KINEMATIC SINGULARITIES OF HELICITY AMPLITUDES. INTEGRAL SPINS

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The general relation between helicity amplitudes and tensor amplitudes, derived in this paper, is used for determining the kinematic singularities (square root branch points, poles and zeros) in the helicity amplitudes for processes of the type $1 + 2 \to 3 + 4$ and $1 \to 2 + 3$, expressed as functions of invariant variables. The case of integral spins is considered.

1. INTRODUCTION

OVER the past few years the method based on the use of relativistic invariance, unitarity, analyticity and universality (crossing symmetry) has occupied a leading place in the theory of strong interactions. At the same time the discovery of a large number of resonances, the application of the method of Regge poles, as well as the intensive investigation of inelastic processes have focused attention on the description of processes in which particles of higher spin participate. The helicity representation for the amplitudes of such processes[1] offers several advantages, but has the disadvantage of possessing relatively complicated analytic properties. It is the objective of the present paper to clarify these properties.

The principle of analyticity requires of the S-matrix elements maximal analyticity, compatible with unitarity, in the components of the 4-momenta. This formulation of the principle suffices if the spin states of the interacting particles are described in terms of the tensor or spinor representation (in general, in terms of any representation in which the amplitude transforms under Lorentz transformations by means of a matrix which does not depend on the momenta). In this case the requirement that the amplitude possess only dynamical singularities (i.e., singularities required by the unitarity condition), does not contradict the transformation properties of the amplitudes. The situation is different when the spin states are described by means of the helicity representation, when the amplitude is transformed by means of a momentum dependent matrix. Maximal analyticity is restricted not only by the unitarity condition but also by the transformation properties. This new restriction implies the presence in the amplitude of poles and zeros in the momentum components.

Owing to invariance, the interaction amplitude for spinless particles depends on invariant quantities, and the analyticity principle implies maximal analyticity in these variables. In a fixed reference system one can consider the helicity amplitudes as a set of functions of the invariant variables, if the momentum components in the given reference system are expressed in terms of these invariants. Since this change of variables involves the taking of square roots, the helicity amplitude, considered as a function of the invariant variables will exhibit also kinematic branch points, in addition to poles and zeros. In the following we call the branch points, poles, and zeros of the indicated origin kinematic singularities. In the present paper the character of such singularities is investigated for amplitudes describing transitions of one particle into two particles and transitions of two particles into two particles, for the case of integral spins. The case of half-integral spins will be considered in a forthcoming paper.

We shall make use of the following method. We first consider the amplitude in the tensor representation and then go over to the helicity representation. Using the analyticity of the tensor amplitude and the explicit form of the transformation to the helicity representation (these functions are derived in Sec. 2) we determine the kinematic singularities of interest.

2. THE TRANSFORMATION FUNCTIONS FROM THE TENSOR REPRESENTATION TO THE HELICITY REPRESENTATION

In the tensor representation the state of a particle of mass m and spin σ (this state will be denoted by $|p, \{i\}\rangle$, where p is the four-momentum and $\{i\}$ the ensemble of tensor indices $i_1, \ldots, i_\sigma$) transforms as a tensor of rank σ subjected to the supplementary conditions:

A) symmetry in the indices:

$$P(i)\,|p, \{i\}\rangle = |p, \{i\}\rangle, \tag{1a}$$

where $p(i)$ is any permutation of indices;

B) tracelessness with respect to any pair of indices:

$$|p,\ \{i\}\rangle g^{i_1 i_2} = 0; \tag{1b}$$

C) transversality condition:

$$|p,\ \{i\}\rangle p^{i_1} = 0. \tag{1c}$$

The condition C) guarantees the vanishing of the time-components in the rest system ($\mathbf{p} = 0$); the conditions A) and B) have the consequence that the 3-tensor $|m, 0, \{\alpha\}\rangle$ ($\alpha = 1, 2, 3$) transforms under rotations according to an irreducible representation of weight σ and has $2\sigma + 1$ independent components.

The conditions (1) can be written in the concise form:

$$|p,\ \{i\}\rangle \theta^{\{i\}}_{\{j\}}(p) = |p,\ \{j\}\rangle, \tag{2}$$

where

$$\theta^{\{i\}}_{\{j\}}(p) = \theta^{\{i\}}_{\{k\}}(p)\,\theta^{\{k\}}_{\{j\}}(p)$$

is a tensor projection operator constructed out of the metric tensor g^i_j and the vector p and satisfying in the ensemble of both indices the conditions (1) [2]. For $\sigma = 1$

$$\theta_j{}^i(p) = g_j{}^i - p^i p_j / m^2.$$

The tensor states are normalized in the following manner:

$$\langle p',\ \{i'\}\,|\,p,\ \{i\}\rangle = 2p^0\delta(\mathbf{p}-\mathbf{p}')\,\theta^{\{i'\}}_{\{i\}}(p). \tag{3}$$

