PENETRATION OF AN ELECTROMAGNETIC WAVE INTO A PLASMA WITH ACCOUNT OF NON-LINEARITY

A. V. GUREVICH

P. N. Lebedev Physics Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor August 22, 1964

J. Exptl. Theoret. Phys. (U.S.S.R.) 48, 701-707 (February, 1965)

The region of reflection of radio waves propagating in an inhomogeneous plasma is considered by taking into account the effect of the alternating electric field of the wave on the dielectric constant of the plasma. It is demonstrated that the reflection point shifts into the interior of the plasma with increasing wave-field amplitude. The magnitude of this shift is calculated. The change in the critical frequency of transversal of the plasma by the radio waves is determined as a function of the amplitude of the wave field.

WE consider the region N(z) of reflection of radio waves propagating in a layered-inhomogeneous plasma normally to the layers ($\mathbf{k} \parallel \mathbf{z}$), with account of the influence of the variable electric field of the wave on the dielectric constant of the plasma. We assume that the frequency ω of the wave is much larger than the electron collision frequency ν. Then the point z_0 of wave reflection is defined in the linear approximation by the well known condition (see, for example, [1])

$$\varepsilon(z_0) = 1-4\pi e^2 N(z_0)/m\omega^2 = 0, \qquad N(z_0) = m\omega^2/4\pi e^2. \tag{1}$$

The electron concentration N(z) usually decreases under the influence of the alternating electric field of the wave[1)]. This leads to a shift of the point of reflection of the wave towards values of z larger than z_0. A very important fact is that the amplitude of the wave field increases strongly in the reflection region (see [1], Secs. 17 and 35), and this should lead to an intensification of the nonlinear effect in the vicinity of the point of reflection. The present article is devoted to an analysis of these questions.

We consider here thermal nonlinear effects, connected with the heating of an electron gas in the alternating electric field of the wave. As shown in [2], these effects play a principal role under the condition that the amplitude of the alternating electric field changes little over the mean free path of the electrons. We assume also, for simplicity, that the plasma is weakly ionized, so that the principal role is played by collisions between the electrons and neutral atoms. When $\omega \gg \nu$ we can usually neglect the absorption of the wave in the vicinity of the reflection point[2)]. The propagation of the wave in the reflection region, with allowance for the thermal nonlinear effects, is then described simultaneously by the wave equation and the equation for the electron temperature:

$$\frac{d^2E}{dz^2} + \frac{\omega^2}{c^2}\varepsilon(z, \Theta)E = 0, \tag{2}$$

$$\Theta + \frac{2kT}{3m\delta}\frac{\Theta+1}{\nu_e(\Theta)}\frac{d}{dz}\left[\frac{\Theta}{\nu_e(\Theta)(\Theta+1)}\frac{d\Theta}{dz}\right] = 1 + \frac{e^2E^2(z)}{3kTm\delta\omega^2}, \tag{3}$$

$$\varepsilon(z, \Theta) = 1 - \frac{4\pi e^2 N(z)}{m\omega^2}\frac{2}{\Theta+1} = \varepsilon_0(z) + \frac{\Theta-1}{\Theta+1}[1-\varepsilon_0(z)]. \tag{4}$$

Here $\Theta = T_e/T$, where T_e is the temperature of the electrons in the wave field and T is the temperature of the unperturbed plasma, $\epsilon_0(z)$

---

1)Under certain conditions the electron concentration can also increase with increasing amplitude of the wave field (see[2]).

2)When account is taken of nonlinear effects, we can neglect the absorption of the wave in the reflection region, if the following condition is satisfied

$$\frac{\nu}{c(d\varepsilon/dz)_0}\left(\frac{E_0}{E_p}\right)^{2/3} \ll 1,$$

where ν is the electron collision frequency, c the velocity of light, $\epsilon(z)$ the dielectric constant of the unperturbed plasma, E_0 the wave amplitude, and E_p the characteristic field (5).

the dielectric constant of the unperturbed plasma, $\nu_e = (2kT_e)^{1/2}/m^{1/2}l$ the frequency of the collisions between the electrons and the neutral atoms, l the mean free path of the electrons, and δ the average fraction of the energy transferred by the electron to the atoms in one collision ($\delta = 2m/M$ for elastic collisions). In Eq. (3) for the electron temperature it is assumed that the amplitude of the wave field changes appreciably in a plane orthogonal to the propagation direction, only over sufficiently large distances $\rho \gg l/\sqrt{\delta}$; because of this, only the derivatives with respect to the direction z are significant in the term describing the thermal conductivity (see [2]).

