SPONTANEOUS MAGNETIZATION OF A PLANE DIPOLE LATTICE

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A comparatively simple method is presented for calculation of the spontaneous magnetic moment of the two-dimensional Ising model.

THE problem of calculating the partition function of a two-dimensional lattice in an external magnetic field has hitherto not been solved. However, the size of the magnetic moment at a vanishingly small value of the field was found by Onsager. He did not publish his method of solution, and therefore interest in the problem has not diminished. The second to solve this problem was Yang;[1] despite the simplicity of the results, the solution was extraordinarily complicated. Subsequently a number of authors[2,3] succeeded in simplifying it. The present paper presents still another method of calculating the spontaneous magnetic moment; it makes use of the achievements of previous authors, but it is more accessible to the understanding.

We consider a plane dipole lattice (the Ising model). With each site (k, l) of this lattice is associated a function σ_{kl}, which can take on the two values ± 1; M is the mean value of this function at a vanishingly small value of the magnetic field. As is known,[2,4,5] the moment can be expressed in terms of the correlation function for two sites separated a long distance from each other,

$$M^2 = \lim_{t\to\infty} \langle \sigma_{11}\sigma_{1t+1}\rangle. \qquad (1)$$

To calculate $\langle \sigma_{11}\sigma_{1t+1}\rangle$, we use the identity $(\sigma_{kl}^2 = 1)$

$$\sigma_{11}\sigma_{11+t} = \sigma_{11}\sigma_{12}\sigma_{12}\sigma_{13}\ldots\sigma_{1t}\sigma_{1t}\sigma_{11+t}.$$

We introduce the notation

$$S\{\alpha_{kl|k'l'}|\sigma_{kl}\} = \prod_{kl k'l'} (1 + \alpha_{kl|k'l'}\sigma_{kl}\sigma_{k'l'}). \qquad (2)$$

For the Ising model

$$\alpha_{kl|k'l'} = \begin{cases} x & \text{for } k=k',\ l=l'+1 \text{ and } k=k'+1,\ l=l' \\ 0 & \text{otherwise} \end{cases}$$

Here $x = \tanh(J/kT)$. Then according to the definition

$$\langle\sigma_{11}\sigma_{1t+1}\rangle = \langle\sigma_{11}\sigma_{12}\sigma_{12}\ldots\sigma_{11+t}S\rangle / \langle S\rangle. \qquad (4)$$

Substitution of (2) and (3) in (4) gives rise to products $\sigma_{1k}\sigma_{1k+1}(1 + x\sigma_{1k}\sigma_{1k+1})$. These may be written in the form

$$x(1 + x^{-1}\sigma_{1k}\sigma_{1k+1}) = x[1 + (x+g)\sigma_{1k}\sigma_{1k+1}], \qquad (5)$$

where $g = 1/x - x$. On substituting this expression in (4), we get

$$\langle\sigma_{11}\sigma_{11+t}\rangle = x^t\langle S\{\beta_{kl|k'l'};\sigma_{kl}\}\rangle / \langle S\{\alpha_{kl|k'l'};\sigma_{kl}\}\rangle, \qquad (6)$$

where

$$\beta_{kl|k'l'} = \alpha_{kl|k'l'} + g\Delta(kl|k'l'), \qquad (7)$$

$$\Delta(kl|k'l') = \begin{cases} 1 & \text{for } k=1;\quad k'=1;\quad l'=l\pm 1,\quad 1\leqslant l\leqslant t \\ 0 & \text{otherwise} \end{cases} \qquad (8)$$

We note that in the calculation of the partition function (cf. [6]), there is associated with the matrix $\alpha_{kl|k'l'}$ a matrix $a_{kl|k'l'}$ (we denote it by A), which is obtained from $\alpha_{kl|k'l'}$ by multiplying individual elements by $e^{\pm i\pi/4}$. Further, we shall use the formula

$$\langle S\{\alpha_{kl|k'l'};\sigma_{kl}\}\rangle = \mathrm{Det}^{1/2}(1 - A). \qquad (9)$$

In the derivation of this formula it was supposed that $\alpha_{kl|k'l'}$ is different from zero only for nearest neighbors; but the values of $\alpha_{kl|k'l'}$ can actually be arbitrary. Therefore formula (9) is correct not only for the denominator of (6) but also for the numerator. The matrix $b_{kl|k'l'}$ (we denote it by B) is constructed from $\beta_{kl|k'l'}$ in the same way as is $a_{kl|k'l'}$ from $\alpha_{kl|k'l'}$ (cf. [6]) and has the form (cf. [2])

$$B = A + gD,$$

where D comes from Δ in (7) and (8).