In the rest system the components of the tensor $|m, 0, \alpha\rangle$ with respect to the canonical basis

$$\mathbf{e}_0 \equiv \mathbf{e}_z,\ \mathbf{e}_\pm = \mp(\mathbf{e}_x \pm i\mathbf{e}_y)/\sqrt{2}$$

are eigenvectors of the spin projection on the z axis; the value $\sigma_z = \lambda$ ($|\lambda| \le \sigma$) corresponds to the state vector

$$|m,\ 0,\ \lambda\rangle \equiv |m,\ 0,\ \{\alpha\}\rangle[\mathbf{e}_0{}^{\sigma-|\lambda|}\mathbf{e}_+{}^{-\lambda}]^{\{\alpha\}}. \tag{4*}$$

Here $\mathbf{e}^\nu$ denotes a tensor of rank ν with components $\prod_{n=1}^{\nu} e^{\alpha_n}$, and for $\lambda > 0$ it is understood that $\mathbf{e}_+^{-\lambda} = (-1)^\lambda \mathbf{e}_-^\lambda$, since $\mathbf{e}_+ \cdot \mathbf{e}_- = -1$ 1).

By definition, the helicity state $|p, \lambda\rangle$ is obtained from the state $|m, 0, \lambda\rangle$ [definition (4)] by applying the operator $\mathscr{H}(p)$:

$$|p,\ \lambda\rangle = \mathscr{H}(p)\,|m,\ 0,\ \lambda\rangle, \tag{5}$$

$$\mathscr{H}(p) = R(\mathbf{p})\,L_z(|\mathbf{p}|/p^0),$$

where $L_z(v)$ is a "boost" (pure Lorentz transformation) along the z axis with velocity v; $R(\mathbf{p})$ is the rotation in the $\mathbf{e}_0$, $\mathbf{e}_p$ plane which transforms $\mathbf{e}_0$ into $\mathbf{e}_p = \mathbf{p}/|\mathbf{p}|$. Under a Lorentz transformation G the state $|p, \lambda\rangle$ transforms according to the equation (cf. [3]):

$$\begin{gathered} G\,|p,\ \lambda\rangle = |p',\ \lambda'\rangle D^{(\sigma)}_{\lambda'\lambda}(R), \\ p' = Gp, \qquad R \equiv R(G,\ p) = \mathscr{H}^{-1}(p)\,G\,\mathscr{H}(p), \end{gathered} \tag{6}$$

where $D^{(\sigma)}_{\lambda'\lambda}(R)$ is the matrix element of the irreducible representation of weight σ of the rotation group.

In order to find the relation between helicity and tensor representations, we operate on both sides of Eq. (4) with the operator $\mathscr{H}(p)$:

$$\begin{aligned} |p,\ \lambda\rangle &= \mathscr{H}(p)\,|0,\ \{\alpha\}\rangle[\mathbf{e}_0{}^{\sigma-|\lambda|}\mathbf{e}_+{}^{-\lambda}]^{\{\alpha\}} \\ &= |p,\ \{i\}\rangle H^{i_1}_{\alpha_1}\dots H^{i_\sigma}_{\alpha_\sigma}\,[\mathbf{e}_0{}^{\sigma-|\lambda|}\mathbf{e}_+{}^{-\lambda}]^{\{\alpha\}} \\ &= |p,\ \{i\}\rangle[h_0{}^{\sigma-|\lambda|}(p)\,h_+{}^{-\lambda}(p)]^{\{i\}}, \end{aligned} \tag{7}$$

where $H^i_\alpha(p)$ are the matrix elements of the transformation $\mathscr{H}$ and we have used the notation

$$h_\lambda{}^i(p) \equiv H_\alpha{}^i(p)\,e_\lambda{}^\alpha, \qquad \lambda = 0, \pm. \tag{8}$$

Thus it follows from (3) and (7) that:

$$\begin{gathered} \langle p',\ \{i\}\,|\,p,\ \lambda\rangle = 2p^0\delta(\mathbf{p}-\mathbf{p}')\,H^{\{i\}}_\lambda, \\ H_\lambda{}^{\{i\}} \equiv \theta^{\{i\}}_{\{j\}}(p)[h_0{}^{\sigma-|\lambda|}(p)\,h_+{}^{-\lambda}(p)]^{\{j\}}. \end{gathered} \tag{9}$$

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*$[\mathbf{e}_0\mathbf{e}_+] = \mathbf{e}_0 \times \mathbf{e}_+$

1)The inverse of Eq. (4) has the form

$$|m,\ 0,\ \{\alpha\}\rangle = \sum_\lambda |m,\ 0,\ \lambda\rangle\,\theta^{\{\alpha'\}}_{\{\alpha\}}(m,\ 0)[(\mathbf{e}_0)^{\sigma-|\lambda|}(\mathbf{e}_+)^\lambda]_{\{\alpha'\}}.\ \ |m,\ 0,\ \lambda\rangle$$