Let us consider here the case of a weak field

$$E^2 \ll E_p^2 = 3kTm\delta\omega^2 / e^2, \tag{5}$$

i.e., we assume that the electron-temperature perturbations are small, $\Theta - 1 \ll 1$. Equation (3) can then be linearized and its solution takes the form $\Theta = 1 + \Delta$, where

$$\Delta = \frac{\sqrt{3\delta}}{2lE_p^2}\left\{e^{-\sqrt{3\delta}z/l}\int_{-\infty}^{\sqrt{3\delta}z/l} E^2(z')\, e^{\sqrt{3\delta}z'/l}\,dz' + e^{\sqrt{3\delta}z/l} \times \int_{\sqrt{3\delta}z/l}^{\infty} E^2(z')\, e^{-\sqrt{3\delta}z'/l}dz'\right\}. \tag{6}$$

It then follows from (4) that

$$\varepsilon(z, \Theta) = \varepsilon_0(z) + \frac{\Delta}{2}[1 - \varepsilon_0(z)] \tag{7}$$

and the system (2)—(4) reduces to a single integro-differential equation

$$\frac{d^2E}{dz^2} + \frac{\omega^2}{c^2}\left\{\varepsilon_0(z) + \frac{\Delta}{2}\right\}E = 0, \tag{8}$$

where the integral term Δ is determined by the expression (6). We have neglected here $\epsilon_0(z)$ compared with unity in the nonlinear term (taking into account the fact that this term becomes essential only in the region where $\epsilon_0(z) \sim \Delta \ll 1$).

In the reflection region, a standing wave is produced as a result of the superposition of the incident and reflected waves. If its length is not very large

$$\lambda \ll l / \sqrt{\delta}, \tag{9}$$

then the presence of regions with periodic increases and decreases of field amplitude in the standing wave has little effect on the electron temperature. The dielectric constant then changes little over the length of the wave, and we can use the geometrical-optics approximation to solve Eq. (2):

$$E^2(z) = \begin{cases} \dfrac{4E_0^2}{\sqrt{\varepsilon}}\sin^2\left(\dfrac{\omega}{c}\displaystyle\int_0^z \sqrt{\varepsilon}\,dz\right), & z \geqslant 0, \\ 0, & z < 0, \end{cases} \tag{10}$$

where we assume that $z = 0$ is the wave reflection point (i.e., that $\epsilon(0) = 0$), and E_0 is its amplitude on the plasma boundary, i.e , in the region where $\epsilon = 1$. Substituting the formal solution of (10) in (6) and (7), and taking into account the fact that the periodic terms in formula (6) average out when condition (9) is satisfied, we obtain in place of (7) the following expression for $\epsilon(z)$

$$\varepsilon(z) = \varepsilon_0(z) + \frac{E_0^2}{E_p^2}\begin{cases} e^{-\sqrt{3\delta}z/l}\displaystyle\int_0^{\sqrt{3\delta}z/l}\frac{e^{\tau}}{\sqrt{\varepsilon}}d\tau + e^{\sqrt{3\delta}z/l}\int_{\sqrt{3\delta}z/l}^{\infty}\frac{e^{-\tau}}{\sqrt{\varepsilon}}d\tau, & z \geqslant 0, \\ e^{\sqrt{3\delta}z/l}\displaystyle\int_0^{\infty}\frac{e^{-\tau}}{\sqrt{\varepsilon}}d\tau, & z < 0, \end{cases} \tag{11}$$

where $\tau = \sqrt{3\delta}z/l$, and $\epsilon_0(z)$ is the dielectric constant of the plasma in the linear approximation, $\epsilon_0(z) = 1 - 4\pi e^2N/m\omega^2$.

The shift of the wave reflection point due to the nonlinearity Δz can be readily determined from (11). In fact, the reflection point $z = 0$ is defined by the condition $\epsilon(0) = 0$. Consequently

$$\varepsilon_0(0) = -\frac{E_0^2}{E_p^2}\int_0^{\infty}\frac{e^{-\tau}}{\sqrt{\varepsilon}}d\tau. \tag{12}$$

Thus, the point of reflection of the wave $z = 0$, with allowance for the nonlinearity, is determined not by the usual condition (1), but by the condition (12). It shifts into the plasma. In the linear approximation as $E_0 \to 0$ the condition (12) coincides of course with (1).

