INTERACTION OF NEGATIVE-ENERGY WAVES IN A WEAKLY TURBULENT PLASMA

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It is shown that statistical equilibrium cannot be established in a gas of quasi-particles (waves) if there are positive-energy and negative-energy quasi-particles. Under these conditions the fact that the quasi-particle entropy must increase means that the number of quasi-particles grows without limit. We consider a concrete example (the interaction of waves in a plasma penetrated by a low-density ion beam) which verifies the conclusion stated above; the rate of growth of the number of quasi-particles is estimated.

1. INTRODUCTION

A weakly turbulent plasma can be regarded as a mixture consisting of a particle gas (background) and a wave gas (quasi-particles). This approach to the investigation of turbulent plasma is fruitful if the interactions between the quasi-particles and the background and between the quasi-particles themselves are weak, as will be the case if the wave amplitudes are small. Under these conditions the dependence of quasi-particle energy ω on momentum $\mathbf{k}$ (the dependence of the frequency on the wave vector[1]) is determined by a dispersion relation obtained from the equations that describe small oscillations of the plasma in the linear approximation. In the next approximation, assuming that the random phase approximation holds for different $\mathbf{k}$, one obtains a quasi-linear equation, which describes the interaction of the quasi-particles with the background, and a kinetic equation which governs the distribution of quasi-particles over momentum $N_{\mathbf{k}}$.[1-3]

In the present work we consider certain features of the interaction between quasi-particles corresponding to longitudinal oscillations of a uniform plasma in the absence of a magnetic field under the assumption that the interaction between the quasi-particles and the particles can be neglected.

Under these conditions the equation for $N_{\mathbf{k}\alpha}$ is written in the form[2]

$$\frac{dN_{\mathbf{k}\alpha}}{dt}=\sum_{\beta\gamma\mathbf{k}'\mathbf{k}''}V^{\alpha\beta\gamma}_{\mathbf{k}\mathbf{k}'\mathbf{k}''}(N_{\mathbf{k}'\beta}N_{\mathbf{k}''\gamma}-N_{\mathbf{k}\alpha}N_{\mathbf{k}'\beta}-N_{\mathbf{k}\alpha}N_{\mathbf{k}''\gamma})\times\delta_{\mathbf{k},\mathbf{k}'+\mathbf{k}''}\delta(\omega_{\mathbf{k}\alpha}-\omega_{\mathbf{k}'\beta}-\omega_{\mathbf{k}''\gamma}). \tag{1}$$

In this equation we retain only the principal nonlinear terms, which are quadratic in $N_{\mathbf{k}}$. Terms proportional to $N_{\mathbf{k}}^3$ and higher are neglected; these appear if one takes account of the higher perturbation-theoretic corrections to the amplitude.

Let us now explain the notation in (1): α, β, and γ enumerate the solutions of the dispersion equation $\omega_{\mathbf{k}\alpha} = \omega_\alpha(\mathbf{k})$

$$\varepsilon(\omega, \mathbf{k}) = 0, \tag{2}$$

$\epsilon(\omega, \mathbf{k})$ is the longitudinal dielectric constant of the plasma. The quantity $V^{\alpha\beta\gamma}_{\mathbf{k}\mathbf{k}'\mathbf{k}''}$ can be written in the form

$$V^{\alpha\beta\gamma}_{\mathbf{k}\mathbf{k}'\mathbf{k}''}=|W^{\alpha\beta\gamma}_{\mathbf{k}\mathbf{k}'\mathbf{k}''}|^2/\varepsilon_{\omega_\alpha}'\varepsilon_{\omega_\beta}'\varepsilon_{\omega_\gamma}',$$

$$\varepsilon_{\omega_\alpha}'=\frac{\partial}{\partial\omega}\varepsilon(\omega,\mathbf{k})\Big|_{\omega=\omega_\alpha(\mathbf{k})}, \tag{3}$$

and satisfies the symmetry relation

$$V^{\alpha\beta\gamma}_{\mathbf{k}\mathbf{k}'\mathbf{k}''}=V^{\beta\alpha\gamma}_{\mathbf{k}'\mathbf{k}\mathbf{k}''}=V^{\alpha\gamma\beta}_{\mathbf{k}\mathbf{k}''\mathbf{k}'}, \tag{4}$$

$$V^{\alpha\beta\gamma}_{\mathbf{k}\mathbf{k}'\mathbf{k}''}=-V^{\alpha\beta\gamma}_{-\mathbf{k}\mathbf{k}'\mathbf{k}''}. \tag{5}$$

An explicit expression for $V^{\alpha\beta\gamma}_{\mathbf{k}\mathbf{k}'\mathbf{k}''}$ for one particular case is given in Sec. 3.

