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ON THE THEORY OF BREMSSTRAHLUNG OF SLOW ELECTRONS ON ATOMS

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The bremsstrahlung of slow electrons on neutral systems is considered. A graph technique is developed which allows one to take account of radiation from the atom as well as from the incident particle. If the elastic scattering cross section is weakly dependent on energy, the results are an extension of known results. A comparison is made for the intensity of the radiation from a gaseous plasma in the infrared region.

At present there exists no satisfactory theory of the bremsstrahlung of slow electrons on neutral atoms. Attempts were made to relate the bremsstrahlung cross section to the elastic scattering cross sections.[1-5] But here the latter was either defined in Born approximation,[1] or only s-scattering was taken into account,[2,4,5] which is completely insufficient in a number of cases. Firsov and Chibisov [5] tried to take account of the radiation from the atom by classical methods, which are, however, not applicable to this process.

In the present paper we investigate the bremsstrahlung from the system slow electron-neutral atom. In the discussion of this problem it is natural to assume that 1) the atoms (molecules) are in the ground state, since the electron energies of interest to us are not sufficient for excitations; and 2) for the same reason the interaction of the charge with the quantized field can be described in the dipole approximation. Here it turns out that the contribution to the radiation from the free particle is mainly determined by the elastic scattering amplitude $f_{a_0a_0}(\mathbf{k}_0, \mathbf{k})$ off the energy shell, where a_0 is the set of quantum numbers describing the neutral system in the ground state, and $\mathbf{k}_0$ and $\mathbf{k}$ are the wave vectors of the free electron before and after the radiation. Estimates of the contribution to the radiation from the atom, with the scattering treated in Born approximation, give a value which is smaller by two orders of magnitude than the radiation from the electron.

1. GRAPH TECHNIQUE

The interaction of the electron and the neutral atom with each other and with the radiation field is conveniently described in the language of Feynman graphs. For a slow particle one must sum over the whole perturbation series for the scattering, since

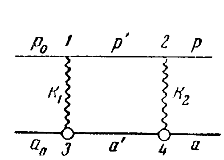

FIG. 1

the probability for multiple scattering is larger than for single scattering. The proposed graph technique is close to the technique developed by Amado[6] for the problem of neutron-deuteron scattering. We shall represent the atom (molecule) by a heavy external line which describes the propagation of the atom as a whole; the index a near the line denotes the quantum numbers for the corresponding state. The free particle is represented by a thin line with momentum p. For definiteness, we shall in the following talk about an atom.

As an example, we consider the graph describing the scattering of an electron with momentum $\mathbf{p}_0$ on an atom in the ground state in second Born approximation (Fig. 1). The vertices 1 and 2 correspond to the point charge of the electron, the vertices 3 and 4 to the form factor of the atom, $F_{a_0a}(K)$, which is equal to

$$F_{a_0a}(K) = e\sum_{s=1}^{Z}\int e^{i\mathbf{K}\mathbf{r}_s}\Psi^*_a\Psi_{a_0}d\tau, \qquad (1)$$

where $\mathbf{K} = \mathbf{k} - \mathbf{k}_0$; Ψ_{a_0} and Ψ_a are the wave functions of the atom in the initial and final states; the summation goes over all atomic electrons. The wavy line corresponds to the Green's function of the virtual photon $4\pi/K^2$. The two parallel internal lines represent the Green's function for the non-interacting electron-atom system. As a result we find for the matrix element

$$M_{a_0a}(k_0, k) = e^4\sum_{a'}\int\frac{d\mathbf{k}'}{(2\pi)^3}\frac{4\pi}{K_1^2}\frac{4\pi}{K_2^2}\frac{F_{a_0a'}(K_1)\,F_{a'a}(K_2)}{E_0-E'},$$

$$E_0 = E_{a_0} + p_0^2/2m,$$

$$E' = E_{a'} + p'^2/2m, \quad K_1 = k' - k_0, \quad K_2 = k - k'. \qquad (2)$$

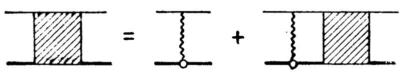

FIG. 2

The sum of all graphs of perturbation theory will be pictured by a graph with a shaded square which corresponds to the exact matrix element defining the scattering amplitude

$$M_{a_0a}(k_0, k) = 2\pi\hbar^2 m^{-1} f_{a_0a}(\mathbf{k}_0, \mathbf{k}). \qquad (3)$$

