# CERENKOV RADIATION OF MAGNETOACOUSTIC WAVES FROM EXTENDED SOURCES

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The emission of magnetoacoustic waves by moving finite sources is studied. Simple examples of such sources are a moving solenoid bearing a constant electric current and a volume source of mass. The fields of disturbances generated in a medium by the solenoid are determined. Integral expressions are obtained for the radiation energy and wave drag force. It is shown that even in the absence of dispersion in the medium the force and energy integrals converge due to interference of the Cerenkov radiation. The double critical region of wave drag of moving sources in a magnetoactive plasma is discussed and a kinematic explanation is proposed.

CERENKOV magnetoacoustic waves from moving sources have been discussed in several publications, such as [1,2]. In these treatments either point sources were considered,[2] or sources extended in only one direction, as in[1] where the investigated waves were excited by a ring bearing direct current. In these investigations the intensity of Cerenkov magnetoacoustic waves and the associated wave drag force acting on the source were derived in the form of divergent integrals over the wave numbers of the disturbances. In order to eliminate these divergences the customary integral cutoff was used in[2] at small wavelengths (i.e., at high frequencies). This procedure is justified physically by the fact that short waves are strongly absorbed in a medium as a result of viscous, thermal, and Joule losses. The cutoff was made at the wavelength for which dispersion in the medium becomes appreciable. In the initial equations for the disturbances in the medium dispersion was neglected in order to simplify the treatment. The cutoff is fully justified physically when the geometric dimensions of the source are considerably smaller than the wavelengths which are appreciably affected by dispersion. This was the case in[2], where we analyzed the Cerenkov magnetoacoustic waves emitted by stars moving in a magnetoactive ionized interstellar gas. The stars were considerably smaller than the characteristic scale of dimensions within which the emitted waves are absorbed strongly.

However, when the geometric size of a moving source exceeds the characteristic distance within which dispersion in the medium becomes effective, it is not physically justifiable to represent the source as a point and to describe it by means of a $\delta$ function; instead, extended sources must be considered. The present article deals mainly with the excitation of magnetoacoustic waves by finite moving sources. It will be shown that the integrals of radiation intensity and of the wave drag force will converge even if dispersion is neglected, because of the interference of Cerenkov radiation from separate parts of the source. The interference of high-frequency Vavilov-Cerenkov waves emitted by moving systems of charges was first considered by Frank[3], and more thoroughly by Bolotovskiĭ.[4] Following these preliminary comments, we proceed to analyze the magnetohydrodynamic disturbances generated in a plasma by moving finite sources.

The system of linear magnetohydrodynamic equations describing small disturbances in a medium in the presence of extraneous sources is[5]

$$\rho_0\frac{\partial \mathbf{v}}{\partial t} = -\operatorname{grad} p + \frac{1}{4\pi}[\operatorname{rot}\mathbf{h}\mathbf{H}_0] - \frac{1}{c}[\mathbf{j}_{\mathrm{extr}}\mathbf{H}_0],$$
$$\frac{\partial\rho}{\partial t} + \rho_0\operatorname{div}\mathbf{v} = Q, \qquad \frac{\partial\mathbf{h}}{\partial t} = \operatorname{rot}[\mathbf{v}\mathbf{H}_0], \qquad p = c_s^2\rho, \tag{1}*$$

where $\rho_0$ and $\mathbf{H}_0$ are the undisturbed density and magnetic field in the medium; p, $\rho$, $\mathbf{v}$, and $\mathbf{h}$ are small disturbances of pressure, density, velocity, and magnetic field, respectively; $c_s$ is the velocity of sound; $j_{\mathrm{extr}}$ and Q are the density of the extraneous electric current and the mass injection rate, respectively.

