INFRARED SINGULARITIES OF THE VERTEX PART IN QUANTUM ELECTRODYNAMICS

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The infrared singularities of the form factors for the charge and for the anomalous magnetic moment of the electron are calculated in perturbation theory to fifth and seventh orders.

1. In quantum electrodynamics the problem of the infrared singularities of the vertex part can be considered from two points of view. If the external momenta of the electrons p_1 and p_2 are on the mass shell, then by introducing the photon mass λ it is possible to investigate the infrared divergences by letting $\lambda \to 0$. No infrared divergence will appear if $p_1^2, p_2^2 \neq m^2$ (where m = electron mass). However, as p_1^2 and p_2^2 tend to m^2 the vertex part $\Gamma^n(p_1, p_2, p_2 - p_1)$ has a singularity. In this note, by calculating in fifth-order perturbation theory all the terms that are singular when $p_1^2 = p_2^2 = p^2 \to m^2$, we establish the infrared asymptotic behavior of the form factor for the charge of the electron, inasmuch as the earlier results[1,2] differ for $|t| \gg m^2$, where $t = (p_2 - p_1)^2$. By calculating the fifth- and seventh-order diagrams we obtain the infrared singularity of the form factor for the additional magnetic moment of the electron. The Green's function of the photon is chosen with a Coulomb gauge.

2. In dealing with the infrared singularity of the vertex part one should take account of the fact that it is a component element of more complicated diagrams. It is consequently operated on from the right and from the left by the matrix factors $(\hat{p}_1 + m)$ and $(\hat{p}_2 + m)$, and this results in the vanishing of a number of terms in the limit as $p_1^2, p_2^2 \to m^2$. To remove the divergence in Γ^n we perform a subtraction at the point

$$p_1 = p_2 = p_0,\quad p_0^2 = m^2 - \delta m^2,\quad 0 < \delta m^2 \ll m^2.$$

The vertex part has matrix structure. The term Γ_1^n which interests us is proportional to γ^n

$$\Gamma_1{}^n(p_1, p_2, p_2 - p_1) = (\hat{p}_2 + m)\gamma^n(\hat{p}_1 + m)F(p^2, t), \quad (1)$$

and we find that as $p^2 \to m^2$ diagrams 1 and 2 (see the Figure) lead to the following expression for the electron charge form factor

$$F_1(p^2, t) + F_2(p^2, t) = 1 + \beta(t)\ln\frac{m^2 - p^2}{2m^2} + \varepsilon(t), \quad (2)$$

$$\beta(t) = \frac{\alpha}{\pi}(2m^2 - t)\int_{4m^2}^{\infty}\frac{dt'}{[t'(t' - 4m^2)]^{1/2}(t' - t - i\varepsilon)} \quad (3)$$

$$\varepsilon(t) = -\frac{\alpha}{\pi}\ln\frac{\delta m^2}{2m^2} + \frac{\alpha t}{4\pi}\int_{4m^2}^{\infty}\left[2(t' - 4m^2)\ln\frac{t' - 4m^2}{2m^2} - 3t' + 8m^2\right] \times \frac{dt'}{[t'(t' - 4m^2)]^{1/2}t'(t' - t - i\varepsilon)} \quad (4)$$

(α is the fine structure constant, δm^2 is fixed), $\epsilon(t)$ is finite for $p^2 = m^2$. If $(-t) \gg m^2$, then $\epsilon(t) = -(\alpha/4\pi)[\ln(-t/m^2)]^2$. For negative t

$$\beta(t) = \frac{\alpha}{\pi}\frac{2m^2 - t}{[t(t - 4m^2)]^{1/2}}\ln\frac{2m^2 - t + [t(t - 4m^2)]^{1/2}}{2m^2}. \quad (5)$$

By calculating to fifth order those terms which have infrared singularities, we find that the contribution of diagram 3 to $F(p^2, t)$ is given by

$$F_3(p^2, t) = \frac{1}{2}\left[\beta(t)\ln\left(\frac{m^2 - p^2}{2m^2}\right)\right]^2 + \beta(t)\varepsilon(t)\ln\left(\frac{m^2 - p^2}{2m^2}\right) + \beta^2(t)\ln\left(\frac{m^2 - p^2}{2m^2}\right). \quad (6)$$

Diagram 4 gives the term

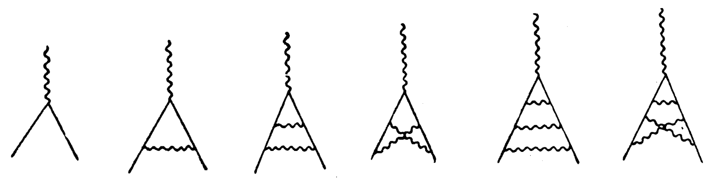
$$F_4(p^2, t) = -\beta^2(t)\ln\left(\frac{m^2 - p^2}{2m^2}\right). \quad (7)$$

The sum of the remaining diagrams of the vertex part is finite for $p^2 = m^2$. Adding (2), (6), and (7) we find that $F_1 + F_2 + F_3 + F_4$ is a sum of the form $1 + x + x^2/2$ of the first terms of the expansion in powers of α of the expression

$$F(p^2, t) = \exp\left\{\beta(t)\ln\frac{m^2 - p^2}{2m^2}\right\}B_1(t), \quad (8)$$

$$B_1(t) = 1 + \varepsilon(t) + \ldots \quad (9)$$

Thus, calculation of all the fifth-order terms



1 2 3 4 5 6

that are singular as $p^2 \to m^2$ leads to the following result concerning the form of the infrared asymptotic behavior of Γ_1^n:

$$\Gamma_1{}^n = (\hat{p}_2 + m)\gamma^n(\hat{p}_1 + m)\left(\frac{m^2 - p^2}{2m^2}\right)^{\beta(t)} B_1(t), \qquad (10)$$

which agrees with the result which Abrikosov[1] obtained for a Coulomb gauge by summing a series of basic diagrams.