Figure 2 shows the graphic representation of the Lippman-Schwinger equation for the scattering amplitude off the energy shell.[7]

In the case when electron exchange is essential, the graphs are modified: the atomic form factor of (1) is replaced by the corresponding expression in the Born-Oppenheimer approximation.[8] The kernel in the integral equation (3) is changed accordingly. In the following we shall not specify the form of the function $F_{aa'}(K)$, assuming that the exchange is included. We note that, instead of the usual procedure of iterating (3) according to Born, we may use the unitarity relation which expresses the imaginary part of the amplitude in n-th order perturbation theory through the $n-1$ st order amplitudes determined earlier, and then establish the entire amplitude in n-th order by application of the dispersion relation. This method has been used successfully in quantum electrodynamics,[9] and it does not appear superfluous to us to call attention to it in the theory of atomic collisions.

If the interaction with the quantized radiation field is included, new graphs occur in which the real photon is represented by a dotted line. In lowest order perturbation theory in the interaction with the radiation field, the graphs of interest to us are shown in Fig. 3. The graphs 1 to 3 correspond to radiation by the free particle, and the graphs 4 to 6 to radiation by the atom. An estimate of the contributions from these graphs is given in the following section.

2. CROSS SECTION OF THE BREMSSTRAHLUNG PROCESSES

Let us assume that the largest contribution to the bremsstrahlung comes from graphs 1 and 2 of Fig. 3. We shall see in the following when this assumption is valid. In this case the differential cross section for the bremsstrahlung with emission of a photon with frequency in the interval $d\omega$ and a wave vector with direction in the interval dO_γ has the form[10]

$$d\sigma^{(1,2)} = \frac{2\pi m^2\omega^2 k\, d\omega\, dO_k\, dO_\gamma\Omega}{\hbar^4 c^3 k_0 (2\pi)^6} \left| A^{k_0k}_{a_0a_0} \right|^2, \qquad (4)$$

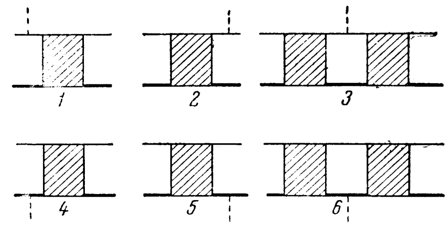

FIG. 3

where $\mathbf{k}_0$ and $\mathbf{k}$ are the wave vectors of the electron before and after the radiation; dO_k is the element of the scattering angle of the electron, Ω is the normalization volume, and m is the electron mass. The quantity $A^{k_0k}_{a_0a_0}$ is the sum of the matrix elements for the graphs 1 and 2 of Fig. 3 in the dipole approximation:

$$A^{k_0k}_{a_0a_0} = \frac{e\hbar}{m}\sqrt{\frac{2\pi\hbar}{\omega\Omega}}\,\frac{M(k_0, k)}{\hbar\omega}(\mathbf{e}, \mathbf{k}_0 - \mathbf{k}). \qquad (5)$$

Here $m(k_0, k)$ is the matrix element corresponding to Fig. 2, and $\mathbf{e}$ is the polarization vector of the photon. The quantity $M(k_0, k)$ can be written in the form

$$M(k_0, k) = \int e^{-i\mathbf{kr}}\Psi^*_{a_0}(\mathbf{R})\, H_{ae}(\mathbf{r}, \mathbf{R})\, \Psi_{k_0}(\mathbf{r}, \mathbf{R})\, d\mathbf{r}\, d\mathbf{R}. \qquad (6)$$

$\Psi_{k_0}(\mathbf{r}, \mathbf{R})$ is the exact wave function of the electron-atom system, whose interaction is described by the Hamiltonian $H_{ae}(\mathbf{r}, \mathbf{R})$. Expanding $\Psi_{k_0}(\mathbf{r}, \mathbf{R})$ in a complete set of atomic functions, one can show that the main contribution to the integral (6) comes from distances $r \sim r_0$, where r_0 is the "effective" range of the interaction. If atomic polarization effects are unimportant, i.e., if r_0 is of the order of the Bohr radius a_0, then the elastic scattering cross section off the energy shell depends weakly on k_0 and k. In this case it can be set equal to the scattering amplitude on the energy shell. For the time being we shall restrict the discussion to this case.