Quasi-particles whose energy ω is related to the momentum $\mathbf{k}$ by (2) will be called plasmons. The plasmon energy in a state characterized by $\mathbf{k}$, α is expressed in terms of the Fourier component of the potential $\varphi_{\mathbf{k}\alpha}$:

$$\mathscr{E}_{\mathbf{k}\alpha}=(8\pi)^{-1}k^2V|\varphi_{\mathbf{k}\alpha}|^2\omega_{\mathbf{k}\alpha}\varepsilon_{\omega_\alpha}' \tag{6}$$

(V is the normalized volume, which we take to be equal to unity). The energy $\mathscr{E}_{\mathbf{k}\alpha}$ is related to the

[1] We take $\hbar = 1$.

[2] The relation in (1) is valid in the classical limit $N_{\mathbf{k}\alpha} \gg 1$. We assume below that the condition $N_{\mathbf{k}\alpha} \gg 1$ is satisfied everywhere.

quantity $N_{\mathbf{k}\alpha}$, which is frequently called the plasmon number, by

$$N_{\mathbf{k}\alpha} = \mathscr{E}_{\mathbf{k}\alpha} / \omega_{\mathbf{k}\alpha}. \tag{7}$$

The number of plasmons is usually taken to be $|\mathscr{E}_{\mathbf{k}\alpha}/\omega_{\mathbf{k}\alpha}|$. This definition, however, would complicate the calculations below. We emphasize at this point that $\omega_{\mathbf{k}\alpha}$ and $N_{\mathbf{k}\alpha}$ are odd functions of **k**:

$$\omega_{\mathbf{k}\alpha} = -\omega_{-\mathbf{k}\alpha}, \quad N_{\mathbf{k}\alpha} = -N_{-\mathbf{k}\alpha}. \tag{8}$$

2. ESTABLISHMENT OF STATISTICAL EQUILIBRIUM IN A QUASI-PARTICLE GAS

Let us investigate (1). Exploiting the properties of the quantities $V^{\alpha\beta\gamma}_{\mathbf{kk'k''}}$, $\omega_{\mathbf{k}\alpha}$, and $N_{\mathbf{k}\alpha}$ [as specified in (5) and (8)] we can write the more symmetric relation

$$\frac{dN_{\mathbf{k}\alpha}}{dt} = \sum_{\beta\gamma\mathbf{k'}\mathbf{k''}} (N_{\mathbf{k'}\beta}N_{\mathbf{k''}\gamma} + N_{\mathbf{k}\alpha}N_{\mathbf{k'}\beta} + N_{\mathbf{k}\alpha}N_{\mathbf{k''}\gamma})\, V^{\alpha\beta\gamma}_{\mathbf{kk'k''}} \times \delta_{-\mathbf{k},\,\mathbf{k'}+\mathbf{k''}}\, \delta(\omega_{\mathbf{k}\alpha} + \omega_{\mathbf{k'}\beta} + \omega_{\mathbf{k''}\gamma}). \tag{9}$$

Dividing (9) by $N_{\mathbf{k}\alpha}$ and carrying out the summation over the subscripts **k** and α we have:

$$\sum_{\mathbf{k}\alpha} \frac{1}{N_{\mathbf{k}\alpha}} \frac{dN_{\mathbf{k}\alpha}}{dt} = \frac{d}{dt} \sum_{\mathbf{k}\alpha} \ln|N_{\mathbf{k}\alpha}| = \sum_{\alpha\beta\gamma\mathbf{kk'k''}} V^{\alpha\beta\gamma}_{\mathbf{kk'k''}} \times N_{\mathbf{k}\alpha}N_{\mathbf{k'}\beta}N_{\mathbf{k''}\gamma} \left(\frac{1}{N_{\mathbf{k}\alpha}^2} + \frac{1}{N_{\mathbf{k}\alpha}N_{\mathbf{k'}\beta}} + \frac{1}{N_{\mathbf{k}\alpha}N_{\mathbf{k''}\gamma}} \right) \times \delta_{-\mathbf{k},\,\mathbf{k'}+\mathbf{k''}}\, \delta(\omega_{\mathbf{k}\alpha} + \omega_{\mathbf{k'}\beta} + \omega_{\mathbf{k''}\gamma}).$$

