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THE RELATION BETWEEN FIELDS AND PARTICLES WITH SPIN

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A method [1] for obtaining the physical quantities characterizing a corpuscular description is extended to the case of particles possessing a spin; the physical quantities are derived from the matrix elements $\langle a | A(x) | a' \rangle$ of a scalar neutral quantum field $A(x)$, which are prescribed in an arbitrary basis α. Transition to a corpuscular description means that one determines a) the spectrum of the masses κ and spins j of all the stable particles; b) the set of quantities $\langle a | 0 \rangle$, $\langle a | \kappa_1 j_1 k_1 m_1 \rangle$, $\langle a | \kappa_1 \kappa_2 j_1 j_2 k_1 k_2 m_1 m_2 \rangle, \ldots$, which constitute the transition matrix from the initial basis to the in-basis in which the particle state is defined by its momentum **k** and spin projection m on the z axis, prescribed at minus infinity in time, and c) the scattering matrix. The method is applicable to local quantities of any physical nature or tensor dimension.

1. Let the operator $A(x)$ of neutral scalar local quantity be specified in an arbitrary basis α by means of its matrix elements $\langle \alpha | A(x) | \alpha' \rangle$. We proceed from this description to the corpuscular description, i.e., we obtain the following: a) the spectrum of the masses κ and of the spins j of all the particles; b) the aggregate of the quantities

$$\langle \alpha | 0 \rangle, \quad \langle \alpha | \varkappa_1 j_1 \mathbf{k}_1 m_1 \rangle, \quad \langle \alpha | \varkappa_1 \varkappa_2 j_1 j_2 \mathbf{k}_1 \mathbf{k}_2 m_1 m_2 \rangle, \ \ldots,$$

which constitute the matrix for the transformation from the initial basis to the in-basis, in which the state is characterized by a momentum **k** and a spin projection m on the z axis, specified at minus infinity in time; c) the S matrix. We admit here of the existence of several species of particles with different masses and spins. As in [1], we assume that a condition of the completeness type is satisfied, consisting of the fact that the single-particle matrix elements $\langle 1 | A(x) | 1 \rangle$ are assumed not to vanish for all the particles. We assume the condition of relativistic invariance to be satisfied, i.e., we assume that we know the 4-moment $M_{\mu\nu}$ and the 4-momentum p_σ satisfying the standard commutation relations. For spinless particles the problem of the transition to the corpuscular description was raised and solved by one of the authors [1]. The result of the present paper is a generalization of the procedure developed in [1] to include the case of particles with spin.

The solution of this problem breaks down naturally into two stages: 1) determination of the particle mass spectrum, and also of the set of transformation matrices of the type

$$\langle \alpha | 0 \rangle, \quad \langle \alpha | \varkappa_1 \mathbf{k}_1 \beta_1 \rangle, \quad \langle \alpha | \varkappa_1 \varkappa_2 \mathbf{k}_1 \mathbf{k}_2 \beta_2 \rangle, \ \ldots, \tag{1}$$

where $\mathbf{k}_1, \mathbf{k}_2, \ldots$ are the momenta of the particles in the in-basis while $\beta_1, \beta_2, \ldots$ are additional discrete variables; 2) determination of the spins of the particles and the construction of the matrices

$$\langle \beta_1 | j_1 m_1 \rangle, \quad \langle \beta_2 | j_1 j_2 m_1 m_2 \rangle, \ \ldots,$$

which effect the transition from the initial basis to the in-basis.

The first stage of the solution is carried out in exactly the same way as for spinless particles. The presence of particle spin is manifest only in the fact that, for example, Eq. (8) of [1], which determines the single-particle states, posseses for a specified momentum not one but several linearly independent solutions, which are numbered in Eq. (1) by the index β_1 and which correspond to the existence of different particle spin states. The matrices (1) are assumed specified, since their derivation was described in [1]. We now proceed to determine the matrices (2).

