THE RENORMALIZED FIELD OPERATOR IN THE HEISENBERG PICTURE

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The general methods developed in a previous paper[1] are applied to the determination of renormalized field operators in the Heisenberg picture. These operators are related to the corresponding free-field operators in a nonunitary way, which, in particular, makes it possible in the scalar case to eliminate the well-known contradiction between the consequences of the Kallen-Lehmann theorem and the canonical commutation relations (c.c.r.) for free fields. At the same time, the use of a "half" S matrix makes it possible to introduce in the interaction picture field operators with which the Heisenberg operators are connected in a unitary way. These operators satisfy the same equations as the free-field operators, but have different c.c.r., the difference being particularly great for fields more complex than the scalar field. These results constitute serious arguments in favor of the thesis that the renormalized Heisenberg and free fields belong to nonequivalent representations of the c.c.r. The question of renormalization of the external lines in the S matrix is also considered, and a unique method for carrying out the procedure using Wick's theorem in the coordinate representation is suggested.

1. INTRODUCTION

IT was shown in a previous paper[1][1)] that the totality of facts established in local quantum field theory leads to the conclusion that the relation between the renormalized operator in the Heisenberg picture $\mathbf{F}(x)$ and the free operator $\mathcal{F}^{\text{in}}(x)$ should be, generally speaking, a nonunitary relation of the form

$$\mathbf{F}(x) = S^{+}T_W(\mathcal{F}^{in}(x)S). \qquad (1)$$

At the same time it follows from the assumption that the interaction picture exists, that for each $\mathbf{F}(x)$ there exists a certain operator $\mathcal{F}^{\text{int}}(x;\sigma)$ in the interaction picture, which is different from $\mathcal{F}^{\text{in}}(x)$ and which is related to $\mathbf{F}(x)$ by a unitary transformation, i.e.,

$$\mathbf{F}(x) = S^{+}(\sigma, -\infty)\mathcal{F}^{\text{int}}(x;\sigma)\ S(\sigma, -\infty), \qquad (2)$$

where

$$S(\sigma, -\infty) = T_D \exp\left\{-i\int_{-\infty}^{\sigma} H_I{}^{\text{int}}(x';\sigma')\,dx'\right\} \qquad (3)$$

represents a "half" S matrix, obtained in I by "slicing" the full S matrix, having first expressed it in terms of the Hamiltonian $H_I^{\text{int}}(x;\sigma)$ and the T_D-product.[2)]

It should also be noted that here, as well as in I, by the interaction picture we mean the in-picture. Therefore the "half" S matrix of type (3) should be written in the notation of Schweber's book[2] as $V_+(t)$, and not at all as the evolution operator $U(t, -\infty)$. Further, although both the operators $\mathcal{F}^{\text{in}}(x)$ and $\mathcal{F}^{\text{int}}(x;\sigma)$ should be expressed in terms of normal products of the operators $\varphi^{\text{in}}(x)$, we shall call the first of them the free operator (or the in-operator), and the second the interaction picture operator. At that $\mathcal{F}^{\text{in}}(x)$ should be chosen by considerations based on analogies with the classical theory, whereas $\mathcal{F}^{\text{int}}(x;\sigma)$ is defined by Eq. (2).[3)] Finally, the use of the name "renormalized" for the operator $\mathbf{F}(x)$ should be understood in the broad sense as taking into account all interactions, including derivatives of any order.

[1)]In the following this paper will be referred to as I.

[2)]In so defining the "half" S matrix we have ignored a possible unitary arbitrary factor of the form $\exp\{i\Phi(\sigma)\}$, which is immaterial from the point of view of the full S matrix.

[3)]It is necessary to emphasize that in the previous papers of the author (and, in particular, in I) only the operator $H_I^{\text{int}}(x;\sigma)$ should have been used (and not $H_I^{\text{in}}(x;\sigma)$). However to indicate the dependence of the operator $H_I^{\text{int}}(x;\sigma)$ on precisely $\varphi^{\text{in}}(x)$ sometimes the notation $H_I^{\text{in}}(x;\sigma)$ was used, which should not be confused with the in-representative of the interaction Hamiltonian.