Integrating (4) over dO_γ and dO_k and summing over the polarizations of the photon, we obtain according to what has been said above

$$d\sigma^{(1,2)} = \frac{2}{3\pi}\frac{\alpha}{m^2c^2}\frac{k}{k_0}\frac{d\omega}{\omega}\int (\mathbf{p}_0 - \mathbf{p})^2 \frac{d\sigma_{el}}{d\Omega}\, d\Omega. \qquad (7)$$

If the photon energy is much smaller than the energy of the free particle, $\hbar\omega \ll E_0$, then $|\mathbf{p}_0| \sim |\mathbf{p}|$ and the integral in (7) is expressed through the transport cross section. For frequencies $\hbar\omega \sim E_0$ expression (7) approaches zero like $\sqrt{E_0 - \hbar\omega}$. Writing the differential cross section $d\sigma_{el}/d\Omega$ as a sum over Legendre polynomials, we can show easily that, if the contribution from the interference terms is small (one of the partial waves predominates over the others), then (7) takes the form

$$d\sigma^{(1,2)} = \frac{4\alpha}{3\pi}\frac{E_0}{mc^2}\sqrt{1-\frac{\hbar\omega}{E_0}}\left(2-\frac{\hbar\omega}{E_0}\right)\sigma_{el}(E_0). \qquad (8)$$

Analogous formulas are found in the literature for the case of s-scattering.[4,5] But as we see from our result, the formula holds not only in this case. In the scattering from oxygen atoms, the main contribution comes from the p wave, starting from the energy 0.5 eV.[11] Here the expression (8) remains valid.

The differential cross section for the radiation corresponding to the graphs 4 and 5 of Fig. 3 is equal to

$$d\sigma^{(4,5)} = \frac{2\pi m^2\omega^2 k\, d\omega\, dO_k\, dO_\gamma \Omega}{\hbar^4 c^3 k_0 (2\pi)^6} \left| B^{k_0 k}_{a_0 a_0} \right|^2, \qquad (9)$$

where $B^{k_0 k}_{a_0 a_0}$ for the sum of graphs 4 and 5 has the form

$$B^{k_0 k}_{a_0 a_0} = -\frac{e}{m}\sqrt{\frac{2\pi\hbar}{\omega\Omega}} \times \sum_{a'}\left[\frac{(\mathbf{ep})_{a_0 a'} M^{k_0 k}_{a' a_0}}{E_{a_0}-E_{a'}-\hbar\omega} + \frac{M^{k_0 k}_{a_0 a'}(\mathbf{ep})_{a' a_0}}{E_{a_0}-E_{a'}+\hbar\omega}\right]. \qquad (10)$$

In the case of graphs 1 and 2, $\langle \mathbf{p} \rangle \neq 0$ for the free particle, and therefore $d\sigma^{(1,2)} \sim d\omega/\omega$ (except for frequencies close to zero, where perturbation theory is inapplicable on account of the infra-red catastrophe). For the atom (graphs 4 and 5) $\langle a_0 | \mathbf{p} | a_0 \rangle = 0$, and neglecting the quantity $\hbar\omega$ compared to $E_{a_0} - E_{a'}$ in the energy denominators in the sum (10) and using

$$(\mathbf{ep})_{if} = im\hbar^{-1}(E_i - E_f)(\mathbf{eR})_{if}, \qquad (11)$$

we obtain

$$B^{k_0 k}_{a_0 a_0} = \frac{e}{m}\sqrt{\frac{2\pi\hbar}{\omega\Omega}}[\langle \Phi_{k a_0} | (\mathbf{eR}) H_{ae} | \Psi^{(+)}_{k_0 a_0} \rangle - \langle \Psi^{(-)}_{k a_0} | H_{ae} (\mathbf{eR}) | \Phi_{k_0 a_0} \rangle] \qquad (12)$$

$\Psi^{(-)}$ is the exact wave function corresponding to converging spherical waves, and Φ is the wave function of the non-interacting system. Since the operators $(\mathbf{eR})$ and H_{ae} commute, (12) vanishes on account of the relation

$$\langle \Phi_f | B H_{ae} | \Psi_i^{(+)} \rangle = \langle \Phi_f | BT | \Phi_i \rangle = \langle \Phi_f | TB | \Phi_i \rangle = \langle \Psi^{(-)} | H_{ae} B | \Phi_i \rangle,$$

where B is an arbitrary operator commuting with H_{ae} and T is the scattering operator ($T = S - 1$).