2. To obtain these matrices it is necessary to express the matrix elements of the operator of the square of the spin j^2 and of one of its projections j_3 in terms of quantities specified in the initial basis α, and to obtain the eigenvalues and the eigenfunctions of these quantities. In the case of one particle, it is simplest to use for this purpose the condition of relativistic invariance. The

sought matrix $\langle \beta_1 | j_1 m_1 \rangle$ is obtained from the equations for the eigenvalues and eigenfunctions of the operator of total intrinsic momentum $j_1^2 = \kappa_1^{-2}\Gamma_\sigma^2$ and the projection of this momentum $j_3 = \kappa_1^{-1}\Gamma_3 - k_3\Gamma_0(\omega_1+\kappa_1)^{-1}\omega_1^{-1}$:

$$\varkappa_1^{-2}\langle\alpha|\Gamma_\sigma^2|\alpha'\rangle\langle\alpha'|\varkappa_1\mathbf{k}_1\beta_1\rangle = j_1(j_1+1)\langle\alpha|\varkappa_1 j_1\mathbf{k}_1\beta_1'\rangle, \quad (3)$$

$$\alpha|\varkappa_1^{-1}\Gamma_3 - k_3\Gamma_0(\omega_1+\varkappa_1)^{-1}\omega_1^{-1}|\alpha'\rangle\langle\alpha'|\varkappa_1 j_1\mathbf{k}_1\beta_1'\rangle = m_1\langle\alpha|\varkappa_1 j_1\mathbf{k}_1 m_1\rangle, \quad (4)$$

where

$$\Gamma_\sigma = -{}^1/_2 i\varepsilon_{\mu\nu\lambda\sigma}M_{\mu\nu}p_\lambda, \quad \omega_1 = (\mathbf{k}_1^2+\varkappa_1^2)^{1/2}$$

(see, for example,[2]).

However, in the case of two or more particles, to obtain a complete classification of the spin states the relativistic invariance condition will no longer suffice. It is necessary to have here some other operators which act on the spin variables in a non-trivial manner. Such operators, which permit the completion of the classification of the corpuscular variables for particles with spin, are the dynamic moments of first rank, introduced in[3,4]

$$\mathscr{D}^1_{i_1\ldots i_n}(x_0) = \frac{1}{n!}\left(1 - x_0\frac{\partial}{\partial x_0}\right) \times \int d^3x x_{i_1}\ldots x_{i_n}\left(\frac{\partial}{\partial x_0}\right)^{n-1}A(x), \quad (5)$$

where $i_1, \ldots, i_n = 1, 2, 3$.

It will be convenient in what follows to express the operators j^2 and j_3 in terms of the dynamic moments also in the single-particle case, without using the quantities $M_{\mu\nu}$ and p_σ. Let us calculate the single-particle matrix element $\langle \kappa jkm | \mathscr{D}_i^1(x_0) | \kappa jk'm' \rangle$ of the operator $\mathscr{D}_i^1(x_0)$. According to[5], the single-particle matrix element $\langle \kappa jkm | A(x) | \kappa jk'm' \rangle$ of the scalar operator $A(x)$ is parametrized, i.e., it is expressed in terms of invariant form factors $f_0(t), f_1(t), \ldots$, in the following fashion

$$\langle\varkappa jkm|A(x)|\varkappa jk'm'\rangle = \exp\{-ix(k-k')\}(2\pi)^{-3}(4\omega\omega')^{-1/2} \times \sum_{m''} D^j_{mm''}(\mathbf{k},\mathbf{k}')\sum_{n=0}^{2j}\langle m''|\{ik_\mu\Gamma_\mu(\mathbf{k}')\}^n|m'\rangle f_n(t), \quad (6)$$

where

$$t = -(k-k')^2 \equiv (\omega-\omega')^2 - (\mathbf{k}-\mathbf{k}')^2,$$

n runs through even values from 0 to 2j in the case of parity conservation, and $D^j_{mm''}(\mathbf{k}, \mathbf{k}')$ is the matrix of three dimensional rotation (see, for example,[6]) with Euler angles α, β, and γ as given in formula (7) of[5]. In our case it is sufficient to know the matrix $D^j(\mathbf{k}, \mathbf{k}')$ for vectors $\mathbf{k}$ and $\mathbf{k}'$, which differ little from each other. Accurate to first order in $\mathbf{k} - \mathbf{k}'$ we have

$$D^j(\mathbf{k},\mathbf{k}') = 1 + i\delta\varphi\mathbf{j}, \qquad \delta\varphi = [\mathbf{k}\mathbf{k}'](2\varkappa)^{-1}(\omega+\varkappa)^{-1}. \quad (7)^*$$

