ACCELERATION OF ELECTRONS IN A PLASMA SITUATED IN A STRONG ELECTRIC FIELD

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We investigate the behavior of a rarefied plasma in a strong uniform and constant electric field. Expressions for the time variations of small plasma perturbations are derived in a linear approximation. The conditions under which initial perturbations (thermal noise) in the plasma practically do not grow with time are determined. Under these conditions, the plasma electrons are accelerated without limit. The possibility of producing a stable strong-current gas betatron using a heavy gas is discussed.

1. INTRODUCTION

In a plasma placed in a strong uniform electric field, satisfying the condition[1] 1)

$$E_0 > E_{cr}^{(1)} \sim 8\pi m\nu_{ei}v_{Te}/e,$$

the velocity acquired by the electron over the free path length greatly exceeds the thermal velocity. Therefore the plasma electrons should be accelerated in such fields without limit. The critical fields, defined by this condition, are quite small. Thus, for $N \sim 10^{12}$ cm^{-3} and $T_e \approx 1$ eV we have $E_{cr}^{(1)} \sim 1$ V/cm. However, such a simplified representation of electron acceleration does not correspond to reality. The point is that in a plasma placed in a strong electric field, under conditions when the directed velocity of the electrons exceeds their thermal velocity, two-stream instabilities develop quite rapidly[2-5] and lead to turbulization of the plasma, so that the acceleration of the electrons is hindered.

V. Shapiro[6] has investigated, on the basis of the adiabatic theory of plasma oscillations in a strong electric field, the electron deceleration due to the development of electrostatic instabilities in a plasma. According to his work, in the absence of an external longitudinal magnetic field the decelerating force in the plasma limits the acceleration of the electrons for practically any value of E_0. On the other hand, in the presence of an external strong magnetic field $\mathbf{B}_0 \parallel \mathbf{E}_0$, when $\Omega_e \gg \omega_{Le}$, in electric fields

$$E_0 > E_{cr}^{(2)} \sim \frac{\sqrt{3}}{2}\left(\frac{m}{2M}\right)^{1/3}\frac{4\pi eNL_\perp}{dX}$$

[$L_\perp$—transverse dimension of the plasma column, $X \sim 10$—20, and $d \sim 1$—2 (for more details see[6])], the decelerating force of turbulent origin is negligibly small and unhindered acceleration of plasma electrons should take place. For $N \sim 10^{11}$—10^{12} cm^{-3} and $m/M \lesssim 10^{-5}$ (plasma in a heavy gas such as Ar, Xe, Kr, Cs or others), we obtain $E_{cr}^{(2)} \sim 10^2$—10^3 V/cm. Such field intensities are presently quite feasible. Thus, even Shapiro's results[6] show the possibility of producing a strong-current plasma accelerator, although this does, to be sure, call for a sufficiently strong longitudinal magnetic field. Thus, for $N \sim 10^{11}$—10^{12} cm^{-3} the required field is $B_0 \gg 10^3$—10^4 Oe.

As already noted above, Shapiro's work[6] is based on the adiabatic approximation. We shall show below that even in relatively weak electric fields E_0 the adiabatic approximation becomes inapplicable, owing to the fast variation of the directed velocity of the electrons. A nonadiabatic analysis of the behavior of a plasma in an electric field shows that unhindered acceleration of plasma electrons can occur both in the presence of a strong longitudinal magnetic field and in its absence. Unhindered acceleration of the electrons is associated with a "jump through" the region where the plasma is unstable relative to potential oscillations[2,5] within such a short time, that the initial thermal disturbances do not have time to grow noticeably and prevent acceleration. Such a "jump through" the instability occurs in electric fields

$$E_0 > E_{cr} \approx eNL_\parallel\sqrt{m/M}, \tag{1.1}$$

where $L_\parallel$ is the longitudinal dimension of the

1) We use the following notation: m – particle mass, e – charge, T – temperature, $v_T = (T/m)^{1/2}$ – thermal velocity, N – density, $\omega_L = (4\pi e^2N/m)^{1/2}$ – Langmuir frequency, $\Omega = eB_0/mc$ – Larmor frequency, $r_D = v_T/\omega_L$ – Debye radius, u – directed velocity. The plasma ions are assumed singly charged, i.e., $e_i = -e$.

plasma pinch. When $N \sim 10^{10}$—10^{11} cm^{-3}, $(M/m)^{1/2} \sim 500$, and $L_{||} \sim 10^2$ cm^{-1} we find from this that the "jump through" the instability region occurs in fields $E_0 \gtrsim 10^2$—10^3 V/cm. This statement pertains to plasma densities $N \lesssim 10^{11}$ cm^{-3}. Apparently we cannot count on accelerating the electrons at larger plasma densities and for a pinch dimension $L_\perp \sim 1$ cm. The point is that the depth of penetration of the quasi-static electric field into the plasma at the initial instant of time is of the order of[2)] c/ω_{Le} (the penetration increases with increasing translational electron velocity). From the requirement that the field be uniform over the cross section of the plasma, it follows therefore that $N \lesssim 3 \times 10^{11}$ cm^{-3}.