Retaining the next term in the expansion of (10) in powers of $\hbar\omega/(E_{a_0} - E_{a'})$, we obtain the following estimate of the quantity (9) in Born approximation for M (after integration over angles and summation over polarizations):

$$\frac{d\sigma^{(4,5)}}{d\omega} \sim \frac{8\alpha m\omega^3}{9c^2 E_0} |\beta_{il}|^2 \ln\frac{k_0+k}{k_0-k}, \qquad (13)$$

where

$$\beta_{il} = -e^2 \sum_{a'}\left[\frac{x^i_{a_0 a'} x^l_{a' a_0}}{E_{a_0}-E_{a'}} + \frac{x^l_{a_0 a'} x^i_{a' a_0}}{E_{a_0}-E_{a'}}\right]$$

is the statistical polarization tensor of the atom. In the same approximation the radiation from the free particle can be written in the form

$$\frac{d\sigma^{(1,2)}}{d\omega} = \frac{256\pi^2 m\alpha^3 E_0}{27\omega} |\langle a_0 | r^2 | a_0 \rangle|^2. \qquad (14)$$

It is interesting to note that (13), as a function of the frequency, reaches its maximal value for $\hbar\omega \sim 0.8\, E_0$ and vanishes for $\hbar\omega$ close to zero or E_0, whereas (14) is large for small $\hbar\omega \ll E_0$. Comparing (13) and (14) for $\hbar\omega \sim 0.8\, E_0$, we find that the cross section for the radiation from the atom is at least two orders of magnitudes smaller than for radiation from the free particle if $\beta_{il} \sim e^2\langle r^2 \rangle / I_{a_0}$. Although the true value of β_{il} may differ significantly from this estimate, the value of the scattering cross section [formula (14)] changes simultaneously in order of magnitude like $\pi |\langle r^2 \rangle|^2 / a_0^2$. Since the estimates are based on the Born approximation, it is hardly justified to use a more precise value for β_{il}.

Finally, we give the expression describing the contribution from graphs 3 and 6 of Fig. 3. For graph 3 we have

$$C^{k_0 k}_{a_0 a_0} = -\frac{e\hbar}{m}\sqrt{\frac{2\pi\hbar}{\omega\Omega}} \sum_{a', a''} \int \frac{d\mathbf{k}'}{(2\pi)^3} \frac{M^{k_0 k'}_{a_0 a'}(\mathbf{ek}') M^{k' k}_{a'' a_0}}{(E_0 - E')(E_0 - E'')} \delta_{a' a''}, \qquad (15)$$

where

$$E' = E_{a'} + \hbar^2 k'^2 / 2m,$$

$$E'' = E_{a''} + \hbar\omega + \hbar^2 k'^2 / 2m.$$

By considerations similar to the preceding ones we can estimate this integral in Born approximation for M. The result is that in our case the contribution from (15) can be neglected, since $C^{k_0 k}_{a_0 a}$ can be written as a gradient with respect to $\mathbf{k}$ of a slowly varying quantity,[24] and also because for $\hbar\omega \ll E_0$ the energy denominators in (15) are finite, whereas (5) contains an extra power of the frequency in the denominator.

For the graph 6 we obtain

$$D^{k_0 k}_{a_0 a_0} = -\frac{e}{m}\sqrt{\frac{2\pi\hbar}{\omega\Omega}}\sum_{a', a''}\int\frac{d\mathbf{k}'}{(2\pi)^3}\,\frac{M^{k_0 k'}_{a_0 a'}\,(\mathbf{ep})_{a'a''}\,M^{k'k}_{a'' a_0}}{(E_0 - E')(E_0 - E'')}\,. \tag{16}$$

For the same reasons as in the case of graph 3, the estimate for (16) gives a value which is negligible compared to the contributions from graphs 1 and 2.

3. INTENSITY OF THE RADIATION FROM A GAS. COMPARISON WITH EXPERIMENT

The intensity of radiation from 1 cm^3 of gas can be expressed in the following way:

$$I_\omega d\omega = N_e N_a \hbar\omega\, d\omega \int\limits_{\sqrt{2\hbar\omega/m}}^{\infty} \frac{d\sigma}{d\omega} v f(E)\, dv, \tag{17}$$

where f(E) is the Maxwell distribution function, and N_e and N_a are the concentrations of the electrons and neutral atoms.