Substituting (6) in (5) and using (7), we obtain for the single-particle dynamic moment of first rank, after simple manipulations, the following expression, which is conveniently written not in matrix but in operator form

$$\mathscr{D}_i^1 = i\frac{f_0(0)}{2\omega}\frac{\partial}{\partial\mathbf{k}_i} - \frac{f_0(0)}{2\omega}\frac{[\mathbf{kj}]_i}{2\varkappa(\omega+\varkappa)}. \quad (8)$$

We see from (8) that although the single-particle operator $\mathscr{D}_i^1$ depends, like the dynamic moments of zero rank $\mathscr{D}_i^0$,[7,4] only on the form factor $f_0(t)$ at zero, it already contains a spin dependence, due to the differentiation of the D^j function, i.e., due to the relativistic spin rotation in the Lorentz transformation. In addition, as follows from the general theory[4], the operator $\mathscr{D}_i^1$ does not depend on the time.

In order to express the spin operator j in terms of $\mathscr{D}_i^1$ in a case of a single particle, it is simplest to multiply the right and left sides of (8) by $a = 2\omega f_0^{-1}(0)$ and to commute $a\mathscr{D}_i^1$ with $a\mathscr{D}_j^1$. As a result we obtain

$$\Delta_l^1 = {}^1/_2\varepsilon_{ijl}[a\mathscr{D}_i^1, a\mathscr{D}_j^1]_- = i\{j_l + k_l(\mathbf{kj}) \times (2\varkappa+\omega)(2\varkappa)^{-1}(\omega+\varkappa)^{-2}\}(2\omega\varkappa)^{-1}, \quad (9)$$

whence

$$\mathbf{j} = 2i\varkappa\{(2\varkappa+\omega)(3\varkappa+\omega)^{-1}(\varkappa+\omega)^{-1}(\mathbf{k}\Delta^1)\mathbf{k} - \omega\Delta^1\}. \quad (10)$$

Thus, the particle spin vector is expressed in our case in terms of quantities that are known in the α basis (we recall that the single-particle masses and momenta from (10) are expressed in terms of the limiting values of a zero-rank dynamic moment), and the transition matrix $\langle \beta_1 | j_1 m_1 \rangle$ has been determined.

3. Proceeding now to an arbitrary number of particles, we shall show that in this case the classification of the spin states can be reduced to the already considered case of a single particle. Calculations analogous to those used in the derivation of (8) show that the single-particle dynamic moments $\mathscr{D}_{ij}^1, \mathscr{D}_{ijl}^1, \ldots$ are expressed in terms of $\mathscr{D}_i^1$ in the following manner (here $v = k\omega^{-1}$):

$$\mathscr{D}_{ij}^1 = v_i\mathscr{D}_j^1 + v_j\mathscr{D}_i^1, \quad \mathscr{D}_{ijl} = v_iv_j\mathscr{D}_l^1 + v_iv_l\mathscr{D}_j^1 + v_jv_l\mathscr{D}_i^1, \ldots. \quad (11)$$

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*$[\mathbf{kk}'] = \mathbf{k} \times \mathbf{k}'$.

In addition, we know that at minus infinity in time the particles become free and, consequently, the asymptotic values of the many-particle dynamic moments go over into time-independent quantities that are additive with respect to the particles. From this and from (11) it follows that the asymptotic values of the total dynamic moments $\mathscr{D}^1_i$, $\mathscr{D}^1_{ij}$, ..., are connected at minus time infinity with the single-particle dynamic moments $\mathscr{D}^1_{iN}$ by the simple relations

$$\mathscr{D}_i^1 = \sum_N \mathscr{D}_{iN}^1, \quad \mathscr{D}_{ij}^1 = \sum_N (v_{iN}\mathscr{D}_{jN}^1 + v_{jN}\mathscr{D}_{iN}^1),$$

$$\mathscr{D}^1_{ijl} = \sum_N (v_{iN}v_{jN}\mathscr{D}_{lN}^1 + v_{iN}v_{lN}\mathscr{D}_{jN}^1 + v_{lN}v_{jN}\mathscr{D}_{iN}^1), \ldots. \tag{12}$$