Now, taking account of (7) we symmetrize the right side of this equation over subscript pairs $\alpha\mathbf{k}$, $\beta\mathbf{k'}$, $\gamma\mathbf{k''}$:

$$\frac{d}{dt} \sum_{\mathbf{k}\alpha} \ln|N_{\mathbf{k}\alpha}| = \frac{1}{3} \sum_{\alpha\beta\gamma\mathbf{kk'k''}} V^{\alpha\beta\gamma}_{\mathbf{kk'k''}} N_{\mathbf{k}\alpha}N_{\mathbf{k'}\beta}N_{\mathbf{k''}\gamma} \times \left(\frac{1}{N_{\mathbf{k}\alpha}} + \frac{1}{N_{\mathbf{k'}\beta}} + \frac{1}{N_{\mathbf{k''}\gamma}} \right)^2 \times \delta_{-\mathbf{k'},\,\mathbf{k'}+\mathbf{k''}}\, \delta(\omega_{\mathbf{k}\alpha} + \omega_{\mathbf{k'}\beta} + \omega_{\mathbf{k''}\gamma}). \tag{10}$$

Then, from (3)—(5)

$$V^{\alpha\beta\gamma}_{\mathbf{kk'k''}} N_{\mathbf{k}\alpha}N_{\mathbf{k'}\beta}N_{\mathbf{k''}\gamma} = |W^{\alpha\beta\gamma}_{\mathbf{kk'k''}}|^2 k^2 k'^2 k''^2 \frac{|\varphi_{\mathbf{k}\alpha}|^2}{8\pi} \frac{|\varphi_{\mathbf{k'}\beta}|^2}{8\pi} \frac{|\varphi_{\mathbf{k''}\gamma}|^2}{8\pi} \geqslant 0,$$

so that (10) yields

$$\frac{d}{dt} \sum_{\mathbf{k}\alpha} \ln|N_{\mathbf{k}\alpha}| \geqslant 0. \tag{11}$$

This inequality, which has been established for the particular case of a three-plasmon interaction, is a consequence of the general law that the entropy must increase. When $N_{\mathbf{k}\alpha} \gg 1$ the entropy of the quasi-particle gas is written in the form (cf. for example[4]):

$$S = \sum_{\mathbf{k}\alpha} \ln|N_{\mathbf{k}\alpha}|.$$

In accordance with the general principles of thermodynamics the entropy must be a non-decreasing function of time

$$dS/dt \geqslant 0, \tag{12}$$

where the equality sign holds only when the quasi-particle gas is in statistical equilibrium. A condition for statistical equilibrium is the equipartition of energy over the degrees of freedom

$$\mathscr{E}_{\mathbf{k}\alpha} = \Theta = \text{const}, \tag{13}$$

or

$$N_{\mathbf{k}\alpha} = \Theta / \omega_{\mathbf{k}\alpha} \tag{14}$$

(the Rayleigh-Jeans law). The constant Θ is the quasi-particle temperature.

In certain systems statistical equilibrium can not be established in a quasi-particle gas. To prove this statement we consider the case in which the dielectric constant $\epsilon(\omega, \mathbf{k})$, treated as a function of ω for fixed **k**, has the form shown in the figure (this is the case, for example, when a current flows through a plasma[5] or when an ion beam moves through a plasma, as in the example considered below).