We note that on account of the symmetry properties of the tensor $\theta^{\{\alpha'\}}_{\{\alpha\}}(m, 0)$ (and analogous properties of the tensor $|m, 0, \{\alpha\}\rangle$) several other combinations of the vectors $\mathbf{e}_\pm$ and $\mathbf{e}_0$ give the same contribution to the right hand side of Eq. (4) as the ones which have been written out. (The form of the tensor included in the square brackets in Eq. (4). has been chosen for convenience in writing.) For instance, in the case of spin 2

$$\theta^{(\alpha_1'\alpha_2')}_{\alpha_1\alpha_2}(m,\ 0)\,e_{+\alpha_1'}\,e_{-\alpha_2'} = \theta^{\alpha_1'\alpha_2'}_{\alpha_1\alpha_2}(m,\ 0)\,e_{-\alpha_1}\,e_{+\alpha_2} = \theta^{\alpha_1'\alpha_2'}_{\alpha_1\alpha_2}(m,\ 0)\,e_{0\alpha_1'}\,e_{0\alpha_2'}.$$

We now determine the quantities $h^i_\lambda(p)$. It follows from the definitions (8) and (5) that:

$$h_0{}^0 = |\mathbf{p}|/m, \quad \mathbf{h}_0 = \mathbf{p}p^0/m|\mathbf{p}|, \quad h_\pm{}^0 = 0. \tag{10}$$

The 3-vectors $\mathbf{h}_\pm$ are obtained from $\mathbf{e}_\pm$ by means of the rotation $R(\mathbf{p})$. We shall call the quantities h^i_λ "helicity vectors," but it should be kept in mind that these are not genuine 4-vectors, since under a Lorentz transformation they undergo the transformation:

$$h^i_\lambda \to G^i_{i'} h^{i'}_{\lambda'} D^{(1)}_{\lambda'\lambda}[R(G,p)].$$

The helicity vectors are defined by the following properties:

$$h_\lambda{}^i p_i = 0, \quad h_\lambda{}^i(p) h_{i\lambda'}(p) = -(\mathbf{e}_\lambda \mathbf{e}_{\lambda'}) = (-1)^{1-\lambda}\delta_{\lambda,-\lambda'}.$$

Hence

$$H_\lambda{}^{\{i\}} H_{\{i\}\lambda'} = (-1)^{\sigma-\lambda}\delta_{\lambda,-\lambda'}, \quad \lambda = 0, \pm. \tag{11}$$

3. KINEMATIC SINGULARITIES OF THE AMPLITUDE OF THE PROCESS 1 + 2 → 3 + 4

We consider the helicity amplitude for the transition of particles 1, 2 into particles 3, 4; the masses and spins of the particles are m_n, σ_n, $n = 1, \ldots, 4$. The helicity amplitude is expressed in terms of the tensor amplitude according to Eq. (9):

$$_{\lambda_3\lambda_4}M_{\lambda_1\lambda_2} = H^*_{\{i_3\}\lambda_3} H^*_{\{i_4\}\lambda_4} {}^{\{i_3\}\{i_4\}}M_{\{i_1\}\{i_2\}} H^{\{i_1\}}_{\lambda_1} H^{\{i_2\}}_{\lambda_2}. \tag{12}$$

Here the indices refering to the incoming particles are written to the right, those for outgoing particles to the left. The tensor amplitude satisfies the supplementary conditions in each of the four ensembles of indices and is analytic in the momentum components. This means that any invariant obtained from the tensor amplitude by contraction with an analytic tensor is free of kinematic singularities with respect to the invariant variables m_n^2, s, t. The reason for the presence of singularities in the helicity amplitude is that the helicity tensors $H(p_n)$ have singularities in the momentum components. We shall fix the reference system, find the form of the quantities $H(p_n)$ in that reference system, and thus derive the singularities of the helicity amplitudes in that system.

We choose the most frequently used center of inertia system, in which $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_2 + \mathbf{p}_3 = 0$ (the C-system). [The analytic properties are obtained in any system by using the transformation formula (6).] In the C-system we choose the z axis along the momentum $\mathbf{p}_1$ and the y axis along the direction $\mathbf{p}_1 \times \mathbf{p}_3$. Following the conventions of [1] we choose different basis vectors in the rest systems of the particles 1 and 2, 3 and 4, which is equivalent to the introduction of a factor $(-1)^{\sigma_2+\sigma_4-\lambda_2-\lambda_4}$ in front of the amplitude.

The quantities $h_\lambda(p_n)$ are related to the amplitude in the following manner (it is convenient to express this connection in the form of an equality between helicity vectors and genuine four-vectors, constructed out of the momenta, and such that these four-vectors coincide with the helicity vectors in the C-system):