It must be noted that the "jump through" the region of two-stream instability of potential plasma oscillations was apparently confirmed experimentally even in relatively weak electric fields ($E_0 \sim 10^2$ V/cm) [7]. The cessation of electron acceleration, observed in this experiment, is due in our opinion to the instability of the spatially-inhomogeneous current-carrying plasma [8].

The instability of the current plasma against non-potential oscillations [4] appears at first glance more dangerous, since the region of such an instability is not "jumped through" with increasing translational electron velocity. It is possible always, however, to indicate conditions when such a two-stream instability in a current plasma cannot develop. When $B_0 = 0$ these conditions take the form [see (2.5) and (2.10)]

$$\frac{T_e}{T_i}\frac{M}{m}\frac{r_{Di}^2}{L_\perp^2} \geqslant \begin{cases} \beta^2/(1-\beta^2), \\ \sqrt{M/m}\,(1-\beta^2)^{1/4}, \end{cases}$$

$$1-\beta^2 \gg (m/M)^{2/5}$$

$$(m/M)^{2/5} \gg 1-\beta^2 \gg (m/M)^{2/3}, \tag{1.2}$$

and when $B_0 \neq 0$

$$(1-\beta^2)^{3/2}\frac{M}{m}\frac{v_A^2}{c^2} > 1. \tag{1.3}$$

Here $\beta = u/c$, and $v_A = (B_0^2/4\pi NM)^{1/2}$ is the ion Alfven velocity.

Conditions (1.2) and (1.3) can be readily satisfied in a plasma with heavy ions in relatively small magnetic fields. Thus, when $M/m \sim (500)^2$, $T_e \sim 1$ eV, and $L_\perp \sim 1$ cm, the first of these conditions is satisfied up to relativistic electron velocities in a plasma with $N \lesssim 10^{11}$ cm^{-3}. The second condition (13), on the other hand, does not depend on the ion mass, and for small electron energies ($\mathscr{E} \sim 1$ MeV), it is satisfied already in fields $B_0 \sim 10^3$ Oe. For the maximally attainable electron energies, when $(1-\beta^2)^{3/2} \sim m/M$, this condition reduces to the form $v_A > c$ and can be satisfied only in very strong magnetic fields $B_0 > 10^4$ Oe.

It follows from all this that electrons with density $N \sim 10^{10}$—10^{11} cm^{-3} can be effectively accelerated in a gas betatron filled with heavy gas to energies ~ 10 MeV, using electric fields $\mathscr{E}_0 \sim 10^2$—10^3 V/cm. The resultant currents reach several kiloamperes.

It must be noted that the problem considered below, that of accelerating electrons in a plasma, pertains strictly speaking to cyclic accelerators, since we have disregarded space-charge effects arising in a linear betatron bounded in the longitudinal direction.

2. ADIABATIC THEORY OF TWO-STREAM INSTABILITY OF A PLASMA IN AN ELECTRON FIELD

Before we proceed with the non-adiabatic analysis of the behavior of a plasma in a strong electric field, we recall the fundamental results of the adiabatic theory. An analysis of these results, pertaining to conditions of plasma instability, greatly facilitates the construction of the nonadiabatic theory. The adiabatic approximation is convenient for the description of oscillations with frequencies considerably exceeding the characteristic times of variation of the directional electron velocity. In the absence of an external magnetic field, the spectra of such plasma oscillations in an electric field are described by the dispersion relation

$$\left(k^2 - \frac{\omega^2}{c^2} + \frac{\omega_{Le}^2 + \omega_{Li}^2}{c^2}\right)\left[1 - \frac{\omega_{Le}^2(1-\beta^2)}{(\omega - \mathbf{uk})^2} - \frac{\omega_{Li}^2}{\omega^2}\right] - \frac{\omega_{Le}^2\omega_{Li}^2}{\omega^2 c^2}\frac{k^2u^2 - (\mathbf{uk})^2}{(\omega - \mathbf{uk})^2} = 0. \tag{2.1}$$

Here $\mathbf{u} = (e/m)\mathbf{E}_0 t$ is the instantaneous value of the translational electron velocity,

$$\omega_{Le}^2 = 4\pi e^2 N_0 m^{-1}(1-\beta^2)^{1/2}, \quad \omega_{Li}^2 = 4\pi e^2 N_0/M,$$

where N_0 is the particle density in the laboratory frame coordinates in which the plasma is regarded as quasi-neutral. Equation (2.1) corresponds to the instant of time when $u \gg v_{Te}$, and to the frequency regions $\omega - \mathbf{u}\cdot\mathbf{k} \gg kv_{Te}$ and $\omega \gg kv_{Ti}$. This has

[2)] In the absence of an external magnetic field, the depth of penetration of a quasi-static electric field can be larger, namely $\approx (c/\omega_{Le})(\omega_{Le}v_{Te}/\omega c)^{1/3} > c/\omega_{Le}$.

enabled us to neglect the thermal motion of the particles.[3)]

In the analysis of Eq. (2.1) we assume that the condition $\omega_{Le}^2(1-\beta^2) \gg \omega_{Li}^2$ is satisfied. This condition will be assumed satisfied also in the nonadiabatic theory. It leads to a limitation of the maximum energy of the electrons to a value $\mathscr{E}_{max} < mc^2(M/m)^{1/3} \sim 30$ MeV [for the case of a heavy gas, when $M/m \sim (500)^2$]. We note, however, that this limitation does not mean at all that it is impossible to accelerate electrons to larger energies, and determines only the region of applicability of the formulas obtained below.