Since the nonunitary nature of the relation (1) causes considerable inconvenience it is of interest to find for each operator $\mathbf{F}(x)$ the corresponding operator $\mathscr{F}^{\text{int}}(x;\sigma)$. In the present paper this program is carried out on the example of the renormalized operators of various fields by making use of the analogue of the Wick theorem established in I, which makes it possible to express the T_W-product in terms of the totality of T_D-products and, consequently, makes it possible to pass from (1) to (2).[4)]

2. THE SCALAR FIELD

In view of (1) the definition of the renormalized operator for the neutral scalar field is of the form

$$\mathbf{A}(x) = S^+T_W(\varphi^{in}(x)S), \tag{4}$$

where $\varphi^{\text{in}}(x)$ is the corresponding free operator satisfying the conditions

$$K_x\varphi^{in}(x) \equiv (\Box_x - m^2)\varphi^{in}(x) = 0; \quad [\dot{\varphi}^{in}(x), \varphi^{in}(y)]\big|_{x^0=y^0} = -i\delta(\mathbf{x}-\mathbf{y}). \tag{5}$$

Our main task is to transform Eq. (4) into the form (2) in renormalizable theories. To this end it is first necessary to obtain, following the rules established in I, the Hamiltonian $H_I^{\text{int}}(x;\sigma)$ which appears in (3). For the primary Lagrangian $L_I^{\text{in}}(x) = g:[\varphi^{\text{in}}(x)]^4:$, which forms part of any theory of scalar fields, the corresponding effective Hamiltonian with the counter terms has the form[5)]

$$H_I^{\text{int}}(x) = -\frac{gZ_1}{Z_3^2}:[\varphi^{in}(x)]^4: + \left(\frac{1}{Z_3^{1/2}} - 1\right):\varphi^{in}(x)K_x\varphi^{in}(x):. \tag{6}$$

We now apply the analogue of Wick's theorem to transform Eq. (4). We find

$$\mathbf{A}(x) = S^+T_D(\varphi^{in}(x)S) + \chi(x), \tag{7}$$

$$\chi(x) = S^+T_D(\overline{\varphi^{in}(x)S}), \tag{8}$$

where the bracket over Eq. (8) signifies "quasicontraction" (see I).

To further transform $\chi(x)$ it is necessary to first express the S matrix in terms of the T_D-product and of $H_I^{\text{int}}(x)$. Then the T_D-product symbol in (8) can be made general. In addition it follows from the formulae for "quasicontraction" obtained in I that in renormalizable theories the only term in $H_I^{\text{int}}(x)$ with which the operator $\varphi^{\text{in}}(x)$ should be "quasicontracted," is the term in (6) containing the second derivative. Therefore

$$\chi(x) = S^+T_D\left[\overline{\varphi^{in}(x)\left(-i\int_{-\infty}^{\infty}H_I^{\text{int}}(y)\,dy\right)}S\right] = (Z_3^{-1/2}-1)S^+T_D(\varphi^{in}(x)S). \tag{9}$$

If one now combines two terms in Eq. (7) one finally obtains for $\mathbf{A}(x)$

$$\mathbf{A}(x) = Z_3^{-1/2}S^+T_D(\varphi^{in}(x)S) \tag{10}$$

$$\equiv S^+(\sigma,-\infty)\varphi^{\text{int}}(x)S(\sigma,-\infty), \quad \varphi^{\text{int}}(x) = Z_3^{-1/2}\varphi^{in}(x). \tag{11}$$

Analogously one can show that

$$\dot{\mathbf{A}}(x) = S^+T_W(\dot{\varphi}^{in}(x)S) = S^+(\sigma,-\infty)\dot{\varphi}^{\text{int}}(x)S(\sigma,-\infty). \tag{12}$$

It is not hard to verify that, with (5) and the unitarity condition of the S matrix taken into account, (10) and (12) lead immediately to canonical commutation relations (c.c.r.) of the form

$$[\dot{\mathbf{A}}(x), \mathbf{A}(y)]\big|_{x^0=y^0} = -iZ_3^{-1}\delta(\mathbf{x}-\mathbf{y}). \tag{13}$$

Thus, the expression (4) proposed by us for the renormalized operator of the scalar field in the Heisenberg picture turns out to be fully sufficient to eliminate the contradiction between Eq. (5) and the well known consequences of the Kallen-Lehman theorem,[3] independently of the question of finiteness of the matrix $S(\sigma,-\infty)$ when the regularizing masses $M_i^2 \to \infty$.