In order to calculate the shift of the reflection point it is necessary to know the function $\epsilon(\tau) = \epsilon(\sqrt{3\delta}z/l)$ in (12), that is, the solution of (11). In the region $z < 0$ the solution of (11) is obvious:

$$\varepsilon(z) = \varepsilon_0(z) - \varepsilon_0(0)\exp(\sqrt{3\delta}z/l). \tag{13}$$

In the region $z > 0$ it is necessary to solve the integral equation (11). It is natural to change over in this equation to the dimensionless variables

$$t = z\sqrt{3\delta}/l, \quad x(t) = \varepsilon(E_0/E_p)^{-4/3}, \quad x_0(t) = \varepsilon_0(E_0/E_p)^{-4/3} \tag{14}$$

We then obtain in lieu of (11)

$$x(t) = x_0(t) + e^{-t}\int_0^t\frac{e^{\tau}d\tau}{\sqrt{x}} + e^t\int_t^{\infty}\frac{e^{-\tau}d\tau}{\sqrt{x}}. \tag{15}$$

Far away from the reflection point, at $z \gg l/\sqrt{3\delta}$, that is, when $t \gg 1$, (15) reduces to the algebraic equation

$$x(t) = x_0(t) + \frac{2}{\sqrt{x_0(t)}}. \tag{16}$$

Figure 1 shows a plot of $x(t)$ vs. $x_0(t)$ as de-

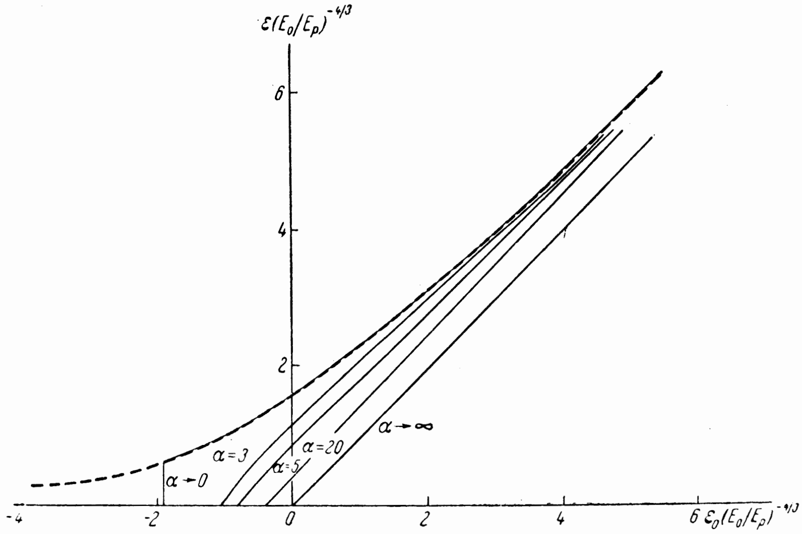

FIG. 1. Dielectric constant of a plasma, with account of the nonlinearity of ϵ as a function of ϵ_0 for different values of the parameter α indicated in the figure.

fined by (16) (dashed curve). We see from the figure that $x(t) \approx x_0(t)$ at large values $x_0(t) \gg 1$. In other words, $\epsilon(z, E) \approx \epsilon_0(z)$ with $\epsilon_0(z) \gg (E_0/E_p)^{4/3}$. On the other hand, if $x_0(t) \lesssim 1$, then the function $x(t)$ differs strongly from $x_0(t)$.

Inasmuch as $\epsilon(z)$ differs significantly from $\epsilon_0(z)$ only in the region of small values $\epsilon_0 \lesssim (E_0/E_p)^{4/3}$, it is natural to use for $\epsilon_0(z)$ the linear approximation

$$\varepsilon_0(z) = \varepsilon_0(0) + \left(\frac{d\varepsilon}{dz}\right)_0 z = \varepsilon_0(0) + \left(\frac{1}{N}\frac{dN}{dz}\right)_0 z \quad (17)$$

or

$$x_0(t) = -x_0 + \alpha t,$$

where

$$x_0 = -\varepsilon_0(0)\,(E_0/E_p)^{-4/3} = \int_0^\infty \frac{e^{-t}}{\sqrt{x(t)}}\,dt, \quad (18)$$

$$\alpha = l(d\varepsilon/dz)_0 (E_0/E_p)^{-4/3}(3\delta)^{-1/2} = l(dN/dz)_0 N^{-1}(0)\,(E_0/E_p)^{-4/3}(3\delta)^{-1/2}. \quad (19)$$