$$h_0(p_1) = \frac{2m_1}{\sqrt{K}}\left(-p_2 + p_1\frac{(p_1p_2)}{m_1^2}\right),$$
$$h_0(p_2) = \frac{2m_2}{\sqrt{K}}\left(-p_1 + p_2\frac{(p_1p_2)}{m_2^2}\right),$$
$$h_0(p_3) = \frac{2m_3}{\sqrt{K'}}\left(-p_4 + p_3\frac{(p_3p_4)}{m_3^2}\right),$$
$$h_0(p_4) = \frac{2m_4}{\sqrt{K'}}\left(-p_3 + p_4\frac{(p_3p_4)}{m_4^2}\right),$$
$$h_\pm = \mp(h_x \pm ih_y);$$
$$h_x(p_1) = h_x(p_2) = \frac{2}{\sqrt{KQ}}(a_1p_1 + a_2p_2 + a_3p_3),$$
$$h_x(p_3) = h_x(p_4) = \frac{2}{\sqrt{K'Q}}(b_1p_1 + b_3p_3 + b_4p_4),$$
$$h_y(p_1) = -h_y(p_2) = h_y(p_3)$$
$$= -h_y(p_4) = [p_1p_2p_3]/\sqrt{Q} \equiv h_y, \tag{13}$$

where the vector $[p_1p_2p_3]$ has the components $[p_1p_2p_3]_i = \epsilon_{ii_1i_2i_3}p_1^{i_1}p_2^{i_2}p_3^{i_3}$, and ϵ is the completely antisymmetric unit tensor.

The quantities a, b, K, K′, and Q in (13) are polynomials in the invariant variables:

$$K = 4((p_1p_2)^2 - m_1^2m_2^2)$$
$$= [s - (m_1 + m_2)^2][s - (m_1 - m_2)^2],$$
$$K' = 4((p_3p_4)^2 - m_3^2m_4^2)$$
$$= [s - (m_3 + m_4)^2][s - (m_3 - m_4)^2];$$
$$a_1 = m_2^2(p_1p_3) - (p_1p_2)(p_2p_3),$$
$$a_2 = m_1^2(p_2p_3) - (p_1p_2)(p_1p_3), \quad a_3 = {}^1/_4 K;$$
$$b_3 = m_4^2(p_1p_3) - (p_1p_4)(p_3p_4),$$
$$b_4 = m_3^2(p_1p_4) - (p_1p_3)(p_3p_4), \quad b_1 = {}^1/_4 K';$$
$$Q = stu - (m_1^2 - m_3^2)(m_2^2 - m_4^2)s$$
$$-(m_1^2 - m_2^2)(m_3^2 - m_4^2)t$$
$$-(m_1^2m_4^2 - m_2^2m_3^2)(m_1^2 - m_2^2 - m_3^2 + m_4^2). \tag{14}$$

The momentum components can be expressed in terms of the invariant amplitudes by means of the equations

$$|\mathbf{p}_1| = |\mathbf{p}_2| = \sqrt{K}/2\sqrt{s}; \qquad |\mathbf{p}_3| = |\mathbf{p}_4| = \sqrt{K'}/2\sqrt{s};$$

$$\sin\theta = 2\sqrt{sQ}/\sqrt{KK'}; \quad \cos\theta = [s(t-u) - (m_1^2 - m_2^2)(m_3^2 - m_4^2)]/\sqrt{KK'} \tag{15}$$

(θ is the angle between the momentum vectors p_1 and p_3).

The tensor amplitude is constructed out of the momenta, the metric tensor g and the pseudotensor ϵ; therefore, for the determination of the kinematic singularities it is necessary to consider all possible products $h_\lambda(p_n)p_m$, $h_\lambda(p_n)\times h'_\lambda(p_m)$ etc.; the singularities of these products give rise to the singularities in the helicity amplitude.

The position of the singularities is clear from Eq. (13): these are the points

$$m_n^2 = 0, \; K = 0, \; K' = 0, \; Q = 0.$$

1) $m_n^2 = 0$. Since the tensor amplitude is subject to the condition $M_i p_n^i = 0$, where i is any index from the set $\{i_n\}$, we can leave out the terms proportional to p_n everywhere in the expressions (13) for the quantities $h_\lambda(p_n)$. Then it follows from (9) and (12) that

$$M_{\lambda_n} \sim m_n^{\sigma_n - |\lambda_n|}. \tag{16}$$

We note that this result is valid for any amplitude and has a simple physical interpretation: all amplitudes with $|\lambda_n| \neq \sigma_n$ vanish for $m_n \to 0$.

We now assume that the masses are fixed. Three cases have to be considered separately: the processes of the two-meson annihilation type, $m_1 = m_2$, $m_3 = m_4$; production of massless particles $m_3 = m_4 = 0$; and the general case. The first two cases are distinguished by the fact that at the same time $Q \sim s$ and $KK' \sim s^2$. We consider the general case first.

2) K = 0 and K′ = 0. We form combinations of the helicity amplitudes which admit a factorization of the singularities at K = 0 and K′ = 0. We introduce the combination of helicity tensors

$$H_\lambda^\pm = {}^1/_2(H_\lambda \pm (-1)^\lambda H_{-\lambda}). \tag{17}$$

Using the expressions (9) and (13) it is easy to show that

$$H_\lambda^+(p_n) = K^{-\sigma_n/2}\hat{H}_\lambda^+(p_n),$$
$$H_\lambda^-(p_n) = K^{(-\sigma_n+1)/2}\hat{H}_\lambda^-(p_n), \quad n = 1, 2, \tag{18}$$

where the quantities $\hat{H}$ have no singularities as $K \to 0$. For n = 3, 4, one should replace K by K′ in (18).