It is easy to show that, assuming $\omega_{Le}^2(1-\beta^2) \gg \omega_{Li}^2$ in the frequency region $\omega \sim \mathbf{u}\cdot\mathbf{k}$, Eq. (2.1) has only stable solutions (i.e., all its roots are real). Unstable solutions are possible in the frequency regions $\omega \ll \mathbf{u}\cdot\mathbf{k}$ and $\omega \gg \mathbf{u}\cdot\mathbf{k}$.

In the first of these regions we obtain from (2.1) the following plasma-oscillation spectrum:

$$\omega^2 = \omega_{Li}^2\left[1+\frac{k^2u^2-(\mathbf{uk})^2}{k^2c^2+\omega_{Le}^2}\frac{\omega_{Le}^2}{(\mathbf{uk})^2}\right]\left(1-\frac{\omega_{Le}^2(1-\beta^2)}{(\mathbf{uk})^2}\right)^{-1/2},$$

$$\omega = \frac{1\pm i\sqrt{3}}{2}(\mathbf{uk})\left\{\frac{\omega_{Li}^2}{2\omega_{Le}^2(1-\beta^2)}\times\left[1+\frac{k^2u^2-(\mathbf{uk})^2}{k^2c^2+\omega_{Le}^2}\frac{\omega_{Le}^2}{(\mathbf{uk})^2}\right]\right\}^{1/3}. \qquad (2.2)$$

The second of the solutions (2.2) is valid in a narrow resonant region, defined by the inequalities

$$1-2\left|\frac{\omega}{\mathbf{uk}}\right| < \frac{\omega_{Le}^2(1-\beta^2)}{(\mathbf{uk})^2} < 1+2\left|\frac{\omega}{\mathbf{uk}}\right|. \qquad (2.3)$$

The first solution (2.2) determines the spectrum of the plasma oscillations outside this region. Recognizing that $|\omega/\mathbf{u}\cdot\mathbf{k}| \sim (m/M)^{1/3} \ll 1$, we can disregard the region (2.3) in the nonadiabatic limit. The point is that the plasma electrons "jump through" the region of instability defined by the first solution of (2.2) within a time that is $(M/m)^{1/3}$ times larger than for the region defined by the inequalities (2.3). At the same time, the growth increment of the oscillations in the first of these regions is only $(M/m)^{1/6}$ times as small as that of the oscillations in the second resonant region. Therefore, in the nonadiabatic limit, if "jump through" the first region of instability takes place, the narrow resonant instability region presents no danger whatever. Taking the foregoing into account, we assume that in the nonadiabatic limit the oscillations grow only in the first region, where $(\mathbf{u}\cdot\mathbf{k})^2 < \omega_{Le}^2(1-\beta^2)$. As soon as $(\mathbf{u}\cdot\mathbf{k})^2 > \omega_{Le}^2(1-\beta^2)$, the oscillations cease to grow and the plasma becomes stable with respect to the oscillations in question. Thus, it is necessary to "jump through" the region of instability within a time in which the amplitudes of the initial thermal perturbations do not have time to grow noticeably.

Finally, we note that formula (2.2) is valid under conditions when the magnetic self-field of the current $H_\perp$ can be neglected. From the inequality

$$\mathbf{uk} \gg eH_\perp/mc \sim \omega_{Le}^2c^{-2}ur_0$$

it follows that this can be done in the instability region at distances

$$r < r_0 \sim k_zc^2/\omega_{Le}^2 \sim L_\parallel^{-1}c^2/\omega_{Le}^2$$

from the pinch axis.

We now consider the frequency region $\omega \gg \mathbf{u}\cdot\mathbf{k}$, corresponding to oscillations propagating almost transversely to the translational velocity of the electron[4]. In this frequency region Eq. (2.1) has unstable solutions under the condition $\omega^2 \ll \omega_{Le}^2$, with

$$\omega^2 = \begin{cases} -\dfrac{\omega_{Li}^2}{1-\beta^2}\dfrac{k^2u^2}{k^2c^2+\omega_{Le}^2}, & \omega^2 \ll \omega_{Le}^2(1-\beta^2) \\ -\omega_{Li}^2\left[\dfrac{\omega_{Le}^2}{\omega_{Li}^2}\dfrac{k^2c^2}{k^2c^2+\omega_{Le}^2}\right]^{1/2}, & \omega_{Le}^2 \gg \omega^2 \gg \omega_{Le}^2(1-\beta^2) \end{cases} \qquad (2.4)$$

The first of these solutions corresponds to the electron energy region $\mathscr{E} \ll mc^2(M/m)^{1/5}$, and the second to the region $(M/m)^{1/5} \ll \mathscr{E}/mc^2 \ll (M/m)^{1/3}$ (ultrarelativism). From (2.4) we see that the growth increment of the oscillations increases with increasing directed velocity of the electrons, and ω^2 has an upper limit $\omega^2 \lesssim \sqrt{M/m}\,\omega_{Li}^2$. It is important that no "jump through" the instability region takes place here, unlike in the frequency region $\omega \ll \mathbf{u}\cdot\mathbf{k}$ considered above. It follows therefore that the oscillations in question can build up only in the nonadiabatic limit.