Formally, however, a different approach is also possible in the case of the scalar field. The point is that the counter term for the self-energy of the scalar field in the effective interaction Lagrangian[4] $L_I^{\text{in}}(x;1)$ is usually written in the two forms

$$\Lambda_1^{in}(x) = \frac{Z_3-1}{2}:\varphi^{in}(x)K_x\varphi^{in}(x): \tag{14}$$

and

$$\Lambda_2^{in}(x) = \frac{Z_3-1}{2}\left(:\left[\frac{\partial\varphi^{in}(x)}{\partial x^\alpha}\right]^2: - m^2:[\varphi^{in}(x)]^2:\right). \tag{15}$$

In the above we have started from Eq. (14). On the other hand the Hamiltonian $\tilde{H}_I^{\text{int}}(x;\sigma)$ corresponding to (15) has the form

$$\tilde{H}_I^{\text{int}}(x;\sigma) = -gZ_1:[\varphi^{in}(x)]^4: - \frac{Z_3-1}{2}:\left[\frac{\partial\varphi^{in}(x)}{\partial x^\alpha}\right]^2: + \frac{(Z_3-1)^2}{2Z_3}:\left(n_\alpha\frac{\partial\varphi^{in}(x)}{\partial x^\alpha}\right)^2: + \frac{Z_3-1}{2}m^2:[\varphi^{in}(x)]^2:, \tag{16}$$

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4)The first communication of these results was made by the author in a report to the All-Union conference on colliding beam accelerators and high-energy particle physics (Novosibirsk, June 1963).

5)The second term of (6) was obtained in I, the first term is obtained in the same way.

which is substantially different from (6). Therefore, if we apply the analogue of Wick's theorem to the field A(x) of the form (4), in which the S matrix depends on $\widetilde{H}_I^{int}(x;\sigma)$ given by (16), we obtain in place of (10) the expression

$$\widetilde{A}(x) = S^+(\sigma, -\infty)\varphi^{in}(x)S(\sigma, -\infty), \qquad (17)$$

because all "quasicontractions" of the field $\varphi^{in}(x)$ with the terms of this Hamiltonian vanish. At the same time

$$\dot{\widetilde{A}}(x) = S^+(\sigma, -\infty)\Big[\dot{\varphi}^{in}(x) + \Big(\frac{1}{Z_3} - 1\Big)\Big(n_\alpha \frac{\partial\varphi^{in}(x)}{\partial x^\alpha}\Big)\Big] S(\sigma, -\infty), \qquad (18)$$

so that if one constructs the c.c.r. for the field $\widetilde{A}(x)$ one obtains a result coinciding with (13).

Thus, in the case of the scalar field for one special choice of the self-energy counter term the generally speaking nonunitary relation between A(x) and $\varphi^{in}(x)$ in the form (4) becomes unitary. However, in spite of this, it is possible to obtain for the field $\dot{\widetilde{A}}(x)$ c.c.r. of the form (13) at the cost of making the unitary operator $S(\sigma, -\infty)$ time dependent, which gives rise to Eq. (18) for $\dot{\widetilde{A}}(x)$. At the same time this possibility is not present for other types of fields. Moreover, as will be shown below, even in the case of the scalar field only one of the two possible types of counter terms $\Lambda_1^{in}(x)$ and $\Lambda_2^{in}(x)$ in fact makes physical sense, namely the one that leads to Eq. (10).