Replacing in (12) H by $H^\pm$ we obtain sixteen quantities ${}^{\epsilon_3\epsilon_4}M^{\epsilon_1\epsilon_2}$, $\epsilon_n = \pm$. It follows from (18) that

$${}^{\varepsilon_3\varepsilon_4}M^{\varepsilon_1\varepsilon_2} = K^{(-\sigma_1-\sigma_2+\delta_1+\delta_2)/2} K'^{(-\sigma_3-\sigma_4+\delta_3+\delta_4)/2}\hat{M}, \tag{19}$$

where $\hat{M}$ has no singularities for $K \to 0$ and $K' \to 0$ and $\delta_n = 0$ for $\epsilon_n = +$ and $\delta_n = 1$ for $\epsilon = -$.

If parity is conserved

$$(-1)^{\lambda_1-\lambda_2-\lambda_3+\lambda_4}{}_{-\lambda_3-\lambda_4}M_{-\lambda_1-\lambda_2} = \eta_g\, {}_{\lambda_3\lambda_4}M_{\lambda_1\lambda_2} \tag{20}$$
$$\eta_g = (-1)^{\sigma_3+\sigma_4-\sigma_1-\sigma_2}\eta_1\eta_2/\eta_3\eta_4,$$

and some of the quantities (19) vanish. Finally

$${}_{\lambda_3\lambda_4}M_{\lambda_1\lambda_2} = K^{-(\sigma_1+\sigma_2)/2}K'^{-(\sigma_3+\sigma_4)/2}({}^+A + {}^-A), \tag{21}$$

where

$${}_{\lambda_3\lambda_4}^{\pm}A_{\lambda_1\lambda_2} = \pm(-1)^{\lambda_3-\lambda_4}{}_{-\lambda_3-\lambda_4}^{\pm}A_{\lambda_1\lambda_2};$$
$${}^+A = {}^+\hat{A}, \quad {}^-A = {}^-\hat{A}(KK')^{1/2} \text{ for } \eta_g = +1, \tag{21'}$$
$${}^+A = {}^+\hat{A}K^{1/2}, \quad {}^-A = {}^-\hat{A}(K')^{1/2} \text{ for } \eta_g = -1,$$

where ${}^\pm\hat{A}$ are functions which do not have singularities at K, K′ → 0.

We note that in the case of elastic scattering ($K = K'$, $\eta_g = +1$, $\sigma_1 = \sigma_3$, $\sigma_2 = \sigma_4$) the amplitude does not have a cut but only a pole of order $\sigma_1 + \sigma_2$ at $K \to 0$. If $\sigma_3 = \sigma_4 = 0$, ${}^-\hat{A} = 0$ and there is no singularity at K′ = 0.

3) Q = 0. This singularity can be found easily by means of a partial wave expansion[1,4]:

$$M = \sum_j (2j+1)F^j d^j_{\lambda\lambda'}(\theta),$$

where $\lambda = \lambda_1 - \lambda_2$, $\lambda' = \lambda_3 - \lambda_4$, and $d^j_{\lambda\lambda'}$ is a generalized spherical function, namely

$$d_{\lambda\lambda'} = (1-\cos\theta)^{|\lambda'-\lambda|/2}(1+\cos\theta)^{|\lambda'+\lambda|/2}P(\cos\theta),$$

where P is a polynomial in cos θ. Since F^j does not depend on t, the only source of singularities for Q = 0 is sin θ. Therefore

$$M \sim Q^{\varkappa/2}, \tag{22}$$
$$\varkappa = \min\{|\lambda - \lambda'|, |\lambda + \lambda'|\} = ||\lambda| - |\lambda'||.$$

The first special case:

$$m_1 = m_2 \equiv m, \quad m_3 = m_4 \equiv m', \quad K = (s - 4m^2)s \equiv ks,$$
$$K' = (s - 4m'^2)s \equiv k's,$$
$$Q = s[tu - (m^2 - m'^2)^2] \equiv sq.$$

The singularities are:

$$k = 0, \quad k' = 0, \quad q = 0 \text{ and } s = 0.$$

The behavior of the amplitude at k = 0, k′ = 0 and q = 0 is described by equations (21) and (22) in which we must replace K by k, K′ by k′, and Q by q. Besides, the quantities $\hat{A}$ have additional zeros for s = 0, so that

$$M \sim s^{\gamma/2}, \qquad (23)$$

the values of γ depend on the quantity $\xi = (-1)^{\Sigma_n(\sigma_n - \lambda_n)}$:

$$\gamma = \begin{cases} 0 \text{ for } \xi = +1, \ \eta_g = +1; \\ 1 \text{ for } \xi = -1, \ \eta_g = +1; \\ 1 \text{ for } \xi = +1, \ \eta_g = -1; \\ 2 \text{ for } \xi = -1, \ \eta_g = -1. \end{cases} \qquad (23')$$

The second special case. As in the preceding case the quantities $\hat{A}$ have zeros in s, however the amplitude as a whole may have both zeros and poles in $\sqrt{s}$:

$$_{\lambda_3\lambda_4}M = K^{-(\sigma_1+\sigma_2)/2} q^{\varkappa/2} s^{(|\lambda|-\sigma_3-\sigma_4)/2} ({}^{+}A + {}^{-}A), \qquad (24)$$

where $|\lambda_3| = \sigma_3$ and $|\lambda_4| = \sigma_4$; the quantities ${}^{\pm}A$ are defined by (21′).