It is appropriate here to make one comment regarding the mechanical means of generating disturbances in a medium. When plasma is injected

*rot = curl

by a source of strength Q, momentum is transferred to the medium; therefore an extraneous force should appear in the first equation of (1). However, the velocity of the ejected gas can be made so small that the role of this additional source of a perturbing force will be considerably smaller than that of the mechanism considered here. From (1) we easily obtain an equation for the velocity vector $\mathbf{v}$:

$$\frac{\partial^2 \mathbf{v}}{\partial t^2} = c_s^2 \,\mathrm{grad\,div}\, \mathbf{v} + c_A^2 [\mathrm{rot\,rot}[\mathbf{v}\mathbf{e}_z],\ \mathbf{e}_z] + \frac{1}{\rho_0 c}\left[\mathbf{H}_0 \frac{\partial \mathbf{j}_{\mathrm{extr}}}{\partial t}\right] - \frac{c_s^2}{\rho_0}\,\mathrm{grad}\, Q, \qquad (2)$$

where $c_A = H_0(4\pi\rho_0)^{-1/2}$ is the Alfvén wave velocity and $\mathbf{e}_z$ is the unit vector along the z axis, which is parallel to the external magnetic field $\mathbf{H}_0$.

We shall first consider magnetohydrodynamic disturbances generated by a moving solenoid of n turns bearing the direct current $I_0 = ni_0$, where $i_0$ is the current in a single turn. We shall also assume that the solenoid moves with the constant velocity $\mathbf{V}_0$ along its axis, which is parallel to $\mathbf{H}_0$ and therefore to the z axis. For convenience we introduce the cylindrical coordinates r, z, $\varphi$, with which the density of the extraneous electric current in the solenoid can be represented by

$$\mathbf{j}_{\mathrm{extr}} = \frac{I_0 \delta(r-a)}{2l} L(z - V_0 t)\,\mathbf{e}_\varphi, \qquad (3)$$

where a is the radius of the solenoid, $2l$ is its length, and $\mathbf{e}_\varphi$ is a unit vector for $\varphi$. The generalized function L is defined by

$$L(z - V_0 t) = \begin{cases} 1 & \text{for} \quad -l \leqslant z - V_0 t \leqslant +l \\ 0 & \text{in the rest of space,} \end{cases} \qquad (4)$$

with

$$\int_0^{+\infty}\int_{-\infty}^{+\infty} j_\varphi \, dr\, dz = I_0. \qquad (5)$$

To solve for the disturbances generated in a medium by a moving solenoid we employ the customary Fourier-Hankel transformation

$$\bar{\psi}_\nu = \frac{1}{2\pi}\int_0^{+\infty}\int_{-\infty}^{+\infty} r\psi(r,\xi)\, e^{i\varkappa\xi} J_\nu(kr)\, dr\, d\xi, \qquad (6)$$

with the inverse transformation

$$\psi = \int_0^{+\infty}\int_{-\infty}^{+\infty} k\bar{\psi}_\nu(k,\varkappa)\, e^{-i\varkappa\xi} J_\nu(kr)\, dk\, d\varkappa, \qquad (7)$$

where $J_\nu$ is the $\nu$-th order Bessel function, and $\psi$ is an arbitrary component of the disturbances or of the source density. Using the new variable $\xi = z - V_0 t$ and applying the transformations (6) and (7) to (1), (2), and (4), we can determine any component of the disturbances as a double integral over the wave numbers k and $\kappa$.

In calculating the wave drag and radiation energy we shall require the radial component of the magnetic field:

$$h_r = \frac{2iaI_0(M_s^2 - 1)}{cl(M_A^2 + M_s^2 - 1)} \times \int_0^{+\infty}\int_{-\infty}^{+\infty} \frac{k \sin \varkappa l J_1(ka) J_1(kr)\, e^{-i\varkappa\xi}}{k^2 + \gamma^2\varkappa^2}\, dk\, d\varkappa, \qquad (8)$$

where we have introduced the notation $M_A = V_0/c_A$ and $M_s = V_0/c_s$ for the magnetohydrodynamic and acoustic Mach numbers, along with

$$\gamma^2 = 1 - \frac{M_A^2 M_s^2}{M_A^2 + M_s^2 - 1} = \frac{(1 - M_A^2)(1 - M_s^2)}{1 - M_A^2 - M_s^2}. \qquad (9)$$

The character of the radial magnetic field $h_r$, and thus of the other disturbances $\rho$, p, $\mathbf{v}$, and $\mathbf{h}$, depends greatly on the sign of the parameter $\gamma^2$.[4] When $\gamma^2 > 0$ Cerenkov radiation is absent and all disturbances are localized near the moving source. On the other hand, when $\gamma^2 < 0$, in which case the