Equations analogous to (13), i.e., for the commutators themselves and not for their vacuum expectation values, have been obtained previously by Kallen[5] by means of canonical quantization of the full renormalized Lagrangian in the Heisenberg picture which is, in our opinion, not a sufficiently clear procedure since, among other things, it gives rise to the same results for any choice of the counter term in the case of the scalar field. In this paper we derive these formulae in a consistent fashion starting from the non-unitary relation between the appropriate operators of the form (4).[6)]

3. CHOICE OF COUNTER TERM AND RENORMALIZATION OF EXTERNAL LINES

The question of the choice of the self-energy counter term for the scalar field in the effective interaction Lagrangian[4] $L_I^{in}(x;g)$ was discussed by us in[7]. It was shown there that although the quasilocal operator $\Lambda^{in}(x, y)$ may be written, by definition, in two equivalent symmetric forms $\Lambda_1^{in}(x, y)$ and $\Lambda_2^{in}(x, y)$, removing the y integration in each of them gives rise to entirely different expressions for $L_I^{in}(x;g)$ and, consequently, for the effective scattering matrix S(g). At the same time for $g \to 1$, the resultant two expressions for $L_I^{in}(x;1)$, which contain respectively $\Lambda_1^{in}(x)$ of the form (14) and $\Lambda_2^{in}(x)$ of the form (15), differ formally by a 4-divergence and in the classical theory may be considered equivalent.

If however one turns to quantum field theory, then the question of equivalence of the theories with $\Lambda_1^{in}(x)$ and $\Lambda_2^{in}(x)$ becomes much more complicated. In particular, if the adiabatic hypothesis is not used it is not possible to prove the vanishing of the integral of the corresponding 4-divergence. However, even if we accept the adiabatic hypothesis and allow that the expressions (14) and (15) themselves are equivalent, this does not yet mean that they give rise to the same expressions for the S matrix. For example, it is known that in the expression for $L_I^{in}(x;1)$ the counter term $\Lambda_1^{in}(x)$ of the form (14) may be set equal to zero. However it must be included in $L_I^{in}(x;1)$, because it enters the S matrix inside the symbol of the nonunitary T_W-product (see I), and its presence is essential to the renormalization program.

Analogously it is not hard to show that even if the 4-divergence of the type $:\partial^2[\varphi^{in}(x)]^2/\partial x^2:$ can be ignored in the $L_I^{in}(x;1)$ itself, it must be taken into account inside the T_W-product (in contrast to the T_D-product). One can, of course, proceed as in I and pass in the S matrix from the T_W-product in $L_I^{in}(x;1)$ to the T_D-product in $H_I^{int}(x;\sigma)$. This will give rise in the cases (14) and (15) to S matrices which depend respectively on different $H_I^{int}(x;\sigma)$ of the form (6) and (16). Inside the T_D-product (which is unitary) and, consequently, in $H_I^{int}(x;\sigma)$ one may omit (see I) terms which vanish as a consequence of the Klein-Gordon equation or the adiabatic hypothesis. Therefore from the entire expression (6) in fact only the first term remains, and from expression (16) the first and the third terms. At that, if the transition in the S matrix from $L_I^{in}(x;1)$ to $H_I^{int}(x)$ of the form (6) simply leads to a transition from additive to multiplicative renormalization, the analogous transition to $H_I^{int}(x;\sigma)$ of the type (16) gives rise only to a certain "renormalization" of $\partial\varphi^{in}(x)/\partial x^\alpha$, while the fields $\varphi^{in}(x)$ themselves turn out to be renormalized. It is therefore to be expected that the

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6)We note by the way, that the idea that the operators A(x) and $\varphi^{in}(x)$, and also $\dot{A}(x)$ and $\dot{\varphi}^{in}(x)$, should be related "unitarily" to within the factor $Z_3^{-1/2}$, was expressed previously by Kaschluhn.[6]

main difference between $\Lambda_1^{\text{in}}(x)$ of the form (14) and $\Lambda_2^{\text{in}}(x)$ of the form (15) lies in the different behavior of these counter terms in so far as the question of the renormalization of the external lines is concerned.