4. SIMPLE EXAMPLES

For some simple examples we find the relation between helicity and tensor amplitudes and follow through the appearance of singularities.

1. $\sigma_1 = 1$, $\sigma_2 = \sigma_3 = \sigma_4 = 0$. The tensor amplitude has the form

$$M_i = A_1 p_{1i} + A_2 p_{2i} + A_3 p_{3i} + iBN_i, \quad N = [p_1 p_2 p_3],$$
$$A_1 m_1^2 = -A_2(p_1p_2) - A_3(p_3p_1).$$

Using Eqs. (13) and (14) we obtain

$$M_{\pm 1} = \sqrt{Q/2K}(\mp A_3 + \sqrt{K}B),$$
$$M_0 = -(2m_1/\sqrt{K})[A_1(p_1p_2) + A_2 m_2^2 + A_3(p_2p_3)].$$

If parity is conserved, for $\eta_g = +1$ we have B = 0 and for $\eta_g = -1$ we have $A_1 = A_2 = A_3 = 0$.

2. $\sigma_1 = 2$, $\sigma_2 = \sigma_3 = \sigma_4 = 0$. The tensor amplitude is

$$M_{ik} = \sum_{m,n} A_{mn} p_{mi} p_{nk} + a g_{ik} + i \sum_m B_m (p_{mi} N_k + p_{mk} N_i);$$
$$m, n = 1, 2, 3.$$

The condition (1a) implies $A_{mn} = A_{nm}$; (1b) implies $4a = -\sum_{m,n} A_{mn}(p_m p_n)$ and (1c) implies equalities, as a result of which the number of independent amplitudes reduces to five:

$$\sum_m A_{mn}(p_1 p_m) + a\delta_{n1} = 0, \quad n = 1, 2, 3,$$
$$m_1^2 B_1 = -B_2(p_1p_2) - B_3(p_3p_1).$$

The helicity amplitudes are

$$M_{\pm 2} = (Q/2K)(A_{33} \pm \sqrt{K} B_3),$$
$$M_{\pm 1} = m_1(\sqrt{2Q}/K)\left[\pm \sum_m A_{m3}(p_m p_2) - \sqrt{K} \sum_m B_m(p_m p_2)\right],$$
$$M_0 = (4m_1^2/K)\left[\sum_{m,n} A_{mn}(p_m p_2)(p_n p_2) + a m_2^2\right].$$

3. $\sigma_1 = \sigma_2 = 1$, $\sigma_3 = \sigma_4 = 0$. The tensor amplitude is

$$M_{ik} = \sum_{m,n} A_{mn} p_{mi} p_{nk} + a g_{ik} + iC\varepsilon_{ikjl} p_{1j} p_{2l}$$
$$+ i \sum_m B_m p_{mi} N_k + i \sum_n B_n' p_{nk} N_i,$$
$$B_3 = B_3', \quad m, n = 1, 2, 3;$$

The subscript i refers to particle 1, the subscript k, to particle 2. Terms of the form $p_{3i}N_k - p_{3k}N_i$ and N_iN_k can be expressed in terms of those already written down and therefore are redundant. It follows from condition (1c) that

$$\sum_m A_{mn}(p_m p_1) + a\delta_{n1} = 0, \quad n = 1, 2, 3;$$
$$\sum_n A_{mn}(p_n p_2) + a\delta_{m2} = 0, \quad m = 1, 2, 3;$$
$$\sum_m B_m(p_m p_1) = 0; \quad \sum_n B_n'(p_n p_2) = 0.$$

We obtain the following expressions for the nine independent amplitudes (note that the rank of the matrix of the first six equations is equal to 5):

$$M_{\pm 1 \mp 1} = \frac{1}{2K}(A_{33}Q + 2aK \pm \sqrt{K}CK),$$
$$M_{\pm 1 \mp 1} = \frac{Q}{2K}(A_{33} \pm \sqrt{K}B_3),$$

$$M_{\pm 1, 0} = m_2\sqrt{2}\frac{\sqrt{Q}}{K}\left[\mp \sum A_{3n}(p_1 p_n) + \sqrt{K}\sum B_n(p_1 p_n)\right].$$

$$M_{0, \pm 1} = m_1\sqrt{2}\frac{\sqrt{Q}}{K}\left[\pm \sum A_{m3}(p_2 p_m) + \sqrt{K}\sum B_m(p_2 p_m)\right],$$
$$M_{0,0} = 4\frac{m_1 m_2}{K}\left[\sum A_{mn}(p_m p_1)(p_n p_2) + a(p_1 p_2)\right].$$

If parity is conserved either $B_m = B_n' = C = 0$, or $A_{mn} = a = 0$.