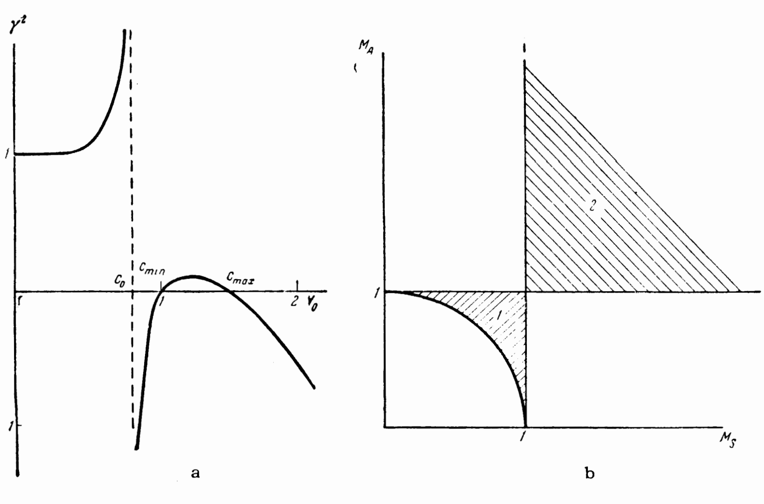


FIG. 1. a – the parameter $\gamma^2$ vs. the velocity $V_0$ of the source; b – in the plane of the numbers $M_A$ and $M_s$ the emission regions of slow and fast magnetoacoustic waves are shaded.

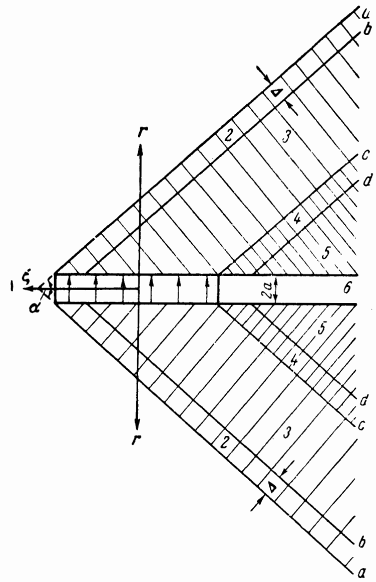


FIG. 2. Magnetohydrodynamic flow around a solenoid bearing a direct current in the presence of Cerenkov radiation. The shaded regions contain disturbances.

integrand in (8) contains poles lying on the real axis, conical magnetoacoustic waves are radiated which are entirely analogous to weak Mach waves occurring in hydrodynamics in the case of supersonic flow around bodies.[6] Figure 1a shows the dependence of $\gamma^2$ on the velocity $V_0$;[1] in Fig. 1b the plane of the parameters $M_A$ and $M_S$ contains two shaded regions 1 and 2 where $\gamma^2 < 0$. Morozov has shown[1] that region 1 corresponds to the emission of a slow magnetoacoustic wave while region 2 corresponds to a fast magnetoacoustic wave. We shall discuss the consequences of the behavior of $\gamma^2$ as the velocity of the source is varied.

When $\gamma^2 > 0$ the double integral in (8) is calculated by familiar procedures:[7,8]

$$h_r = \frac{a\gamma I_0}{cl(1-M_A{}^2)\sqrt{ar}}\left\{Q_{1/2}\left[\frac{\gamma^2(r^2+a^2)+(l-\xi)^2}{2\gamma^2 ar}\right]\right.$$
$$\left.-Q_{1/2}\left[\frac{\gamma^2(r^2+a^2)+(l+\xi)^2}{2\gamma^2 ar}\right]\right\}, \tag{10}$$

where $Q_{1/2}$ is a spherical (Legendre) function of the second kind (a special toroidal function[9]). When $V_0 = 0$, Eq. (10) gives the magnetic field component $h_r$ of a fixed current-bearing solenoid.