It was already shown by Dyson[8] that in the momentum representation the introduction of self-energy counter terms into the internal lines leads to the transition from $D^c(x-y)$ to $Z_3^{-1}D^c(x-y)$. At the same time, if they are introduced into external lines there results an undetermined expression of the form $(p^2-m^2)^{-1}(p^2-m^2)\delta(p^2-m^2)$, which takes on any value between 0 and Z_3^{-1}. On the basis of considerations of preserving the unitarity of the S matrix after renormalization Dyson[8] proposed the value $Z_3^{-1/2}$ for the above expression, which is equivalent to the assumption that upon renormalization the field $\varphi^{\text{in}}(x)$ goes over into $\varphi^{\text{int}}(x)$, although this result does not follow directly from his work.

An analysis of this problem in the coordinate representation, starting from $\Lambda_1^{\text{in}}(x)$ in the form (14) and $\Lambda_2^{\text{in}}(x)$ in the form (15), shows that from the point of view of the internal lines both counter terms give rise to identical results coinciding with those of Dyson,[8] owing to the equivalence of $\Lambda_1^{\text{in}}(x, y)$ and $\Lambda_2^{\text{in}}(x, y)$.[7] The question of the renormalization of the external lines must however be considered separately on the basis of the Wick theorem, which makes sense precisely in the coordinate representation.

To this end it is necessary in the case of $\Lambda_1^{\text{in}}(x)$ to sum the following series:

$$\varphi^{in}(x)+i\int T_W(\varphi^{in}(x)\Lambda_1^{in}(y))\,dy + \frac{i^2}{2!}\int T_W(\varphi^{in}(x)\Lambda_1^{in}(y)\Lambda_1^{in}(z))\,dydz+\ldots \tag{19}$$

Let us develop the second term of this series according to the usual Wick's theorem:

$$i\int T_W(\varphi^{in}(x)\Lambda_1^{in}(y))\,dy = {}^1/_2(Z_3-1)\int dy\,\{i{:}\varphi^{in}(x)\varphi^{in}(y)K_y\varphi^{in}(y){:} + \varphi^{in}(y)K_yD^c(x-y)+D^c(x-y)K_y\varphi^{in}(y)\}. \tag{20}$$

If one now takes into account the fact that within the normal product the field $\varphi^{\text{in}}(x)$ satisfies (5), and also makes use of the expression for $K_yD^c(x-y)$, then

$$i\int T_W(\varphi^{in}(x)\Lambda_1^{in}(y))\,dy = -\frac{1}{2}(Z_3-1)\varphi^{in}(x). \tag{21}$$

The remaining terms in the series (18) may be transformed in an analogous manner with the result that the sum is equal to $\varphi^{\text{int}}(x)$ in the form (11). Thus, proceeding in the coordinate representation and making use of the counter term $\Lambda_1^{\text{in}}(x)$ one can get rid of the nonuniqueness in the renormalization of the external lines. This nonuniqueness is in fact due to the usually assumed freedom of transferring derivatives in the second and third terms in (20) between $\varphi^{\text{in}}(y)$ and $D^c(x-y)$, whereas according to the above considerations one should, generally speaking, take into account the substitution of the limits.

A series analogous to (19) can also be formed in the case of $\Lambda_2^{\text{in}}(x)$. Applying Wick's theorem to the second term of that series leads to

$$i\int T_W(\varphi^{in}(x)\Lambda_2^{in}(y))\,dy = \frac{Z_3-1}{2}\int dy\Big\{i{:}\varphi^{in}(x)\Big[\Big(\frac{\partial\varphi^{in}(y)}{\partial y^\alpha}\Big)^2 - m^2(\varphi^{in}(y))^2\Big]{:} + 2\Big[\frac{\partial\varphi^{in}(y)}{\partial y^\alpha}\frac{\partial D^c(x-y)}{\partial y^\alpha} - m^2\varphi^{in}(y)D^c(x-y)\Big]\Big\}. \tag{22}$$

It is not hard to see that without an integration by parts Eq. (22) cannot, in general, be summed with $\varphi^{\text{in}}(x)$ because it contains terms of different structure. On the other hand a simple transfer of the derivative from $\varphi^{\text{in}}(y)$ to $D^c(x-y)$ or vice versa gives rise to a completely undetermined expression which violates the requirements of unitarity. Consequently, the term $\Lambda_2^{\text{in}}(x)$ should be excluded from considerations since when it is used if one proceeds in a consistent manner it is not possible to satisfy simultaneously the requirements of renormalization (which are guaranteed only in the Lagrangian approach) and the requirements of unitarity (which are guaranteed only in the Hamiltonian approach).