4. $\sigma_1 = \sigma_3 = 1$, $\sigma_2 = \sigma_4 = 0$. It is convenient to write the tensor amplitude in the form

$$^kM_i = \sum A_{mn} p_{mi} p_{nk} + a g_{ik} + iC\varepsilon_{ikjl} p_{1j} p_{3l}$$
$$+ iN_k \sum B_m p_{mi} + iN_i \sum B_n' p_{nk};$$
$$B_2 = B_4', \quad m = 1, 2, 3, \quad n = 1, 3, 4.$$

The consequences of (1c) are analogous to the preceding case. The helicity amplitudes have the form

$$_{\pm 1}M_{\pm 1} = {}^1/_2(1+\cos\theta)$$
$$\times [A_{33}pp'(1-\cos\theta) - a \pm C(E_1p' - E_3p)],$$
$$_{\mp 1}M_{\pm 1} = {}^1/_2(1-\cos\theta)$$
$$\times [-A_{33}pp'(1-\cos\theta) - a \pm C(E_1p' + E_3p)].$$

Here E_1, p and E_3, p′ are the energies and momenta of particles 1 and 3, respectively and θ is the angle between the momenta (equations (15)). The apparent pole in the coefficient of C is absent in reality, since the numerator vanishes for s = 0.

Further,

$$_0M_{\pm 1} = \sqrt{2}\, m_3 \sqrt{\frac{Q}{KK'}}\Big[\mp \sum A_{3n}(p_np_4) + \sqrt{K}\sum B_n'(p_np_4)\Big];$$

$$_{\pm 1}M_0 = \sqrt{2}\, m_1 \sqrt{\frac{Q}{KK'}}\Big[\mp \sum A_{m3}(p_mp_2) + \sqrt{K'}\sum B_m(p_mp_2)\Big];$$

$$_0M_0 = 4\frac{m_1m_3}{\sqrt{KK'}}\Big[\sum A_{mn}(p_mp_2)(p_np_4) + a(p_2p_4)\Big].$$

We remark that it is essential to take into account the transversality condition in determining the mass singularities.

5. THE KINEMATIC SINGULARITIES OF THE AMPLITUDE OF THE PROCESS 1 → 2 + 3

We consider the helicity amplitude of the decay 1 → 2 + 3 in the rest system of the decaying particle, $p_1 = p_2 + p_3 = 0$. We direct the z axis in this system along the momentum of particle 2. The choice of the x and y axes is irrelevant for the amplitudes. The helicity vectors have the form

$$h_0(p_1) = \frac{2m_1}{\sqrt{K}}\Big(p_2 - p_1\frac{(p_1p_2)}{m_1^2}\Big);$$

$$h_0(p_2) = \frac{2m_2}{\sqrt{K}}\Big(-p_3 + p_2\frac{(p_2p_3)}{m_2^2}\Big);$$

$$h_0(p_3) = \frac{2m_3}{\sqrt{K}}\Big(-p_2 + p_3\frac{(p_2p_3)}{m_3^2}\Big);$$

$$h_x(p_1) = h_x(p_2) = h_x(p_3); \quad h_y(p_1) = h_y(p_2) = -h_y(p_3);$$

$$K = m_1^4 + m_2^4 + m_3^4 - 2m_1^2m_2^2 - 2m_1^2m_3^2 - 2m_2^2m_3^2. \tag{25}$$

As for the four-line diagram treated in Sec. 3, the singularities at $m_n^2 = 0$ are described by Eq. (16).

There is also a singularity at K = 0. The tensor amplitude has the following form:

$$M = \sum_n A_n T^{(n)},$$

where A_n are analytic functions of m^2, and $T^{(n)}$ are all possible tensors which can be formed out of the momenta and g and ϵ. The helicity amplitude is

$$_{\lambda_2\lambda_3}M_{\lambda_1} = \sum_n A_n (H_{\lambda_2}^* H_{\lambda_3}^* T^{(n)} H_{\lambda_1}). \tag{26}$$

We note that $h(p_m)p_n \sim \sqrt{K}$ for $n \neq m$ and $\epsilon h_+h_-p_1p_2 \sim \sqrt{K}$; the remaining combinations of h, p, g and ϵ either vanish, or have no singularities as $K \to 0$. Thus

$$_{\lambda_2\lambda_3}M_{\lambda_1} = K^{(N-\nu)/2}\hat{M}, \tag{27}$$

where N is the minimal total power of the momenta in the tensors $T^{(n)}$ and $\nu = 0$ if $\eta\zeta = +1$, $\nu = 1$ if $\eta\zeta = -1$; $\eta = \eta_2\eta_3\eta_1^{-1}$; $\zeta = (-1)^{\sigma_1-\sigma_2-\sigma_3}$.