We note, however, that neglect of the thermal motion of the particles $\omega^2 \gg k^2v_{Te}^2$ imposes limitations on the plasma parameters under which such an instability is possible

[3)] It must be noted that in (2.1) we have neglected completely the directed motion of the ions. This is legitimate if $u_i = eE_0t/M \ll v_{Te}$. When $M/m \approx (500)^2$ and $T_e \approx 1$ eV, this yields as an upper limit for the energy of the accelerated electrons $\mathscr{E} < (M/m)mc^2v_{Te}/c \approx 200$ MeV.

$$\frac{\beta^2}{1-\beta^2} \gg \frac{T_e}{T_i}\frac{M}{m}k^2 r_{Di}^2\left(1+\frac{\omega_{Le}^2}{k^2c^2}\right) \text{ for } 1-\beta^2 \gg \left(\frac{m}{M}\right)^{2/5},$$

$$\sqrt{1-\beta^2} \gg \frac{M}{m}\frac{T_e^2}{T_i^2}k^4 r_{Di}^4\left(1+\frac{\omega_{Le}^2}{k^2c^2}\right) \text{ for }$$

$$\left(\frac{m}{M}\right)^{2/5} \gg 1-\beta^2 \gg \left(\frac{m}{M}\right)^{2/3}. \tag{2.5}$$

If conditions (2.5) are violated, then the Landau damping makes the plasma in the electric field stable against the oscillations in question. Recognizing that $k \sim 1/L_\perp$, the condition for such plasma stability can be written in the form (1.2) (we recall that $k^2c^2 \gtrsim \omega^2_{Le}$). The neglect of the self-field of the current, with account of the limitations (2.5), leads to the following conditions for the applicability of (2.4):

$$\frac{c^2}{v_{Te}^2} > \begin{cases} \left(\frac{M}{m}\right)^2\frac{(1-\beta^2)^3}{\beta^2}, & 1-\beta^2 \gg \left(\frac{m}{M}\right)^{2/5}, \\ \frac{M}{m}(1-\beta^2)^{1/2}, & \left(\frac{m}{M}\right)^{2/5} \gg 1-\beta^2 \gg \left(\frac{m}{M}\right)^{2/3} \end{cases} \tag{2.6}$$

We have assumed above that there is no constant magnetic field in the plasma. The presence of an external longitudinal magnetic field greatly affects the spectrum of the possible plasma oscillations. In the general case the adiabatic-approximation dispersion equation for the oscillation spectrum of a plasma in an electric field is very complicated in form (see, for example[5]) and is difficult to analyze. There is no need for this. In the frequency region $\omega - \mathbf{u}\cdot\mathbf{k} \gg \Omega_e$, $\omega \gg \Omega_i$, this equation coincides of course with (2.1), which was obtained in the absence of a magnetic field. It is therefore meaningful to analyze only the region of frequencies $\omega - \mathbf{u}\cdot\mathbf{k} \ll \Omega_e$, thus greatly simplifying the problem.

We begin the analysis of the oscillation equation with the case of potential oscillations having a dispersion equation

$$k_\perp^2\left[1-\frac{\omega_{Le}^2}{(\omega-\mathbf{uk})^2-\Omega_e^2}-\frac{\omega_{Li}^2}{\omega^2-\Omega_i^2}\right] + k_z^2\left[1-\frac{\omega_{Le}^2(1-\beta^2)}{(\omega-\mathbf{uk})^2}-\frac{\omega_{Li}^2}{\omega^2}\right]=0. \tag{2.7}$$

The solutions of this equation in the nonresonant regions of frequencies correspond to unstable oscillations only if $\omega \ll \mathbf{u}\cdot\mathbf{k}$, and are determined by the following expressions: when $k_\perp = 0$

$$\omega^2 = \omega_{Li}^2\left[1-\frac{\omega_{Le}^2(1-\beta^2)}{(\mathbf{uk})^2}\right]^{-1}, \tag{2.8a}$$

when $k_\perp \neq 0$, $\mathbf{uk} \ll \Omega_e$, $\omega \ll \Omega_i$

$$\omega^2 = \omega_{Li}^2\left[1+\frac{k_\perp^2}{k_z^2}\left(1+\frac{\omega_{Li}^2}{\Omega_i^2}\right)-\frac{\omega_{Le}^2(1-\beta^2)}{(\mathbf{uk})^2}\right]^{-1}, \tag{2.8b}$$

when $k_\perp \neq 0$, $\mathbf{uk} \ll \Omega_e$, $\omega \gg \Omega_i$

$$\omega^2 = \frac{k^2}{k_z^2}\omega_{Li}^2\left[1+\frac{k_\perp^2}{k_z^2}\left(1+\frac{\omega_{Le}^2}{\Omega_e^2}\right) - \frac{\omega_{Le}^2(1-\beta^2)}{(\mathbf{uk})^2}\right]^{-1}. \tag{2.8c}$$