Since earlier we considered σ_{el} a slowly varying function, we can take it out from under the integral sign and replace it by some average value. Integrating (17), we have

$$I_\omega d\omega = 10^{-2}\frac{\hbar}{m^{3/2}c^2}\,\tilde{\sigma} N_e N_a \hbar\omega (kT)^{-3/2}\, d\omega\Big[(kT)^2 K_1\Big(\frac{\hbar\omega}{2kT}\Big) + \frac{\hbar\omega}{4} kT K_0\Big(\frac{\hbar\omega}{2kT}\Big)\Big] e^{-\hbar\omega/2kT} \tag{18}$$

where $K_n(x)$ is the modified Bessel function of the second kind, or Basset function.[12]

Taking account of the asymptotic form of $K_n(x)$, we obtain

1) for $\hbar\omega/2kT \ll 1$, $I_\omega = 2\cdot 10^{-2}\hbar m^{-3/2} c^{-2}\tilde{\sigma} N_e N_a (kT)^{1/2}$;

2) for $\hbar\omega/2kT \gg 1$,

$$I_\omega = 4.4\cdot 10^{-3}\hbar m^{-3/2} c^{-2}\tilde{\sigma} N_e N_a (\hbar\omega)^{1/2} e^{-\hbar\omega/kT}.$$

For $\hbar\omega/kT \sim 1$ the result of the integration according to Firsov and Chibisov[5] may differ from (18) by an order of magnitude.

Taking into account that

$$k_\nu = \frac{I_\omega c^2}{4\hbar\nu^3}\left(e^{h\nu/kT} - 1\right), \tag{19}$$

we have for the absorption coefficient

$$k_\nu = 8\cdot 10^{-4}\frac{h}{m^{3/2}c^2}\tilde{\sigma} N_e N_a (kT)^{-3/2}\sinh\Big(\frac{h\nu}{2kT}\Big) \times\Big[(kT)^2 K_1\Big(\frac{h\nu}{2kT}\Big) + \frac{h\nu}{4} kT K_0\Big(\frac{h\nu}{2kT}\Big)\Big]. \tag{20}$$

It is interesting to compare these results with the experimental data.

The contribution from free-free transitions in neutral systems is important in the infra-red region for large concentrations of electrons and neutrals. In air the electrons are mainly formed on account of the ionization of NO molecules, and at large pressures (of the order of 10 atm) the free-free radiation mechanism may play an important role. The first calculations of the quantity I_λ (W/cm^3 sr μ) determined by this mechanism were carried out in [13]. The authors assumed that for T ~ 8000°K and pressures p ~ 30 atm bremsstrahlung on neutral oxygen atoms plays the main role. However, the agreement with experiment[14] which these authors claim, is due to a value of the scattering cross section $\sigma \sim 8\pi a_0^2$ which is too high compared with the experimental data.[15] Actually, the most important contribution to the radiation under these conditions comes from the N_2 molecules.[16] 1)

Figure 4 shows the results of experiment[16] and the calculations with the help of formulas of the type (18) for the quantity I_λ. For comparison, the results of calculations[16] on the basis of formulas analogous to the Kramers formula are also shown. The elastic cross sections for O, N, and N_2 were taken from[15,17,18].

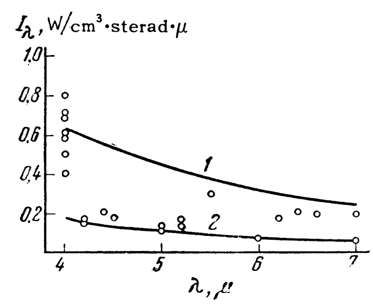

FIG. 4. Intensity of radiation in air. o – experimental points obtained by Taylor,[16] 1 – curve derived from the Kramers formula in[16], 2 – present work.

It seems to us that it is hardly worthwhile to use such formulas for the computation of the process under consideration.

In particular, Taylor[16,19] arrives, on the basis of the Kramers formula, at the wrong conclusion that the contribution from the neutral N atoms to the bremsstrahlung is larger than from the O atoms.