Then Eq. (15) can be rewritten in the form

$$x(t) = x_0(e^t - 1) + \alpha t + e^{-t}\int_0^t \frac{e^{\tau}}{\sqrt{x}}\,d\tau - e^t \int_0^t \frac{e^{-\tau}}{\sqrt{x}}\,d\tau. \quad (20)$$

It follows therefore that for small $t \ll 1$

$$x(t) \approx (\alpha + x_0)t - \frac{8}{3\sqrt{\alpha + x_0}}\,t^{3/2} + \ldots \quad (21)$$

When $t \gg 1/\alpha$ we have

$$x(t) \approx x_0(t) = \alpha t. \quad (22)$$

The form of the function $x(t)$ depends significantly on the magnitude of the parameter α, that is, on the electron concentration gradient in the unperturbed plasma. If

$$\alpha \gg 1, \quad (23)$$

then we can approximately put $x(t) \approx \alpha t$ in the integral terms of (20), and obtain

$$x_0 = \int_0^\infty e^{-t} x^{-1/2}\,dt = \sqrt{\pi/\alpha}, \quad (24)$$

$$x(t) = \alpha t - \sqrt{\pi/\alpha} + \sqrt{\pi/\alpha}\, e^t[1 - \Phi(\sqrt{t})] + \sqrt{\pi/\alpha}\, e^{-t} f(\sqrt{t}), \quad (25)$$

where

$$\Phi(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-y^2}\,dy, \qquad f(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{y^2}\,dy.$$

Thus, when condition (23) is satisfied, the dielectric constant of the unperturbed plasma at the reflection point is equal to

$$\varepsilon_0(0) = -\sqrt{\pi/\alpha}\,(E_0/E_p)^{4/3}. \quad (26)$$

Recognizing that $\epsilon_0(0) = -(d\epsilon/dz)_0 \Delta z$ we find that in the case in question the point of wave reflection has shifted into the plasma, as a result of the nonlinearity, by an amount

$$\Delta z = \left(\frac{E_0}{E_p}\right)^2 \left[\frac{\pi\sqrt{3\delta}}{l(d\varepsilon/dz)_0{}^3}\right]^{1/2}. \quad (27)$$

With increasing $(d\epsilon/dz)_0$, the shift of the point of reflection increases. By virtue of condition (23), however, Δz remains smaller than $l/\sqrt{\delta}$.

If the electron concentration gradient in the unperturbed plasma is small, then a condition opposite to (23) is satisfied:

$$\alpha \ll 1. \quad (28)$$

Let us rewrite for convenience (20) in differ-

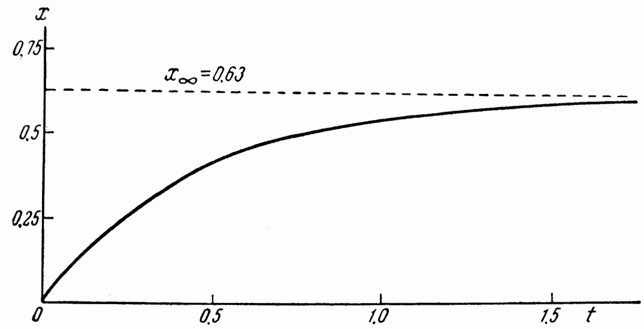

FIG. 2. Dielectric constant $\epsilon = (E_0/E_p)^{4/3}\, x$ of the plasma as a function of the distance $z = tl/(3\delta)^{1/2}$ from the point of reflection of the wave for $\alpha = 0$.

ential form. Differentiating the function $x(t)$ twice, we get

$$d^2x\,/\,dt^2 = x + x_0 - 2\,/\,\sqrt{x} - \alpha t. \qquad (29)$$

The boundary conditions take the form

$$x(0) = 0, \quad dx\,/\,dt\,|_{t\to 0} = \alpha + x_0, \quad dx\,/\,dt\,|_{t\to\infty} = \alpha, \qquad (30)$$

where the constant x_0 is defined, as before, by (18).