We see from these formulas that when $k_\perp = 0$, as in the absence of an external magnetic field, the region of two-stream instability of the plasma is determined by the inequality $(\mathbf{u}\cdot\mathbf{k})^2 \ll \omega^2_{Le}(1-\beta^2)$. On the other hand, in the case when $k_\perp \gg k_z$, the region of two-stream instability of the plasma relative to the potential oscillations is determined by the transverse dimension $k_\perp \sim 1/L_\perp$, namely $u^2k_\perp^2 < \omega^2_{Le}(1-\beta^2)$. In this case the "jump through" the region of instability is attained in much smaller electric fields than for the case when $k_\perp = 0$, and this is the consequence of the narrowing down of the instability by a factor $k_\perp/k_z \sim L_\parallel/L_\perp$. Formulas (2.8) are applicable only in the nonresonant regions of frequencies, when the denominators of these expressions do not vanish. As already noted, the resonant frequency regions are so narrow that they can be disregarded under condition of "jump-through" the nonresonant regions of instabilities. By neglecting the self-field of the current we make formulas (2.8) valid at distances $r < r_0$ from the axis of the plasma pinch (see above).

Let us consider now non-potential plasma oscillations in an electric field, propagating almost transversely to the external magnetic field, i.e., $\omega \gg \mathbf{u}\cdot\mathbf{k}$. Such oscillations are quite dangerous, for the velocity instability regions are not bounded from above with respect to such oscillations. The spectra of such plasma oscillations are determined by the relations[4] 4)

$$\omega^2 = k^2v_A^2\left[1-\frac{mu^2}{Mv_A^2(1-\beta^2)^{3/2}}\right]\left[1+\frac{v_A^2}{c^2}\right]^{-1}$$

when $\omega^2 \ll \Omega_i^2$, $k^2c^2 \ll \omega_{Le}^2(1-\beta^2)$ and (2.9a)

$$\omega^2 = \omega_{Li}^2\left[1-\frac{m}{M}\frac{u^2}{v_A^2}(1-\beta^2)^{-1/2}+\frac{\omega_{Le}^2(1-\beta^2)}{k^2c^2}\times\left(1-\frac{m}{M}\frac{u^2}{v_A^2}(1-\beta^2)^{-3/2}\right)\right]\left[1+\frac{\omega_{Le}^2}{k^2c^2}(1-\beta^2)\right.$$

4)We note that in deriving (2.9) it is necessary to take into account the directed motion of the ions in the plasma.

$$+\frac{m}{M}\frac{c^2}{v_A^2}(1-\beta^2)^{1/2}\left(1+\frac{\omega_{Le}^2}{k^2c^2}\right)\Bigg]^{-1}$$

when $\omega^2 \ll k^2c^2,\ \omega_{Le}^2,\ \Omega_i^2 \ll \omega^2 \ll \Omega_e^2$. (2.9b)

It is seen from these expressions that instability is possible when the directed electron velocities exceed the Alfven velocity, i.e., when $u > v_A\sqrt{M/m}(1-\beta^2)^{3/4}$, and no "jump through" the instability region will take place. However, if

$$(1-\beta^2)^{3/2}\frac{M}{m}\frac{v_A^2}{c^2} > 1, \tag{2.10}$$

this instability cannot develop in the plasma at all. In strong magnetic fields, $B_0 \gtrsim 10^4$ Oe, this condition can be satisfied up to electron energies ~ 10 MeV and plasma densities $N \lesssim 10^{11}\ \mathrm{cm}^{-3}$. We note also that in the nonadiabatic limit it is sufficient to confine oneself in the investigation of the instability in question to the nonrelativistic approximation.

3. NONADIABATIC THEORY OF BEHAVIOR OF A PLASMA IN A STRONG ELECTRIC FIELD

In the preceding section we presented the results of the adiabatic theory of oscillations of a plasma in an electric field, and indicated the conditions under which such oscillations are unstable. If the plasma is situated in a sufficiently strong electric field, then the results of the adiabatic theory are, generally speaking, invalid. The analysis in the section that follows, however, greatly facilitates the construction of nonadiabatic theory of oscillations.

We describe the behavior of a cold plasma in external electric and magnetic fields $E_0 \parallel B_0$ by a system of relativistic equations of two-fluid magnethydrodynamics[9]:

$$\partial n/\partial t + \operatorname{div} n\mathbf{v} = 0,$$

$$\frac{\partial}{\partial t}\frac{\mathbf{v}}{\sqrt{1-v^2/c^2}} + (\mathbf{v}\nabla)\frac{\mathbf{v}}{\sqrt{1-v^2/c^2}} = \frac{e}{m}\left\{\mathbf{E}+\frac{1}{c}[\mathbf{vB}]\right\},$$

$$\operatorname{div}\mathbf{E} = 4\pi\rho, \qquad \operatorname{div}\mathbf{B} = 0,$$

$$\operatorname{rot}\mathbf{B} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}+\frac{4\pi}{c}\mathbf{j}, \qquad \operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} \tag{3.1*}$$

Here v—velocity of any one component of the plasma (electrons or ions), n—particle density, and ρ and j—charge and current densities induced in the plasma by the fields E and B.