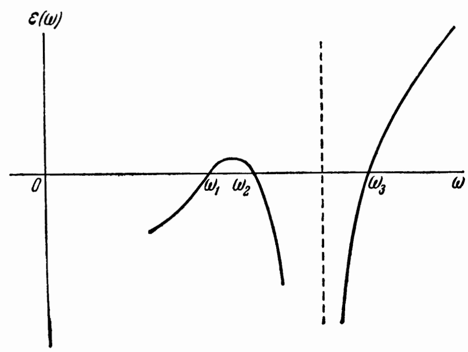
The proof is by induction. Let us assume that the interaction between quasi-particles leads to the establishment of an equilibrium distribution (13). We then have from (4)

$$|\varphi_{\mathbf{k}\alpha}|^2 = 8\pi\Theta/\omega_{\mathbf{k}\alpha}k^2\varepsilon'_{\omega_\alpha}. \tag{15}$$

The figure shows that $\omega_{\mathbf{k}\alpha}\epsilon'_{\omega\alpha} > 0$ when $\alpha = 1.3$ and $\omega_{\mathbf{k}\alpha}\epsilon'_{\omega\alpha} < 0$ when $\alpha = 2$. According to (15) these inequalities must lead to a contradiction: if $\Theta > 0$ then the quantity $|\varphi_{\mathbf{k}2}|^2$ is negative; however, if we assume that $\Theta < 0$ then the quantity $|\varphi_{\mathbf{k}1,3}|^2$ is negative.

Let us now examine the meaning of this result. The sign of the energy of the quasi-particles of type α is the same as the sign of the expression $\omega_{\mathbf{k}\alpha}\epsilon'_{\omega\alpha}$. In the case considered above the energy of the different quasi-particles is different in sign and this is incompatible with the sign of the distribution (13).

From the foregoing considerations it is clear that the following general statement is valid: if a system contains quasi-particles with positive energy and quasi-particles with negative energy it is impossible to establish statistical equilibrium in the quasi-particle gas.



If statistical equilibrium cannot be established, the entropy must be an increasing function of time:

$$dS/dt > 0.$$

This increasing function will always have a limit, which may be finite or infinite. Evidently the first possibility corresponds to the establishment of equilibrium in the quasi-particle gas and does not apply in the case being considered. Consequently

$$\sum_{\mathbf{k}\alpha} \ln|N_{\mathbf{k}\alpha}| \longrightarrow +\infty,$$

that is to say, the number of quasi-particles, at least those of one kind, grows without limit.[3)] However, from energy conservation we have

$$\mathscr{E} = \sum_{\mathbf{k}\alpha} \mathscr{E}_{\mathbf{k}\alpha} = \text{const}$$

from which it follows that the increase in the number of quasi-particles with energy of one sign means a simultaneous increase in the number of quasi-particles with energy of the other sign.

3. NONLINEAR WAVE INTERACTION IN A PLASMA CONTAINING AN ION BEAM

As an example we consider the interaction of waves in a quasi-neutral plasma through which an ion beam moves; the beam moves in the direction of the magnetic field. Assume that the temperatures of the plasma ions and the beam ions are small compared with the electron temperature T_e and that the ion density in the beam n' is much smaller than the plasma density n. We assume that the magnetic field H is large: $H^2/8\pi \gg Mnc^2/2$. Under these conditions the dielectric constant of the plasma (neglecting Coulomb collisions between particles) for the low-frequency irrotational oscillations of interest here ($\omega \lesssim \omega_{pi}$):[5]

$$\varepsilon(\omega, \mathbf{k}) = 1 - \frac{k_z}{k^2} \sum_{\mu} \frac{\omega_{p\mu}^2}{n_\mu} \int \frac{\partial f_\mu(\mathbf{v}, \mathbf{u})/\partial v_z}{\omega - k_z v_z} d\mathbf{v}, \quad (16)$$

where μ = e, i, i′ is a subscript denoting the plasma component (electrons, plasma ions or beam ions), $\omega_{p\mu} = 4\pi n_\mu e^2/m_\mu$, f_μ is the distribution function, **u** is the mean velocity of the beam ions, k_z and v_z are the components of these vectors along the magnetic field.

The spectrum $\omega_{\mathbf{k}\alpha} = \omega_\alpha(\mathbf{k})$ for waves whose phase velocities satisfy the condition $\sqrt{T_e/m} \gg \omega/k_z \gg \sqrt{T_i/M}$, is given by

$$\varepsilon(\omega, \mathbf{k}) \equiv 1 - \frac{\omega_{pi}^2}{k^2} k_z^2 \left[\frac{1}{\omega^2} + \frac{n'/n}{(\omega - k_z u)^2} \right] + \frac{\omega_{pi}^2}{k^2 c_s^2} = 0,$$

$$c_s = \sqrt{\frac{T_e}{M}}. \quad (17)$$

This equation has four solutions. The solutions are easily found from the condition $u^2 = c_s^2[1 + 3(n'/n)^{1/3}]$ (at the stability limit):

$$\omega_{1,2} = k_z u (1 - \eta) \mp \left(\frac{\eta}{3}\right)^{1/2} \frac{|\mathbf{k}| k_z u^2}{\omega_{pi}},$$

$$\omega_3 = k_z u (1 + \eta/2), \quad \omega_4 = -k_z u, \quad \eta = (n'/n)^{1/3}. \quad (18)$$