The expression for $h_r$ is considerably more complicated in the presence of radiation, i.e., when $\gamma^2 = -\gamma_1^2 < 0$. In this case the double integral in (8) and the functions that it determines are discontinuous. Upon integrating with respect to the wave number $\kappa$ with pole indentations in the upper half-plane and then integrating with respect to the radial wave number k, we obtain the corresponding expressions for $h_r$. Figure 2 shows schematically the separate perturbation regions 1—6 bounded by the surfaces a, b, c, and d on which the function $h_r(r, \xi)$ becomes discontinuous. These discontinuity surfaces are defined by the equations

$$l-\xi=\gamma(r-a), \qquad l-\xi=\gamma(r+a),$$
$$-l-\xi=\gamma(r-a), \qquad -l-\xi=\gamma(r+a) \tag{11}$$

The radial components $h_r$ of the magnetic field in these regions are

$$h_{r1}=0,$$
$$h_{r2}=\frac{\pi a\gamma_1 I_0}{cl(M_A{}^2-1)\sqrt{ar}}P_{1/2}\left[\frac{\gamma_1^2(r^2+a^2)-(l-\xi)^2}{2\gamma_1^2 ar}\right]$$
$$h_{r3}=\frac{2a\gamma_1 I_0}{cl(M_A{}^2-1)\sqrt{ar}}Q_{1/2}\left[-\frac{\gamma_1^2(r^2+a^2)-(l-\xi)^2}{2\gamma_1^2 ar}\right],$$
$$h_{r4}=h_{r3}-\frac{\pi a\gamma_1 I_0}{cl(M_A{}^2-1)\sqrt{ar}}P_{1/2}\left[\frac{\gamma_1^2(r^2+a^2)-(l+\xi)^2}{2\gamma_1^2 ar}\right],$$
$$h_{r5}=h_{r3}-\frac{2a\gamma_1 I_0}{cl(M_A{}^2-1)\sqrt{ar}}Q_{1/2}\left[-\frac{\gamma_1^2(r^2+a^2)-(l+\xi)^2}{2\gamma_1^2 ar}\right], \tag{12}$$

where $P_{1/2}$ is a sphérical (Legendre) function of the first kind. Expressions for $h_{r2}$, $h_{r3}$, $h_{r4}$, and $h_{r5}$ were obtained for $r \geq a$. For region 6, the wakes of the moving source for $r \leq a$ of expression $h_r$ are of no special interest and are not given here.

It is easily seen from the asymptotic forms of $P_{1/2}$ and $Q_{1/2}$ that a finite discontinuity of $h_r$ occurs on the discontinuity surfaces a and c, whereas on the surfaces b and d it approaches infinity logarithmically. Thus for the sharply bounded source considered here two conical waves occur (regions 2 and 4) with the vertex angle $\alpha = 2\tan^{-1}(1/\gamma_1)$. The width $\Delta$ of both waves is given by

$$\Delta = 2a\gamma_1(1+\gamma_1^2)^{-1/2}. \tag{13}$$

The character of the disturbances resembles to a considerable degree the head and tail Mach shock waves appearing in supersonic flow around bodies of rotation. A more detailed analysis of the fields of the disturbances will not be given here; in this analysis it is more convenient to represent spherical Legendre functions of half-integral order in terms of elliptic integrals.[9]

We can now determine the radiative deceleration force acting on a moving solenoid in the presence of Cerenkov magnetoacoustic waves. For this purpose we first determine the total force exerted on the electric current (3) by the magnetic field disturbances which it had generated:

$$\mathbf{F}=\frac{1}{c}\int_0^{+\infty}\int_{\infty}^{+\infty}\int_0^{2\pi} r[\mathbf{j}_{\text{extr}}\mathbf{h}^*]\,dr\,dz\,d\varphi, \tag{14†}$$

†$[\mathbf{jh}] = \mathbf{j}\times\mathbf{h}$.

where * denotes the complex conjugate quantity. Substituting j from (3) and $h_r$ from (8) in (14), and integrating over the radial wave numbers k and the coordinates r, z, $\varphi$, we finally obtain for the projection of the force **F** on the direction of motion:

$$F_z = -\frac{4\pi^2 a^2 I_0^2 |M_s^2 - 1|}{c^2 l^2 (M_A^2 + M_s^2 - 1)} \int_0^{+\infty} \frac{\sin^2 \varkappa l}{\varkappa} J_1^2(\gamma_1 a \varkappa)\, d\varkappa. \tag{15}$$