For the exponent of the infrared singularity Blank and Shirkov[2] obtained a result which goes over into (10) for small $|t|$ and differs from it when

$$(\alpha / 3\pi)\ln(-t/m^2) \sim 1, \quad t < 0.$$

If one expands the corresponding expression from Blank and Shirkov[2] in a power series, then it appears that in addition to $F_3 + F_4$, the function F(t) must contain a term of fourth order in e,

$$-\frac{1}{6}\left(\beta(t) + \frac{\alpha}{2\pi}\right)^2 \ln\frac{m^2 - p^2}{2m^2}.$$

As we have seen, calculations of fifth-order diagrams do not give such a term.

3. That term of the vertex part Γ_2^n whose matrix structure corresponds to the additional magnetic moment of the electron

$$\Gamma_2{}^n = \frac{1}{4m}(\hat{p}_2 + m)[\hat{p}_2 - \hat{p}_1, \gamma^n]_-(\hat{p}_1 + m)G(p^2, t), \qquad (11)$$

where $G(p^2, t)$ is the magnetic moment form factor, also exhibits infrared singularities. To third order, (diagram 2), we have[1)]

$$G_2(m^2, t) = \frac{\alpha m^2}{\pi}\int_{4m^2}^{\infty}\frac{dt'}{[t'(t' - 4m^2)]^{1/2}(t' - t - i\varepsilon)}. \qquad (12)$$

The sum of diagrams 2, 3, 5 and 6 gives the following contribution to $G(p^2, t)$, which is proportional to $G_2(m^2, t)$:

$$G_2(m^2, t)\left\{1 + \beta(t)\ln\frac{m^2 - p^2}{2m^2} + \varepsilon(t) + \frac{1}{2}\left[\beta(t)\ln\frac{m^2 - p^2}{2m^2}\right]^2 + \beta(t)\varepsilon(t)\ln\frac{m^2 - p^2}{2m^2}\right\}. \qquad (13)$$

Fifth-order diagrams add to the additional magnetic moment a correction $G_3(t)$ which is finite when $p^2 = m^2$[4]. Diagrams to seventh order, which are obtained from the fifth-order diagrams by the addition of photon lines that have both of their ends on external electron lines, lead to the term

$$G_3(t)\left[1 + \beta(t)\ln\frac{m^2 - p^2}{2m^2} + \varepsilon(t)\right]. \qquad (14)$$

Formulas (10), (13), and (14) enable one to draw the conclusion that as $p^2 \to m^2$ the vertex part is

$$\Gamma^n(p^2, t) = \left(\frac{m^2 - p^2}{2m^2}\right)^{\beta(t)}(\hat{p}_2 + m) \times\left\{B_1(t)\gamma^n + \frac{1}{4m}B_2(t)[\hat{p}_2 - \hat{p}_1, \gamma^n]_-\right\}(\hat{p}_1 + m), \qquad (15)$$

where $B_1(t)$ and $B_2(t)$ are power series in α and their first terms are represented in (9) and (12). The following spectral representations of them are valid:

$$B_1(t) = \frac{t}{\pi}\int_{4m^2}^{\infty}\frac{b_1(t')dt'}{t'(t' - t - i\varepsilon)} + B_1(0), \qquad (16)$$

$$B_2(t) = \frac{1}{\pi}\int_{4m^2}^{\infty}\frac{b_2(t')dt'}{t' - t - i\varepsilon}. \qquad (17)$$

We note that in the limit as $p^2 \to m^2$ the matrix structure of the vertex part reduces to two terms of formula (15).

4. If $p_1^2 = p_2^2 = m^2$, then to remove the divergence in the region of small virtual-photon momenta we introduce the small photon mass λ. In[5,6] it was shown that the infrared divergences which arise as $\lambda \to 0$ separate into exponential factors. Direct low-order diagram calculations lead to the same results which can be obtained exactly from formulas (1)—(16) by the change of variable

$$m^2 - p^2 \to 2m\lambda, \quad (\hat{p}_2 + m) \to \bar{u}(p_2), \quad (\hat{p}_1 + m) \to u(p_1)$$

and confirm the conclusion regarding the separation into exponential factors.

If the vertex part is included in the matrix element for the scattering of an electron by an external field, then in the physical region $t < 0$ the quantity $\beta(t)$ is positive. Formula (15) then leads to the vanishing of the matrix element as $\lambda \to 0$. Physically this corresponds to the impossibility of scattering an electron through a non-zero angle without radiating real photons.

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1)Concerning formulas (2) – (4) and (12) see the survey[3].

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