In Fig. 5 we show the results of the measurement of the radiation intensity in a shock wave in pure nitrogen[16] for T = 8000°K and p = 35 atm and also our values computed in the same way as for air.

1)This was pointed out to us by L. M. Biberman.

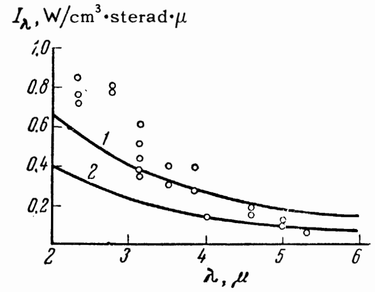

FIG. 5. Intensity of radiation in pure nitrogen. o – experimental points obtained by Taylor,[16] 1 – curve derived from the Kramers formula in[16], 2 – present work.

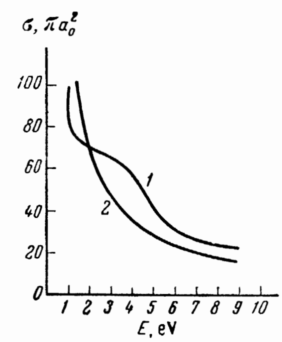

FIG. 6.

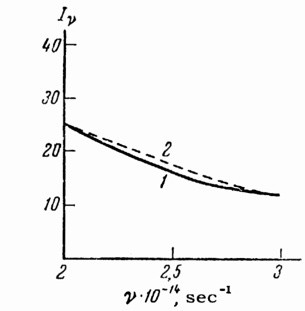

FIG. 7.

FIG. 6. Cross section for elastic scattering of electrons on Hg atoms. 1 – curve from[18], 2 – proposed approximation.

FIG. 7. Intensity of radiation in Hg in relative units 1 – measurement of Rossler,[22] 2 – present work.

Taking into account the inaccuracy of the experimental determination of the elastic scattering cross section and the contribution from other processes (molecular bands, free-free transitions in ionic fields) we may regard the agreement with experiment as satisfactory.

We have also made estimates of the maximum of the emissivity of an oxygen and nitrogen plasma for the temperatures 10 500 and 13 000°K on the basis of the evaluation of the experimental data of Boldt[20] by Biberman and Norman.[21] Under these conditions, the main contribution comes from the recombination radiation and bremsstrahlung on positive ions and also from the radiation due to the formation of negative ions. For the temperature 10 500°K and frequency $\nu \sim 5 \times 10^{14}$ sec^{-1} the emissivity due to the bremsstrahlung is of the order 10^{-10} erg/cm^3 rad for the nitrogen plasma and about twice as large for the oxygen plasma. For T = 13 000°K and the same frequencies the results are three times larger in both cases. These data are not in disagreement with the measured total radiation intensity.

We have made a comparison of the proposed method of calculation with the results of the measurement of the relative intensity from a mercury plasma for T = 7 500°K and frequencies of 2 to 3×10^{14} sec^{-1}.[22] We note that at these energies the scattering cross section can hardly be considered a slowly varying function of energy.[18] Assuming that, nevertheless, formula (18) remains valid, we integrated (17) over the Maxwell distribution, taking for the cross section $\sigma \sim 140\pi a_0^2/E$. This approximation for the cross section is in satisfactory agreement with the data of Massey and Burhop,[18] cf. Fig. 6.

In Fig. 7 we show the results on the frequency dependence of the radiation intensity from a mercury plasma. As normalization value we have taken the value of the intensity at $\nu = 3 \times 10^{14}$ sec^{-1}. We note that the experimental data lie above the computed values in the region $\nu < 2 \times 10^{14}$ sec^{-1}.

CONCLUSION

The proposed method of calculation may be extended to the case of bremsstrahlung on ions. Thus, for example, the account of graphs 1 to 3 of Fig. 3 for the bremsstrahlung of an electron on a proton leads to the Sommerfeld formula.[23]

If polarization effects play an important role in the scattering, the assumption of a slow variation of the scattering amplitude may be wrong. Then the graphs 3 to 6 of Fig. 3 may give a significant contribution. In particular, if a Ramsauer minimum or a sharply defined maximum is observed in the cross section in the energy region of interest, then the contribution of the radiation from the atom may evidently become comparable with the radiation from the free particle. The authors hope to clarify these questions in a future publication.

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40