4. OTHER TYPES OF FIELDS

Passing now to fields of other types we turn first to the case of the electromagnetic field, where an incorrect definition of the renormalized Heisenberg operator $A_\mu(x)$ in terms of the T_D-product (instead of the T_W-product) results in it having an unphysical dependence on the surface σ. This latter is due to the fact that as a result of gauge invariance requirements the part of the Hamiltonian responsible for the renormalization of the electromagnetic field has the form [cf. (6)]:

$$H_I^{\text{int}}(x) = \Big(1-\frac{1}{\sqrt{Z_3}}\Big){:}\Big[A_n^{in}(x)\Box_x A_n^{in}(x) + A_m^{in}(x)\frac{\partial^2 A_n^{in}(x)}{\partial x^n\partial x^m}\Big]{:}. \tag{23}$$

As a result, if we apply to the renormalized Heisenberg operator $A_\mu(x)$ the analogue of Wick's theorem we get

$$A_\mu(x) = S^+T_W(A_\mu^{in}(x)S) = S^+T_D(A_\mu^{in}(x)S) + \chi_\mu(x;\ \sigma), \tag{24}$$

where, in contrast to (10), each term on the right side of (24) depends on the surface σ, because of the dependence on σ of the T_D-product (see I). Continuing with transformations of $\chi_\mu(x;\sigma)$ analogous to (8) and (9) we find that the operator corresponding to $A_\mu(x)$ according to Eq. (2) is given by

$$A_\mu^{\text{int}}(x;\sigma) = Z_3^{-1/2}[A_\mu^{in}(x) + (Z_3^{1/2} - 1)n_\mu n_\alpha A_\alpha^{in}(x)], \tag{25}$$

where in contrast to $\varphi^{\text{int}}(x)$ of the form (11) the operator $A_\mu^{\text{int}}(x;\sigma)$ in the interaction picture itself depends on σ.

Before constructing the c.c.r. for the field $A_\mu(x)$ we note that, in contrast to (12)

$$\dot{A}_\mu(x) = S^+T_W(\dot{A}_\mu^{in}(x)S) = S^+(\sigma, -\infty)\tilde{A}_\mu^{\text{int}}(x;\sigma)S(\sigma, -\infty), \tag{26}$$

where

$$\tilde{A}_\mu^{\text{int}}(x;\sigma) = \dot{A}_\mu^{\text{int}}(x;\sigma) + (1 - Z_3^{-1/2})\left(n_\alpha\frac{\partial A_\alpha^{in}(x)}{\partial x^\mu} + n_\mu\frac{\partial A_\alpha^{in}(x)}{\partial x^\alpha} - 2n_\mu n_\alpha n_\beta\frac{\partial A_\beta^{in}(x)}{\partial x^\alpha}\right), \tag{27}$$

where $\dot{A}_\mu(x)$, as well as $A_\mu(x)$, are independent of σ as a consequence of their definition.

Taking (25) and (27) into account one obtains for the corresponding c.c.r.

$$[\dot{A}_\mu(x), A_\nu(y)]\big|_{x^0=y^0} = [\tilde{A}_\mu^{\text{int}}(x;\sigma), A_\nu^{\text{int}}(y;\sigma)]\big|_{x^0=y^0} = iZ_3^{-1}\xi^{\mu\nu}\delta(\mathbf{x}-\mathbf{y}), \tag{28}$$

where

$$\xi^{\mu\nu} = g^{\mu\nu} + (Z_3 - 1)n_\mu n_\nu. \tag{29}$$

Thus, an interesting fact appears. Although the operators $A_\mu(x)$ and $\dot{A}_\mu(x)$ are themselves independent of σ, their commutator, taken on the surface σ, depends on that surface.

