing time derivatives]. We have

$$\frac{1}{t}\frac{\partial F^{0\alpha}t}{\partial t} \sim \frac{1}{t^{1+p_3}} \qquad (2.9)$$

and from this,

$$F^{\alpha 0} = A^{\alpha 0}t^{-1} + B^{\alpha 0}t^{-p_3}, \qquad (2.10)$$

where $A^{\alpha 0}$ and $B^{\alpha 0}$ are functions of the coordinates.

Lowering the indices, we find the covariant components of the electromagnetic field tensor

$$F_{\alpha 0} = A_{\alpha 0}t^{-1+2p_3} + B_{\alpha 0}t^{p_3}. \qquad (2.11)$$

We recall that the exponents p_i in the anisotropic solution are functions of the coordinates.

From the equations (2.4) we can now find the components $F_{\alpha\beta}$

$$F_{\alpha\beta} = A_{\alpha\beta}t^{2p_3}\ln t + B_{\alpha\beta}t^{1+p_3}\ln t. \qquad (2.12)$$

According to (2.4) the functions $A_{\alpha\beta}$ and $B_{\alpha\beta}$ can be expressed in terms of the functions $A_{\alpha 0}$ and $B_{\alpha 0}$; for example,

$$p_3A_{\alpha\beta} + A_{0\alpha}(p_3)_{,\beta} + A_{\beta 0}(p_3)_{,\alpha} = 0. \qquad (2.13)$$

Equations (2.11) and (2.12) determine the main terms in the time dependence of the components of the electromagnetic field tensor for the maximum possible current.

We can now calculate the law of variation with time of the components of the energy-momentum tensor T_i^k of the electromagnetic field:

$$T_i^k = (F_{il}F^{kl} - {}^1/_4\delta_i^kF_{lm}F^{lm}) / 4\pi. \qquad (2.14)$$

According to (2.11) and (2.12) we have

$$T_0^0 \sim F_{\alpha 0}F^{\alpha 0} \sim t^{-2(1-p_3)}, \quad T_\alpha^0 \sim t^{-1+2p_3}, \quad T_\alpha^\beta \sim t^{-2(1-p_3)}. \qquad (2.15)$$

But the energy ϵ varies with time according to precisely the same kind of law as T_0^0 and T_α^β. Accordingly, the electromagnetic field gets inscribed into the anisotropic solution without changing its character, and all of the functions that characterize the electromagnetic field come in with the arbitrariness inherent in them.

It is entirely obvious that the free electromagnetic field will also behave in an analogous way. This can be seen from the fact that the equation (2.9) without its right member has the solution

$$F^{\alpha 0} \sim t^{-1}$$

and thus all further calculations for the free field reduce to the previous results. This was indeed to be expected, since a free field must be described by the equation of state $\epsilon = 3p$, and consequently all asymptotic forms for the free electromagnetic field must coincide with the analogous forms for matter with the ultrarelativistic equation of state.

Let us now consider other possible conductivity mechanisms of collapsing matter. Suppose it is ideally conducting. As is well known, the current is connected with the conductivity λ in a relativistically invariant theory by the relation

$$j^i = \lambda F^{ik}u_k. \qquad (2.16)$$

For infinite conductivity ($\lambda \to \infty$) we get from this the condition of being "frozen in," which is well known in magnetohydrodynamics,

$$F^{ik}u_k = 0. \qquad (2.17)$$

We use the relation (2.6), which gives

$$u_0 \sim u_\alpha t^{-p_3}.$$

We get from (2.17) a connection between the components of the electromagnetic field tensor

$$F_{\alpha\beta} \sim F_{\alpha 0}t^{p_3}. \qquad (2.18)$$

We then find from the Maxwell equations (2.4) and (2.17) the law of variation of the components F_{ik} with time, for the given metric of the anisotropic solution,

$$F_{\alpha\beta} = A_{\alpha\beta} + B_{\alpha\beta}t^{1-p_3}\ln t, \qquad (2.19)$$

$$F_{\alpha 0} = A_{\alpha 0}t^{-p_3}, \qquad (2.20)$$

where $A_{\alpha\beta}$, $B_{\alpha\beta}$, and $A_{\alpha 0}$ are functions of the coordinates connected by the relation

$$(1 - p_3)R_{\alpha\beta} - A_{0\alpha}(p_3)_{,\beta} - A_{\beta 0}(p_3)_{,\alpha} = 0. \qquad (2.21)$$

Comparing (2.19) and (2.20) with (2.11) and (2.12), we can verify that in the case of ideal conductivity the components of the tensor F_{ik}, and also the components of the current j^i, change more rapidly than is allowed by the maximal asymptotic laws. Therefore even if at some earlier stage of the collapse there is ideal conductivity, as t decreases we shall finally reach the greatest allowable value of the current, and its further variation with time will occur according to the law (2.8). Because of this there is no point in examining the inverse effect of the electromagnetic field on the collapse in the case of ideal conductivity.

Finally, let us consider one further possible mechanism for the development of an electromagnetic field in the collapsing matter. What we have in mind is the relativistic analog of the Batchelor mechanism of the production of a spontaneous field in the turbulent motion of a conducting fluid. In the nonrelativistic case, in the turbulent motion of a conducting fluid a magnetic field **H** arises which is proportional to curl **v** (**v** is the velocity). It is easy to show that the relativistic analog of this relation is of the following form:

$$F_{ik} \sim \frac{\partial w u_i}{\partial x^k} - \frac{\partial w u_k}{\partial x^i}. \qquad (2.22)$$

(w is the thermal function[7]). Thus in the relativistic case there is not only a spontaneous magnetic field, but also a spontaneous electric field. According to (2.6) the law of increase of these fields with the time in anisotropic collapse is

$$F_{\alpha 0} \sim \frac{\partial}{\partial t} w u_\alpha \sim t^{-1-3(1-p_3)/2}, \qquad (2.23)$$

$$F_{\alpha\beta} \sim t^{-3(1-p_3)/2} \ln t. \qquad (2.24)$$

In this case, just as in the preceding one, the law of increase of the components of the tensor F_{ik} with the time is more rapid than is allowed by the maximum current. Therefore the electromagnetic fields produced by the Batchelor mechanism must change, as t decreases further, from the time law (2.23), (2.24) to the formulas (2.11) and (2.12) which are the maximum asymptotically allowable.

In conclusion I would like to express by deep gratitude to E. M. Lifshitz for his constant interest and valuable comments, and also to V. L. Ginzburg and A. S. Kompaneets for stimulating discussions.

[1] E. M. Lifshitz and I. M. Khalatnikov, UFN 80, 391 (1963), Soviet Phys. Uspekhi **6**, 495 (1964).

[2] Lifshitz, Sudakov, and Khalatnikov, JETP 40, 1847 (1961), Soviet Phys. JETP **13**, 1298 (1961).

[3] A. S. Kompaneets and A. Chernov, JETP 47, 1939 (1964), Soviet Phys. JETP **20**, 1303 (1965).

[4] E. M. Lifshitz and I. M. Khalatnikov, JETP 39, 800 (1960), Soviet Phys. JETP **12**, 558 (1961).

[5] V. L. Ginzburg and L. M. Ozernoĭ, JETP 47, 1030 (1964), Soviet Phys. JETP **20**, 689 (1965).

[6] L. D. Landau and E. M. Lifshitz, Teoriya polya (Field Theory), Fourth edition, Fizmatgiz, 1962.

[7] I. M. Khalatnikov, JETP 27, 529 (1954).

Translated by W. H. Furry
36