When condition (28) is satisfied, the last term in (29) can be neglected. Then the equation is simple to integrate and we obtain

$$dx\,/\,dt = ([x + x_0]^2 - 8\sqrt{x})^{1/2}. \qquad (31)$$

From the last condition of (30) it follows that if $a = 0$ the function $x(t)$ tends to a constant value x_∞ as $t \to \infty$. Consequently, as $x \to x_\infty$ the right side of (31) should vanish in proportion to $x - x_\infty$. This leads to two algebraic equations

$$[x_\infty + x_0]^2 - 8\sqrt{x_\infty} = 0, \qquad x_\infty^{3/2} + x_0\sqrt{x_\infty} - 2 = 0,$$

from which we can easily determine the constants x_0 and x_∞:

$$x_0 = 1.89, \quad x_\infty = 0.63. \qquad (32)$$

Knowing the constant x_0, we can readily integrate (31). The corresponding function $x(t)$ is shown in Fig. 2. The variation of the function $x(t) = \epsilon(E_0/E_p)^{-4/3}$ with $x_0(t) = -x_0 + \alpha t = \epsilon_0(z) \times (E_0/E_p)^{-4/3}$ is shown in Fig. 1 for different values of the parameter α. It is seen from the figure that the shift of the point of reflection increases with decreasing α and is largest as $\alpha \to 0$, when $x_0 = 1.89$, that is,

$$\varepsilon_0(0) = -1.89(E_0\,/\,E_p)^{4/3}.$$

Consequently, the point of reflection shifts in this case by a distance

$$\Delta z = 1.89(E_0\,/\,E_p)^{4/3}\,/\,(d\varepsilon\,/\,dz)_0. \qquad (33)$$

By virtue of condition (28), we always have here $\Delta z \gg l/\sqrt{\delta}$.

It is very important that the point of reflection is shifted by large amount as $(d\epsilon/dz)_0 \to 0$. This case is realized, in particular, when a radio wave propagates in a plasma layer and the wave frequency is close to the critical frequency ω_c of the layer, where $\omega_c = \sqrt{4\pi e^2 N_{max}/m}$ is the Langmuir frequency for the maximum electron concentration in the layer. In the linear approximation, the condition for the wave to pass through the layer of plasma has, naturally, the form $\omega \geq \omega_c$. Taking the nonlinearity into account, this condition should be written in the form

$$\omega \gtrsim \omega_c[1 + 1.89(E_0\,/\,E_p)^{4/3}]^{-1/2} \approx \omega_c[1 - 0.95(E_0\,/\,E_p)^{4/3}].$$

Thus, the frequency of the waves that can pass through a plasma layer is reduced by the nonlinearity by an amount

$$\Delta\omega = 0.95\omega_c(E_0/E_p)^{4/3} = 0.95\,(e^2E_0^2/3kTm\delta)^{2/3}\frac{1}{\omega_c^{1/3}}. \qquad (34)$$

The frequency shift $\Delta\omega$ increases like $E_0^{4/3}$ with increasing field amplitude E_0.

The above-mentioned passage of a wave with frequency $\omega < \omega_c$ through the layer is due to the heating of the plasma in the region perturbed by the electric field of the wave. The concentration of the electrons in the heated region decreases, and the wave passes through the layer. It is clear that large-amplitude waves will greatly disturb the electron concentration, forming "holes" in the uniform distribution of the electrons in the layer.

We note also that a solution of Eq. (8) was obtained above only for the case when the length λ of the standing wave is not very large, so that condition (9) is satisfied. When the inverse condition is satisfied

$$\lambda \gg l\,/\,\sqrt{\delta} \qquad (35)$$

the averaging in formula (6) does not play an appreciable role, and the expression for Δ, accurate to small terms of order $l/\sqrt{\delta}$, assumes the simple form $\Delta = E^2/E_p^2$. Consequently, we obtain here in place of (8)

$$\frac{d^2E}{dz^2} + \frac{\omega^2}{c^2}\left\{\varepsilon_0(z) + \frac{E^2}{2E_p{}^2}\right\}E = 0. \qquad (36)$$

In dimensionless variables, taking into account the linear approximation (17) for the function $\epsilon_0(z)$, Eq. (36) takes the form

$$d^2y\,/\,dt^2 + (t + y^2)\,y = 0. \qquad (37)$$

The same dimensionless equation describes also the propagation in a plasma, near the reflection point, of narrow beams with transverse dimension $\rho \ll l$, when the nonlinearity is due not to thermal but to striction effects (see, for example,[3]). Therefore numerical integration of (37) with boundary conditions corresponding to a standing wave (see, for example, [1], Sec. 17), would be of appreciable interest.

The author is grateful to V. L. Ginzburg for a useful discussion.

[1] V. L. Ginzburg, Rasprostranenie élektromagnitnykh voln v plazme (Propagation of Electromagnetic Waves in Plasma), Fizmatgiz, 1960.

[2] A. V. Gurevich, Geomagnetizm i aéronomiya 5, (1965).

[3] A. V. Gaponov and M. A. Miller, JETP **34**, 242 (1958), Soviet Phys. JETP **7**, 168 (1958).

Translated by J. G. Adashko
90