The modulus in the numerator of the expression for $F_z$ appears because when a slow magnetoacoustic wave is emitted ($M_A^2 + M_s^2 > 1$, $M_A$ and $M_s < 1$) the solution for the field $h_r$ must be taken in the form of advanced-phase waves.[1,10] The radiation energy $\mathscr{E}$ of magnetoacoustic waves has a simple relation to the wave drag:[4]

$$d\mathscr{E} / dt = F_z V_0. \tag{16}$$

The integral in (15) is convergent and is calculated by a familiar procedure:

$$\int_0^{\infty} \frac{\sin^2 \varkappa l}{\varkappa} J_1^2(\gamma_1 a \varkappa)\, d\varkappa = \begin{cases} \dfrac{E(\zeta) - (1 - \zeta^2) K(\zeta)}{\pi \zeta^2} & \text{for } 0 < \zeta \leqslant 1, \\[2ex] \dfrac{E(1/\zeta)}{\pi \zeta} & \text{for } 1 \leqslant \zeta < \infty \end{cases} \tag{17}$$

where $\zeta = a\gamma_1/l$, and K and E are complete elliptic integrals of the first and second kinds. We shall show that for $l \to 0$, Eq. (15) becomes the divergent integral over $\kappa$ that was derived by Morozov.[1]

The convergence of the integrals in the expressions for the wave drag and, therefore, for the radiation energy of an extended solenoidal source, is not accidental, as will be shown by a second example. The problem regarding the excitation of magnetoacoustic disturbances by a moving source of variable mass can be solved very similarly. We shall assume $j_{extr} = 0$ and shall represent the volume source of mass Q in the continuity equation (1) by a generalized function:

$$Q(r, z, t) = \begin{cases} Q_0/4\pi l a & \text{for } |z - V_0 t| \leqslant l, r \leqslant a \\ 0 & \text{in the rest of space,} \end{cases} \tag{18}$$

where $2l$ and a are, respectively, the length and transverse radius of the region into which plasma is injected by a mass source of strength $Q_0$. It would be of little value to present the complicated expressions for disturbances in the medium. These are similar in general to those already considered here; we shall therefore limit ourselves to the expression for the radiation energy of magnetoacoustic waves:[5]

$$\frac{d\mathscr{E}}{dt} = -\int_S \left\{ p\mathbf{v} + \frac{1}{4\pi} [\mathbf{h}[\mathbf{v}\mathbf{H}_0]] \right\} d\mathbf{s}. \tag{19}$$

Selecting as the integration surface a cylindrical surface surrounding the trajectory of the mass source, and substituting into (19) the appropriate values of p, v, and h obtained from (1) and (2), we finally obtain

$$\frac{d\mathscr{E}}{dt} = -\frac{Q_0^2 V_0}{\pi \rho_0 a^2 l^2} \frac{M_A^2}{|M_s^2 - 1| (M_A^2 + M_s^2 - 1)} \times \int_0^{\infty} \frac{\sin^2 \varkappa l}{\varkappa^3} J_1^2(\gamma_1 \varkappa a)\, d\varkappa. \tag{20}$$

The expression for the reactive force of radiative deceleration is (16). It is easily seen that in this case the integral over the wave numbers $\kappa$ converges, and that its upper limit is given by [1)]

$$\int_0^{+\infty} \frac{\sin^2 \varkappa l}{\varkappa^3} J_1^2(\gamma_1 a \varkappa)\, d\varkappa \leqslant l^2 \int_0^{+\infty} J_1^2(\gamma_1 \varkappa a) \frac{d\varkappa}{\varkappa} = \frac{l^2}{2}$$

$$\text{for} \quad \gamma_1 a > 0. \tag{21}$$

For $l \to 0$ and $a \to 0$, Eq. (20) becomes the expression obtained in[2] for a point mass source.

It thus follows directly from (15), (16), and (20) that interference of Cerenkov radiation eliminates the divergence of the integrals for the wave drag and radiation energy. We can anticipate a similar result for all extended finite sources. It should be noted, however, that a divergence remains in the expressions for the energy and drag at the threshold of slow magnetoacoustic wave radiation, when

$$M_A^2 + M_s^2 - 1 = 0,$$

$$\text{i.e., for} \quad V_0 = c_0 = \frac{c_A c_s}{(c_A^2 + c_s^2)^{1/2}}. \tag{22}$$

This divergence can easily be removed by taking into account the wave absorption in the medium, i.e., thermal, viscous, and Joule losses. We also note that here in both cases we are analyzing disturbances generated by sources that do not excite an Alfvén wave.