Let us recall that the operator for the free electromagnetic field satisfies c.c.r. of the form[4]

$$[\dot{A}_\mu^{in}(x), A_\nu^{in}(y)]\big|_{x^0=y^0} = ig^{\mu\nu}\delta(\mathbf{x}-\mathbf{y}). \tag{30}$$

Comparing with (28) we note that the zeroth component of the electromagnetic field is not renormalized. Consequently it is as if excluded from the interaction. Thus in the present case the renormalization does not amount to a simple multiplication by a constant (possibly a divergent one), but involves a characteristic change in the metric.

Up till recently most authors[2] used in formulas analogous to (28) instead of $\xi^{\mu\nu}$ the symbol $g^{\mu\nu}$, and only in[5], where quantization was performed of the full renormalized Lagrangian in the Heisenberg picture with gauge invariance taken into account, a formula of type (28) was obtained. In our formalism it arises naturally, starting from the supposition of a nonunitary connection of the type (24) between the operators of the electromagnetic field.

For the case of a spinor field a full analogy with the scalar case is observed, so that after the application of the analogue of Wick's theorem to a formula of type (4) one obtains the operator $\psi^{\text{int}}(x) = \psi^{in}(x)/Z_2^{1/2}$, whose connection with the renormalized Heisenberg operator is already unitary.

Let us remind the reader that until now we were mainly concerned with showing that the inclusion in the effective Lagrangian $L_I^{in}(x;1)$ of self-energy counter terms gives rise in renormalizable theories to a nonunitary relation of the type (4) between the corresponding field operators. Further, there are examples[9] in renormalizable theories where this nonunitary relation can be demonstrated already on the primary Lagrangian $L_I^{in}(x)$. To these belong the neutral vector field theory and the scalar electrodynamics in the Duffin-Kemmer formalism.

In the first case

$$L_I^{in}(x) = g : \bar{\psi}^{in}(x)\gamma^\alpha\psi^{in}(x)U_\alpha^{in}(x) : \; = \; : j_\alpha^{in}(x)U_\alpha^{in}(x) :. \tag{31}$$

Following I, one finds easily in this case $H_I^{\text{int}}(x;\sigma)$ and transforms the formula for $U_\alpha(x)$ from the form which gives a nonunitary relation to a unitary one. It then turns out that

$$U_\alpha^{\text{int}}(x,\sigma) = U_\alpha^{in}(x) - m^{-2}n_\alpha n_\beta j_\beta^{in}(x). \tag{32}$$

Consequently the operator $U_\alpha^{\text{int}}(x;\sigma)$, which should be called the interaction picture operator, differs in this case from the free field operator $U_\alpha^{in}(x)$ by terms that have even a different operator structure.

Application of the same procedure to scalar electrodynamics in the Duffin-Kemmer formalism leads to the expression

$$\psi^{\text{int}}(x;\sigma) = :\{1 + iem^{-1}[1 - (n_\beta\Gamma_\beta)^2]\Gamma_\alpha A_\alpha^{in}(x)\}^{-1}\psi^{in}(x):, \tag{33}$$

where Γ_α are the five-by-five Duffin-Kemmer matrices, so that the operator $\psi^{\text{int}}(x;\sigma)$ differs from the free field operator $\psi^{in}(x)$ by an infinite series in the operators $A_\alpha^{in}(x)$.

It is interesting to note that although the formulae (32) and (33) could also be derived directly from the results in[9], up to now the possibility of a nonunitary connection between the renormalized Heisenberg operators and the corresponding free operators was accepted only in exotic local field theories with high derivatives. This omission, in all likelihood, was due to the fact that the main purpose of the authors in[9] and other papers was the construction of $H_I^{int}(x;\sigma)$ [not coinciding with $H_I^{in}(x;\sigma)$], and for this operator [see (2)] the indicated connection is, by definition, unitary.