In order to determine the quantities N, one must distinguish two cases: the case when the values of the spins, σ_1, σ_2, σ_3 satisfy triangular inequalities, and the opposite case. We denote the difference between the largest spin and the sum of the two others by Δ. It is easy to see that

$$N - \nu = \begin{cases} \Delta & \text{for } \Delta > 0,\ \eta\zeta = +1; \\ \Delta + 1 & \text{for } \Delta > 0,\ \eta\zeta = -1; \\ 0 & \text{for } \Delta \leqslant 0,\ \eta = +1; \\ 1 & \text{for } \Delta \leqslant 0,\ \eta = -1. \end{cases} \tag{27'}$$

The obtained result agrees with the threshold behavior known from nonrelativistic mechanics, since the power exponent of $\sqrt{K}$ is the smallest possible for the given angular momenta and orbital parity[5].

6. KINEMATIC SINGULARITIES OF PARTIAL WAVE AMPLITUDES

We now return to the study of the four-line diagram and write down the expansion in states of determined total angular momentum in the following form:

$$M = \sum_j (2j+1)F^jd^j$$
$$= \sum (2j+1)(F^{j+} + F^{j-})(d^{j+} + d^{j-})$$
$$= \sum (2j+1)(F^{j+}d^{j+} + F^{j-}d^{j-})$$
$$+ \sum (2j+1)(F^{j+}d^{j-} + F^{j-}d^{j+}),$$
$$_{-\lambda_3-\lambda_4}F^{j\pm}_{\lambda_1\lambda_2} = \pm\, _{\lambda_3\lambda_4}F^{j\pm}_{\lambda_1\lambda_2}, \quad d^{j\pm}_{\lambda,-\lambda'} = \pm\, d^{j\pm}_{\lambda\lambda'}. \tag{28}$$

The decomposition of the amplitude into two terms corresponds to the two possible parities of the intermediate state; the amplitude $F^{j\pm}$ describes an intermediate state with the parity

$$\eta_{j\pm} = \pm\eta_3\eta_4(-1)^{j-\sigma_3-\sigma_4}. \tag{29}$$

The kinematic singularities of the partial wave amplitudes follow from the analytic properties of the total amplitude and those of the function d^j. We note that the decomposition of the total amplitude into two terms in Eq. (28) is identical with the decomposition in (21) and each of these terms possesses simple properties. On the other hand, since

$$d^j_{\lambda,-\lambda'}(\theta) = (-1)^{j+\lambda'} d^j_{\lambda\lambda'}(\pi+\theta),$$

depending on the sign $(-1)^{j+\lambda'}$ the functions $d^{j\pm}_{\lambda\lambda'} = \frac{1}{2}(d^j_{\lambda\lambda'} \pm d^j_{\lambda,-\lambda'})$ are even or odd with respect to the substitution $\cos\theta \to -\cos\theta$, $\sin\theta \to -\sin\theta$, i.e., $\sqrt{KK'} \to -\sqrt{KK'}$. Besides, it follows from the asymptotic form of the function d^j that for $KK' \to 0$

$$(KK')^{j/2} d^{j+} \sim \begin{cases} 1, & \text{if } (-1)^{\lambda'} = +1, \\ \sqrt{KK'}, & \text{if } (-1)^{\lambda'} = -1; \end{cases}$$

$$(KK')^{j/2} d^{j-} \sim \begin{cases} \sqrt{KK'}, & \text{if } (-1)^{\lambda'} = +1, \\ 1, & \text{if } (-1)^{\lambda'} = -1. \end{cases}$$

From Eqs. (21), (28), and (30) we obtain

$$F^{j\pm} \sim K^{l/2} K'^{l'/2},$$

where l and l' are the smallest orbital angular momenta of the initial and final states respectively, which are possible for given values of j and $\eta_{j\pm}$.

As for the three-line diagram, this result is in agreement with the nonrelativistic limit.

The partial waves also have singularities at s = 0.

1. $(m_1 - m_2)(m_3 - m_4) > 0$. In this case $1 - \cos\theta \sim s$ and $d_{\lambda\lambda'} \sim s^{|\lambda-\lambda'|/2}$ for $s \to 0$, and since the total amplitude has no singularities, it follows that

$$F^j \sim s^{-|\lambda-\lambda'|/2}. \tag{32}$$

2. $(m_1 - m_2)(m_3 - m_4) < 0$, $1 + \cos\theta \sim s$, $d_{\lambda\lambda'} \sim s^{|\lambda+\lambda'|/2}$, and

$$F^j \sim s^{-|\lambda+\lambda'|/2}. \tag{32'}$$

In those cases, when

$$(m_1 - m_2)(m_3 - m_4) = 0,$$

the behavior for s = 0 is modified in analogy to the behavior of the total amplitude.

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Note added in proof (January 6, 1965). There has recently appeared a paper by Y. Hara, Phys. Rev. 136, B507 (1964) devoted to the same problem. The author of the indicated paper has obtained amplitudes which are free of kinematic branch points and poles, but possess zeros of kinematic origin.

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