The external fields E_0 and B_0 are directed along the z axis. Then the ground state of the plasma is characterized by velocity $\mathbf{u} \parallel \mathbf{z}$:

$$\mathbf{u}/\sqrt{1-\beta^2} = (e/m)\mathbf{E}_0 t. \tag{3.2}$$

The electron velocity increases much faster than the ion velocity. The electrons become ultrarelativistic before the ions can be appreciably accelerated. This enables us to neglect the motion of the ions in the ground state of the plasma. We note that the self-field of the plasma current is also neglected (see above).

We now consider a small deviation of the plasma state from equilibrium, and trace its development in time. The system of equations for small deviations is of the form

$$\left[\frac{\partial}{\partial t}+(\mathbf{u}\nabla)\right]\frac{\mathbf{v}+\mathbf{u}(\mathbf{uv})/c\sqrt{1-\beta^2}}{\sqrt{1-\beta^2}} = \frac{e}{m}\left\{\mathbf{E}+\frac{1}{c}[\mathbf{uB}]+\frac{1}{c}[\mathbf{vB}_0]\right\},$$

$$\partial n/\partial t + \operatorname{div} n\mathbf{u} + \operatorname{div} N_0\mathbf{v} = 0, \quad \operatorname{rot}\mathbf{B} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}+\frac{4\pi}{c}\mathbf{j},$$

$$\operatorname{div}\mathbf{E} = 4\pi\Sigma ne, \qquad \mathbf{j} = \Sigma e(N_0\mathbf{v}+n\mathbf{u}),$$

$$\mathbf{B} = \operatorname{rot}\mathbf{A}, \qquad \mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} - \nabla\Phi, \tag{3.3}$$

where v, E, B, φ, A, and n characterize the deviations of the plasma from the equilibrium state. We shall henceforth use for the ions the nonrelativistic equations.

The solutions of the system (3.3) can be sought, in view of the inhomogeneity of the plasma in space, in the form $f(t)\exp(i\mathbf{k}\cdot\mathbf{r})$. Taking this into account we can obtain from the system (3.3), in the simplest case of potential oscillations ($\mathbf{E} = -\nabla\Phi$, curl $\mathbf{E} = 0$) in the absence of an external magnetic field, the following equation for the time variation of the nonequilibrium ion density:

$$\frac{\partial^2 n_i}{\partial t^2} + \frac{\omega_{Li}^2}{1-\omega_{Le}^2(1-\beta^2)/k^2u_e^2}n_i = 0. \tag{3.4}$$

In the derivation of this equation we have used the results of the adiabatic theory. Namely, we took into account the fact that the potential oscillations of a plasma with current are stable only in the frequency region $\omega \ll \mathbf{u}_e\cdot\mathbf{k}$, and neglected the time variation of the electronic quantities. In addition, we have confined ourselves for simplicity to an examination of waves propagating along the directional velocity ($\mathbf{k} \parallel \mathbf{u}$). As can be seen from (2.1) and (2.2), the presence of the transverse component $k_\perp \neq 0$ does not influence qualitatively the spectrum of the plasma oscillations in the frequency region $\omega \ll \mathbf{u}_e\cdot\mathbf{k}$, and can lead quantitatively to a correction of the order of unity in the spectrum. In the absence of a magnetic field, the waves that

*[v B] ≡ v × B, rot ≡ curl.

can be potential are, strictly speaking, those of the electrons propagating along the beam. As was already noted many times, Eq. (3.4) is not valid in the resonant region of velocities when $k^2u_e^2 \approx \omega_{Le}^2(1-\beta^2)$. However, this region can be disregarded because it is extremely narrow.

At velocities $k^2u_e^2 > \omega_{Le}^2(1-\beta^2)$, Eq. (3.4) has only those solutions which correspond to non-growing plasma oscillations with frequency $\sim\omega_{Li}$. On the other hand, at velocities $k^2u_e^2 < \omega_{Le}^2(1-\beta^2)$, the general solution of (3.4) is written in the form[5)]

$$n_i = t^{1/2}\left\{C_1 I_{1/4}\left(\frac{t}{2\tau_1}\right) + C_2 I_{-1/4}\left(\frac{t}{2\tau_1}\right)\right\} \qquad (3.5a)$$

when $u_e \ll c$ and

$$n_i = t^{1/2}\left\{C_1 I_{1/5}\left(\frac{2}{5}\left(\frac{t}{\tau_2}\right)^{5/2}\right) + C_2 I_{-1/5}\left(\frac{2}{5}\left(\frac{t}{\tau_2}\right)^{5/2}\right)\right\} \qquad (3.5b)$$

when $u_e \approx c$ (ultrarelativism). Here

$$\tau_1 = \left(\frac{m}{M}k^2u_e^2\right)^{-1/2}, \qquad \tau_2 = \left(\frac{m}{M}\frac{k^2}{c}\left(\frac{e}{m}E_0\right)^3\right)^{-1/5},$$

and I_ν is a Bessel function of imaginary argument of order ν. Recognizing that $I_\nu(x) \sim x^\nu$ when $x \ll 1$, we obtain from (3.5)

$$n_i = C_1 + C_2 t \qquad (3.6)$$

regardless of whether $t/\tau_1 \ll 1$ and $u_e \ll c$, or whether $t/\tau_2 \ll 1$ and $u_e \approx c$ (ultrarelativism). During this stage the initial disturbances can increase to only double their value.