5. DISCUSSION

In this paper a concrete form is proposed for the nonunitary connection of the form (4) between the renormalized field operator in the Heisenberg picture and the free field operator, which leads to results in agreement with the consequences of the Kallen-Lehmann theorem.[3] The contradictions usually encountered in this problem were apparently due to the fact that the proper distinction was not always made between the interaction picture operator $\varphi^{int}(x)$, whose relation to A(x) is by definition unitary, and the free operator $\varphi^{in}(x)$. At that, although the operators A(x) and $\varphi^{int}(x)$ satisfy different equations, they have the same c.c.r. of type (13). At the same time, although the operators $\varphi^{int}(x)$ and $\varphi^{in}(x)$ satisfy the same equations, their c.c.r. (13) and (5) are different, with Z_3^{-1} tending to infinity fastest as $M_i^2 \to \infty$. In other words, the field operator is fixed not only by the equations of motion but also by the concrete form of the c.c.r. In particular the operator $A_\mu^{int}(x;\sigma)$ differs from $A_\mu^{in}(x)$ not simply by a numerical factor but by the character of the metric ($g^{\mu\nu}$ is replaced by $\xi^{\mu\nu}$). Even more serious differences between the operators $\mathscr{F}^{int}(x;\sigma)$ and $\mathscr{F}^{in}(x)$ occur in the case of the neutral vector field and in the case of scalar electrodynamics in the Duffin-Kemmer formalism.

Thus the present results once more emphasize the fact (see also[5,9,10]) that the operator $\mathscr{F}^{int}(x;\sigma)$ in the interaction picture and the free operator $\mathscr{F}^{in}(x)$ are substantially different. At the same time a serious confirmation was obtained of the previously expressed opinion[10] that not only nonrenormalizable local theories[4] but also renormalizable local theories with counter terms (and even, as shown above, certain theories without counter terms) are in a certain sense nearer to nonlocal field theories than to trivial local theories without counter terms.

Of course, the question of passing to the limit $M_i^2 \to \infty$ in the matrix $S(\sigma, -\infty)$ of type (3) with all its properties preserved, especially unitarity, still remains unresolved. In this connection it may be hoped that this transition will lead to a finite in a certain sense quantity, i.e., that it will be possible to obtain the coefficient functions of the matrix $S(\sigma, -\infty)$ as generalized functions on a certain class of regular functions, since after the resolution (for M_i^2 = const) of the problem of "surface" divergences[7] and liquidation of the contradictions with the consequences of the Kallen-Lehman theorem[3] obtained in this paper, the last reasons that have caused some theorists to consider the "half" S matrix as being infinite in this sense are in fact eliminated.

Finally, we consider one more point. Very often the problem raised here is placed in close connection with the question raised by Haag[11], outside the framework of perturbation theory, about the existence in quantum field theory of nonequivalent representations of c.c.r. In particular, in view of Haag's theorem proved in[12], if certain field operators $A_1(x)$ and $A_2(x)$ are connected by a unitary Euclidean-invariant transformation V(t), and the vacuum states corresponding to these operators are connected by the formula $V(t)\Phi_{01} = \Phi_{02}$, then all the Wightman functions in the two theories coincide. A rash application of this theorem to operators related by formulas of type (10) could also lead to an erroneous result such as that $S \equiv 1$.

At that the analysis carried out above of expressions for operators of different fields in the interaction picture shows that they do not simply differ from the free field operators by a multiplicative factor of the type $Z_3^{-1/2}$. This could serve as a strong argument in favor of the hypothesis that the renormalized Heisenberg field operator and the corresponding free operator belong to nonequivalent representations of the c.c.r. As regards the scalar field, the problem there may be solved analogously after passing to the limit $M_i^2 \to \infty$, when Z_3^{-1} actually becomes infinite. However even for a finite value of $Z_3^{-1/2}$ the conclusions of the Haag theorem are not necessarily applicable. The point is, as was shown in[13], that in spite of its formal unitarity the matrix $S(\sigma, -\infty)$ is not a "truly unitary" operator but rather "pseudo-unitary" (or "improperly unitary"), i.e., such that when it is applied to the vacuum state it produces a state of infinite norm, which does not belong to the originally introduced Hilbert space. It is precisely this feature that should be kept in mind, in our opinion, when it is said that the transformation produced by the matrix $S(\sigma, -\infty)$ is infinite. It is not hard to see that for such a "pseudo-unitary" operator one of the conditions of the Haag theorem is not fulfilled, namely the requirement of equivalence of the vacuum states, and consequently the formulas of type (10) cannot give rise to any embarassing conclusions.

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