We shall now consider how the wave drag and radiation energy of magnetoacoustic wave sources depend on the velocity. For this purpose we return to Fig. 1 and to the formulas (15) for wave drag and (20) for radiation energy. At low velocities, when

$$V_0 < c_A c_s / (c_A^2 + c_s^2)^{1/2} \tag{23}$$

1)The integral in (21) could not be expressed in terms of known functions.

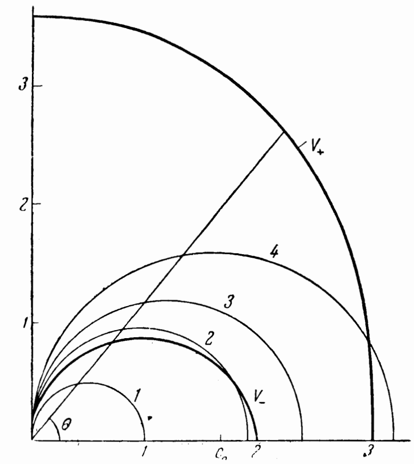


FIG. 3. Phase velocities of slow and fast magnetoacoustic waves vs. the angle $\theta$ between the wave vector and the external magnetic field. The semicircles 1 – 4 define the projections of four different source velocities on the direction $\theta$.

radiation is absent ($\gamma^2 > 0$) and the wave drag force vanishes. When (22) is fulfilled, in which case a slow magnetoacoustic wave is emitted, the energy and drag approach infinity if we neglect the wave dissipation processes in real media. Furthermore, with increasing velocity $F_z$ and $d\mathscr{E}/dt$ vary in a complex manner, depending on the properties of the medium and the nature of the interaction between the source and this medium. However, with the further growth of velocity we have a region where

$$\min(c_A, c_s) < V_0 < \max(c_A, c_s), \tag{24}$$

in which case $\gamma^2 > 0$, when radiation is again absent. Upon fulfillment of the condition

$$V_0 > \max(c_A, c_s) \tag{25}$$

a fast magnetoacoustic wave is emitted, and the wave drag again increases from zero to some finite level.

By analogy with the critical region of wave drag in the aerodynamics of transsonic velocities, in the considered case of a moving source of magnetoacoustic disturbances two critical regions of wave drag occur as the velocity is increased.

We shall now present a relatively simple kinematic explanation of the variation of wave drag and radiation intensity with increasing velocity of the sources of magnetohydrodynamic disturbances. It is well known that Cerenkov radiation exists in an isotropic medium when the velocity of the source equals or exceeds the phase velocity of the normal wave in the medium. Then there always exists a direction in which the phase velocity of the wave equals the projection of the source velocity on this direction. In the present case of an anisotropic medium the phase velocities of the fast and slow magnetoacoustic waves excited by the source depend on the angle $\theta$ between the wave vector and the external magnetic field $H_0$.

The heavy curves in Fig. 3 represent the known angular dependences of the phase velocities of the fast ($V_+$) and slow ($V_-$) waves:[5]

$$V_\pm^2 = \frac{1}{2}\{c_A^2 + c_s^2 \pm [(c_A^2 + c_s^2)^2 - 4c_A^2 c_s^2 \cos^2\theta]^{1/2}\}. \tag{26}$$

The figure also includes four semicircles representing $V_0 \cos\theta$, i.e., the projections of four different source velocities $V_0$ on the direction $\theta$. Curve 1 corresponds to the velocity region (23), when radiation is absent. For the semicircle 2 there is a direction along which $V_{02}\cos\theta = V_-$ and a slow magnetoacoustic wave is emitted.

Curve 3 was plotted for the velocity associated with the region of (24). In this case the phase velocity of the slow wave is seen to be everywhere smaller, and that of the fast wave is everywhere greater, than the projection of $V_0$; therefore, radiation is again absent. Finally, semicircle 4, which intersects the curve of $V_+$, indicates the existence of a fast magnetoacoustic wave for the velocity region defined by (25). Since the double critical region of wave drag in magnetohydrodynamics is of purely kinematic character, it should accompany any dynamic interaction between a medium and moving sources of magnetohydrodynamic disturbances. Specifically, a double critical region of wave drag exists when an electroconducting magnetoactive plasma flows around various bodies.

In conclusion, I wish to thank A. A. Andronov for a valuable discussion of the present results.


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