On the other hand, for long periods of time ($x \gg 1$), when we can use in (3.5) the asymptotic form of the Bessel function $I_\nu \sim e^x/\sqrt{2\pi x}$, the ion density begins to grow exponentially (in accordance with the adiabatic theory), and

$$n_i \sim \begin{cases} t^{-1/2}e^{t/2\tau_1} & \text{for } u_e \ll c,\ t/\tau_1 \gg 1 \\ t^{-3/4}\exp\{2/5(t/\tau_2)^{5/2}\} & \text{for } t/\tau_2 \gg 1,\ u_e \approx c \end{cases} \qquad (3.7)$$

During this stage, the disturbances increase exponentially in the plasma. Therefore, in order to "jump through" the region of two-stream instability, it is necessary to violate the condition $(\mathbf{k}\cdot\mathbf{u}_e)^2 < \omega_{Le}^2(1-\beta^2)$ before solution (3.5) can be treated asymptotically. From this requirement we obtain the condition for the stability of the plasma in a strong electric field against potential oscillations when $k_{min} \approx 2\pi/L_{||}$:

$$E_0 > eNL_{||}\sqrt{m/M}. \qquad (3.8)$$

Accurate to a factor smaller than or of the order of unity, this condition is obtained in both the non-relativistic ($u_e \ll c$) and ultrarelativistic limit ($u_e \approx c$). As was already noted above, when $N \lesssim 10^{11}$ cm^{-3}, in the case of a sufficiently heavy gas with $\sqrt{M/m} \sim 500$, condition (3.8) can be satisfied in fields $E_0 \sim 10^2$–10^3 V/cm.

Let us consider now disturbances that propagate transversely to the directional velocity of the electrons in the plasma in the absence of an external magnetic field. In the non-relativistic case ($u_e \ll c$), the equation for the time variation of the disturbed quantities is of the form

$$K\chi'' - k^2u_e^2\frac{m}{M}\chi = 0, \qquad K = 1 + \frac{k^2c^2}{\omega_{Le}^2}, \qquad (3.9)$$

where $\chi = v_x/u_e$ and v_x is the deviation of the component of the electron velocity from equilibrium. A solution of this equation is the function

$$\chi = t^{1/2}\left\{C_1 I_{1/4}\left(\frac{t}{2\tau_1 K^{1/2}}\right) + C_2 I_{-1/4}\left(\frac{t}{2\tau_1 K^{1/2}}\right)\right\}. \qquad (3.10)$$

When the argument is larger than unity, the Bessel functions of imaginary argument increase exponentially, and this means that small deviations of the quantities from equilibrium will increase without limit with time, and the instability in this case cannot be "jumped through." At large values of the argument, the function takes the form

$$\chi \sim t^{-1/2}\exp\{t/2\tau_1\sqrt{K}\}. \qquad (3.11)$$

In the ultrarelativistic limit, the equation of small oscillations propagating transverse to the directional velocity of the electrons turns out to be an equation with constant coefficients. The spectrum of unstable oscillations is then described, as in the adiabatic theory, by expression (2.4), corresponding to an aperiodic growth of small disturbances with time. As was already noted in the preceding section, an account of the thermal motion of the electrons (Landau damping), and also of the self-field of the current, narrows down the region of instability against such transverse oscillations, and we can always point to conditions under which they cannot develop in a current-carrying plasma [see (2.5) and (2.6)].

The character of the oscillations of the current plasma can change markedly if the plasma is situated not only in an electric field but also in an external longitudinal magnetic field $\mathbf{B}_0$. We have shown in the preceding section that transverse oscillations in a current-carrying plasma are excited at velocities u_e that exceed the Alfven velocity of the electrons, and the latter can be smaller than the velocity of light. In this connec-

[5)]Analogous solutions for the case of a nonrelativistic electron-positron plasma in an external electric field were obtained in [10].

tion, it is sufficient to confine oneself when investigating such oscillations to an examination of the nonrelativistic case ($u_e \ll c$).

The magnetic field can appreciably influence the processes in a current-carrying plasma only if the cyclotron frequencies exceed in magnitude the relative time variations of the characteristic plasma values. In the opposite case, the magnetic field causes practically no change in the plasma dynamics. In the presence of an external magnetic field, it is possible to separate two main time variation regions in which growing oscillations are possible: a) the relative change of the quantities with time is small compared with the ion cyclotron frequency Ω_i; b) the relative change of the quantities is small compared with Ω_e, ω_{Le} and kc, but large compared with Ω_i.

In case a) we obtain for the component A_y of the vector potential of the field the equation

$$A_y'' + k^2c^2\frac{1 - mu_e^2/Mv_A^2}{1 + c^2/v_A^2}A_y = 0. \tag{3.12}$$

In the derivation of this equation we have assumed that $k^2c^2 \ll \omega_{Le}^2$.