We introduce the notation: $\epsilon = e^{i\pi/4}$, $\bar\epsilon = e^{-i\pi/4}$, $\nu = 1, 2, 3, 4$, an index connected with the direction of passage from site to site. In this notation, we get for the matrix elements $\langle kl\nu|D|k'l'\nu'\rangle$ for $1 \le l \le t$

$$\langle 1l\nu|D|1\, l-1\, 1\rangle = (1, \bar{\varepsilon}, 0, \varepsilon),$$

$$\langle 1l\nu|D|1l+1 3\rangle = (0, \varepsilon, 1, \bar{\varepsilon}), \qquad (10)$$

and the remaining matrix elements vanish. Two types of supplementary term occur because of the two ways of passing over the chain of bonds, 1, 2, 3, . . . t + 1.[6] We have

$$\langle kl\nu|D|k'l'\nu'\rangle = \delta_{k1}\delta_{k'1}(\alpha_{\nu\nu'}\delta_{l-1,\, l'} + \beta_{\nu\nu'}\delta_{l+1,\, l'}), \qquad (10')$$

where δ_{ij} is the Kronecker symbol, $\alpha_{11} = 1$, $\alpha_{21} = \bar{\epsilon}$, $\alpha_{41} = \epsilon$, $\beta_{23} = \epsilon$, $\beta_{33} = 1$, $\beta_{43} = \bar{\epsilon}$, and the remaining $\alpha_{\nu\nu'}$'s and $\beta_{\nu\nu'}$'s vanish.

For M^2 we get the expression

$$M^2 = \lim_{t\to\infty} \langle \sigma_{11}\sigma_{11+t}\rangle = \lim_{t\to\infty} x^t \frac{\mathrm{Det}^{1/2}(1-B)}{\mathrm{Det}^{1/2}(1-A)}. \qquad (11)$$

This expression solves the problem of calculating M^2, since the infinite summation over σ_{kl} has been accomplished and the matrices A and B resemble the ordinary operators of mathematical physics.

If we substitute in (11) the following expressions for 1-B,

$$(1-B) = 1 - A - gD = (1-A)[1 - g(1-A)^{-1}D],$$

then instead of (11) we get

$$M^2 = \lim_{t\to\infty} x^t \mathrm{Det}^{1/2}[1 - g(1-A)^{-1}D]. \qquad (12)$$

The matrix A is a shift matrix with constant coefficients. As is known, such matrices are diagonalized by a finite Fourier transformation, a transition from the k, l- representation to a p, q-representation with the matrix $\exp\{2\pi i(kp + lq)/N\}$ (N^2 is the number of sites in the square lattice). To find the matrix $(1 - A)^{-1}$, it is necessary to go over from the k, l- to the p, q-representation. We then get the matrix (cf. [6])

$$1 - A_{\nu\nu'}(p, q)$$

$$= \begin{pmatrix} 1 - x\bar{\alpha}_p & -x\bar{\varepsilon}\bar{\beta}_q & 0 & -x\varepsilon\beta_q \\ -x\varepsilon\bar{\alpha}_p & 1 - x\bar{\beta}_q & -x\bar{\varepsilon}\alpha_p & 0 \\ 0 & -x\varepsilon\bar{\beta}_q & 1 - x\alpha_p & -x\bar{\varepsilon}\beta_q \\ -x\bar{\varepsilon}\bar{\alpha}_p & 0 & -x\varepsilon\alpha_p & 1 - x\beta_q \end{pmatrix}. \qquad (13)$$

It is then necessary to find the matrix inverse to (13), and once more to go over to the k, l-representation. We get

$$\langle kl\nu|(1-A)^{-1}|k'l'\nu'\rangle$$

$$= \frac{1}{(2\pi)^2}\int\int e^{i[p(k-k')+q(l-l')]}\frac{d_{\nu\nu'}(p, q)}{d(p, q)}\, dp\, dq. \qquad (14)$$

Here d(p, q) is the determinant of the matrix (13), and $d_{\nu\nu'}(p, q)$ is a minor of the same matrix.

If in the relation

$$\langle kl\nu\,|\,(1-A)^{-1}D\,|\,k'l'\nu'\rangle = \sum_{k''l''\nu''}\langle kl\nu\,|\,(1-A)^{-1}\,|\,k''l''\nu''\rangle$$

$$\times \langle k''l''\nu''\,|\,D\,|\,k'l'\nu'\rangle$$

we substitute (14) and (10'), we get for the non-vanishing matrix elements

$$\langle kl\nu|\,(1-A)^{-1}D\,|1l'1\rangle$$

$$= \frac{1}{(2\pi)^2}\int\int e^{i[p(k-1)+q(l-l'-1)]}\frac{d_{\nu1} + \bar{\varepsilon}d_{\nu2} + \varepsilon d_{\nu4}}{d}\, dp\, dq,$$

$$\langle kl\nu|\,(1-A)^{-1}D\,|1l'3\rangle$$

$$= \frac{1}{(2\pi)^2}\int\int e^{i[p(k-1)+q(l-l'+1)]}\frac{\varepsilon d_{\nu2} + d_{\nu3} + \bar{\varepsilon}d_{\nu4}}{d}\, dp\, dq. \qquad (15)$$

We introduce the matrices P_{11}, P_{13}, P_{31}, P_{33} with matrix elements

$$P_{\nu\nu'} = \langle 1l\nu\,|\,1 - g(1-A)^{-1}D\,|\,1l'\nu'\rangle.$$

The matrix $1 - g(1 - A)^{-1}D$ will have the form shown in the figure. In the crosshatched part the matrix elements are different from zero, but their form is unimportant, since they do not contribute to the determinant.