It is obvious that for any specified number of particles we can, by writing out a sufficient number of equations (12), express alll the single-particle operators $\mathscr{D}^1_{iN}$, and consequently all the single-particle spin vectors, in terms of the total dynamic moments $\mathscr{D}^1_i$, $\mathscr{D}^1_{ij}$, ... For example in the case of two particles we have a system of equations

$$\mathscr{D}_i^1 = \mathscr{D}_{i1}^1 + \mathscr{D}_{i2}^1,$$

$$\mathscr{D}_{ij}^1 = v_{i1}\mathscr{D}_{j1}^1 + v_{j1}\mathscr{D}_{i1}^1 + v_{i2}\mathscr{D}_{j2}^1 + v_{j2}\mathscr{D}_{i2}^1, \tag{13}$$

from which it follows that

$$\mathscr{D}_{i1}^1 = \mathbf{w}^{-2}(w_jG_{ij}^1 - w_iG_{jj}^1/2),$$

$$\mathscr{D}_{i2}^1 = \mathscr{D}_i^1 - \mathscr{D}_{i1}^1, \tag{14}$$

where

$$\mathbf{w} = \mathbf{v}_1 - \mathbf{v}_2, \quad G_{ij}^1 = \mathscr{D}_{ij}^1 - v_{i2}\mathscr{D}_j^1 - v_{j2}\mathscr{D}_i^1.$$

4. The matrices (2) obtained in the preceding section were determined by us not uniquely, but also accurate to phase factors of the form $\exp\{i\chi(\mathbf{k}_1, \mathbf{k}_2, \ldots)\}$, where $\chi(\mathbf{k}_1, \mathbf{k}_2, \ldots)$ are certain functions of the momenta, which can depend in non-trivial fashion on the spin variables. This ambiguity affects the form of the operators which do not commute with the single-particle momenta. Thus, when calculating the single-particle matrix elements of the operators $\mathscr{D}^1_{iN}$ in the new basis we obtain an expression that differs from (8) in that it contains in place of $i\partial\chi/\partial\mathbf{k}_N$

$$i\partial/\partial\mathbf{k}_N + \partial\chi/\partial\mathbf{k}_N. \tag{15}$$

It is obvious that this ambiguity can be eliminated by stipulating that all the single-particle operators $\mathscr{D}^1_{iN}$ be in the form (8). Only after eliminating the ambiguity do the single-particle operators M and N of infinitesimally small three-dimensional rotations and Lorentz transformations assume the standard form given by Eq. (11) of [2]:

$$\mathbf{M} = -i\left[\mathbf{k}\frac{\partial}{\partial\mathbf{k}}\right] + \mathbf{j}, \quad \mathbf{N} = i\omega\frac{\partial}{\partial\mathbf{k}} - \frac{[\mathbf{jk}]}{\omega+\varkappa} + \frac{i}{2}\frac{\mathbf{k}}{\omega}. \tag{16}$$

We note that the ambiguity exists also in the case of spinless particles, analyzed in [1], where the ambiguity cannot be eliminated. In addition, for particles with spin there exists an additional ambiguity connected with the fact that the eigenfunctions of the operator j_3 are determined only accurate to a phase factor that depends on j_3. To eliminate this ambiguity it is sufficient to stipulate additionally that, for example, the operators j_1 have a standard form (see, for example, [6]) in the new basis.

Thus, the problem of obtaining the matrix of transition from the α basis to the in-basis has been solved. By replacing in all the equations the limiting transition $x_0 \to -\infty$ by $x_0 \to +\infty$ we obtain a matrix $\langle \alpha | \kappa_1 j_1 \mathbf{k}_1 m_1, \ldots \rangle$ for transition to the out-basis, and by the same token the S matrix. This completes the solution of the problem stated at the beginning of the paper.

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