In case b) we obtain for the quantity Ψ, which is connected with the component A_y by

$$A_y = \Psi \exp\left\{\frac{k^2u_e^2}{\omega_{Le}^2 + k^2c^2 + (\Omega_e/\omega_{Le}^2)^2(k^2c^2 + \Omega_i^2)}\right\}$$

the following equation

$$\Psi'' + k^2v_A^2\frac{1 - mu_e^2/Mv_A^2}{K + (\Omega_e/\omega_{Le}^2)^2(k^2c^2 + \Omega_i^2)}\Psi = 0. \tag{3.13}$$

When $1 > mu_e^2/Mv_A^2$, Eqs. (3.12) and (3.13) describe non-growing transverse oscillations. When the opposite conditions $(m/M)v_A^{-2}(eE_0t/m)^2 > 1$ holds true, these equations correspond to instability, and all the quantities characterizing the deviation of the plasma from equilibrium increase exponentially with time. We have in case a)

$$A_y \sim t^{-1/2}e^{t/2\tau_3} \tag{3.14a}$$

and in case b)

$$\Psi \sim t^{-1/2}e^{t/2\tau_4}, \tag{3.14b}$$

where

$$\tau_3 = \left(\frac{m}{M}\frac{k^2u_e^2}{1 + v_A^2/c^2}\right)^{-1/2},$$

$$\tau_4 = \left(\frac{m}{M}\frac{k^2u_e^2}{K + \Omega_e^2\omega_{Le}^{-4}(k^2c^2 + \Omega_i^2)}\right)^{-1/2}$$

The condition $v_A^2 > mc^2/M$, the satisfaction of which ensures stability of a plasma with respect to transverse oscillations, can be readily realized in rather strong yet perfectly feasible magnetic fields.

Let us consider now the potential oscillations of a plasma in the presence of an external magnetic field. As above, we shall analyze two principal regions of time variation of the nonequilibrium quantities: a) the relative change of the quantities with time is small compared with Ω_i; b) the relative change of the quantities is small compared with Ω_e, but large compared with Ω_i. We confine ourselves to an examination of the nonrelativistic limit $u_e \ll c$. We note, however, that the conditions obtained below for "jumping through" the instability region (3.18) remain in force also in the relativistic limit.

In case a) we obtain for the quantity φ, which is connected with the scalar potential of the field Φ by

$$\Phi = \varphi\left[\frac{k_\perp^2}{k_z^2}\left(1 + \frac{c^2}{v_A^2}\right) + 1 - \frac{\omega_{Le}^2}{(\mathbf{ku}_e)^2}\right]^{-1},$$

the equation

$$\left[\frac{k_\perp^2}{k_z^2}\left(1 + \frac{c^2}{v_A^2}\right) + 1 - \frac{\omega_{Le}^2}{(\mathbf{ku}_e)^2}\right]\varphi'' + \omega_{Li}^2\varphi = 0. \tag{3.15}$$

In case b), the equation for the time variation of the ion density is

$$\left[\frac{k_\perp^2}{k_z^2}\left(1 + \frac{m}{M}\frac{c^2}{v_A^2}\right) + 1 - \frac{\omega_{Le}^2}{(\mathbf{ku}_e)^2}\right]n_i'' + \frac{k^2}{k_z^2}\omega_{Li}^2 n_i = 0. \tag{3.16}$$

We see from (3.15) and (3.16) that at small values of the directional velocity u_e, the plasma disturbances increase with time, so that the plasma becomes unstable. Solutions of these equations in the instability region are written in the form

$$\varphi = t^{1/2}\left\{C_1I_{1/4}\left(\frac{t}{2\tau_5}\right) + C_2I_{-1/4}\left(\frac{t}{2\tau_5}\right)\right\},\quad \tau_5 = \left(\frac{m}{M}(\mathbf{ku}_e)^2\right)^{-1/2},$$

$$n_i = t^{1/2}\left\{C_1I_{1/4}\left(\frac{t}{2\tau_5}\left|\frac{k}{k_z}\right|\right) + C_2I_{-1/4}\left(\frac{t}{2\tau_5}\left|\frac{k}{k_z}\right|\right)\right\}. \tag{3.17}$$

With increasing directional velocity the oscillations described by (3.15) and (3.16) become stable when $\mathbf{k}\cdot\mathbf{u} > \omega_{Le}$, i.e., a "jump through" the region of instability of the plasma takes place.

From (3.17) we readily obtain conditions under which the region of instability can be "jumped through" without any noticeable increase in the initial perturbations. For cases a) and b) these conditions are, respectively,

$$E_0 > eNL_\parallel\sqrt{m/M}\Big/\left[1 + \frac{k_\perp^2}{k_z^2}\left(1 + \frac{c^2}{v_A^2}\right)\right], \tag{3.18a}$$

$$E_0 > eNL_\parallel\sqrt{\frac{m}{M}\frac{k^2}{k_z^2}}\Big/\left[1 + \frac{k_\perp^2}{k_z^2}\left(1 + \frac{m}{M}\frac{c^2}{v_A^2}\right)\right]. \tag{3.18b}$$

When $k_\perp = 0$, these conditions go over into (3.8) and are valid also in the relativistic limit.

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