On taking the expression (13) for the matrix (1 - A) and calculating the integrand in (15), we get

$$\langle l\,|\,P_{11}\,|\,l'\rangle = \delta_{ll'} - \frac{g}{(2\pi)^2}\int\int e^{iq(l-l')}e^{iq}$$

$$\times \frac{(1-x^2) - e^{-iq}x[2\cos(2\pi p/N) + 1 - x^2]}{d(p, q)}\, dp\, dq,$$

$$\times \langle l\,|\,P_{33}\,|\,l'\rangle = \delta_{ll'} - \frac{g}{(2\pi)^2}\int\int e^{iq(l-l')}e^{-iq}$$

$$\times \frac{(1-x^2) - e^{iq}x[2\cos(2\pi p/N) + 1 - x^2]}{d(p, q)}\, dp\, dq,$$

$$d(p, q) = (1+x^2)^2 - 2x(1-x^2)\left(\cos\frac{2\pi p}{N} + \cos\frac{2\pi q}{N}\right). \qquad (16)$$

For P_{13} and P_{31}, there are obtained under the integral sign expressions odd in p. After the integration over p we get zero. The term $\delta_{ll'}$, in this case is absent; therefore $P_{13} = P_{31} = 0$.

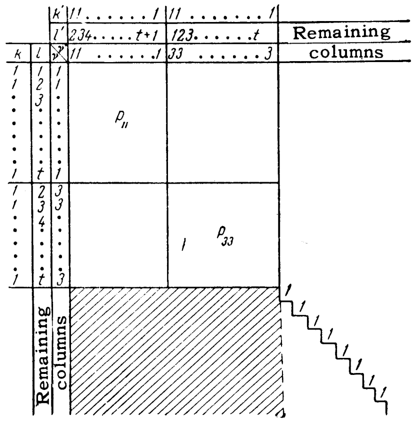
Thus it is found that

$$M^2 = \lim_{t\to\infty} x^t\, \mathrm{Det}^{1/2} P_{11}\, \mathrm{Det}^{1/2} P_{33}.$$

Now if in P_{11} we replace q by −q and l by l', we get P_{33}. This means that their determinants are equal, and

$$M^2 = \lim_{t\to\infty} x^t\, \mathrm{Det}\, P_{11} = \lim_{t\to\infty} \mathrm{Det}(c_{ll'}),$$

where



$$c_{ll'} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega(l-l')} f(\omega)\,d\omega, \qquad (17)$$

$$f(\omega) = \frac{1}{2\pi}\int_{-\pi} dp\Big[x - gxe^{i\omega}\Big\{1 - x^2 - e^{i\omega}x \times \Big[1 - x^2 + 2\cos\frac{2\pi p}{N}\Big]\Big\}\Big/ d(p,q)\Big]. \qquad (18)$$

On substituting here g = 1/x − x and integrating over p, we get—introducing the notation $x^* = (1-x)/(1+x)$—

$$f(\omega) = [(xx^*e^{i\omega} - 1)(xe^{i\omega} - x^*) / (e^{i\omega} - xx^*)(x^*e^{i\omega} - x)]^{1/2}. \qquad (19)$$

The elements $c_{ll'}$ depend only on the difference of the indices, $c_{ll'} = c(l - l')$. Such matrices are called Toeplitz forms. From the theory of Toeplitz forms, the following theorem is known (cf. [2,3,7]). If $c_{ll'}$ is given by formula (17), then

$$\lim_{t\to\infty}\Big\{D_t(c_{ll'}) \Big/ \Big[\exp\Big\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln f(\omega)\,d\omega\Big\}\Big]^{t+1}\Big\} = \exp\Big\{\sum_{1}^{\infty} nK_nK_{-n}\Big\}, \qquad (20)$$

where

$$K_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega n}\ln f(\omega)\,d\omega,$$

$D_t(c_{ll'})$ is the Toeplitz determinant of order t.

On taking for $f(\omega)$ the expression (19), we get for $x^* < x$

$$K_n = \frac{1}{2n}\Big[\Big(\frac{x^*}{x}\Big)^n - (xx^*)^n\Big] = -K_{-n},$$

and for $x^* > x$

$$K_n = \frac{(-1)^n}{n} - \frac{(x/x^*)^n + (xx^*)^n}{2n} = -K_{-n},$$

$$\int_{-\pi}^{\pi}\ln f(\omega)\,d\omega = 0.$$

On substituting these expressions in (20), we get finally: for $x^* < x$,

$$M^2 = \exp\Big\{\frac{1}{4}\ln\Big[1 - \Big(\frac{1/x - x}{2}\Big)^4\Big]\Big\} = \Big\{1 - \Big(\frac{1/x - x}{2}\Big)^4\Big\}^{1/4};$$

for $x^* > x$,

$$M^2 = e^{-\infty} = 0.$$

The point $x = x^* = \sqrt{2} - 1$ is the phase-transition point.

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