In obtaining these solutions we have assumed that $k \ll \omega_{pi} \eta^{1/2}/u$.

Suppose that at the initial time the distribution function for the plasmons differs from zero for the branches α = 1, 2, 3 for $k_x = k_y = 0$, $|k_z| \leq k_0 \ll \omega_{pi}\eta^{1/2}/u$. Let us now trace the behavior of the plasmon distribution function taking account of the nonlinear interactions described by (9). It follows from (9) that if plasmons characterized by k_x, $k_y \neq 0$ are not present initially then they will not appear. We therefore omit the subscript z on k_z hereinafter.

We now obtain the value of $\varepsilon'_{\omega_\alpha}$ for the spectrum in (18):

$$\varepsilon'_{\omega_{1,2}} = \pm \frac{6\omega_{pi}}{(ku)^2} (3\eta)^{-1/2} \operatorname{sign} k,$$

$$\varepsilon'_{\omega_3} = \frac{18\omega_{pi}^2}{(ku)^3} \quad \varepsilon'_{\omega_4} = -\frac{2\omega_{pi}^2}{(ku)^3}. \quad (19)$$

The quantity $V^{\alpha\beta\gamma}_{\mathbf{k}\mathbf{k}'\mathbf{k}''}$ for one-dimensional plasmons is given, for example, in[5]:

$$V^{\alpha\beta\gamma}_{kk'k''} = \frac{8\pi^2}{\varepsilon'_{\omega_\alpha}\varepsilon'_{\omega_\beta}\varepsilon'_{\omega_\gamma}} \left[\sum_\mu \frac{4\pi e_\mu^3}{m_\mu^2} \right.$$

$$\left. \times \int dv \frac{\partial f_\mu/\partial v}{(\omega_{k\alpha} - kv)(\omega_{k'\beta} - k'v)(\omega_{k''\gamma} - k''v)} \right]^2. \quad (20)$$

Analysis of the conservation relations

[3)]Actually, the growth of the number of quasi-particles of a given kind continues as long as the quasi-particle energy is less than the particle energy; when this point is reached it is no longer meaningful to make a distinction between the particles and the quasi-particles.

$$\omega_{k\alpha} = \omega_{k'\beta} + \omega_{k''\gamma}, \qquad k = k' + k'' \tag{21}$$

for the initial conditions on $N_{k\alpha}$ chosen above shows that only those interactions are possible in which plasmons from three different branches of the dispersion curve participate $\omega_{\mathbf{k}\alpha} = \omega_\alpha(\mathbf{k}) : N_{\mathbf{k}1}$, $N_{\mathbf{k}2}$, and $N_{\mathbf{k}3}$; each triplet of interacting plasmons is independent of other triplets. The conservation laws do not forbid interaction with plasmons of the fourth branch but estimates of the quantity $V^{\alpha\beta\gamma}_{\mathbf{kk'k''}}$ show that these are much weaker than the 1-2-3 interactions. Hence the infinite set of equations (1) splits up into independent equation triplets. We write such a triplet for the case $k > 0$:

$$\dot{N}_{k1} = \frac{2}{3u\eta} V^{123}_{kk'k''}(N_{k1}N_{k''3} - N_{k1}N_{k'2} - N_{k'2}N_{k''3}),$$

$$\dot{N}_{k'2} = -\frac{2}{3u\eta} V^{123}_{kk'k''}(N_{k1}N_{k''3} - N_{k1}N_{k'2} - N_{k'2}N_{k''3}),$$

$$\dot{N}_{k''3} = \frac{2}{3u\eta} V^{123}_{kk'k''}(N_{k1}N_{k''3} - N_{k1}N_{k'2} - N_{k'2}N_{k''3})\frac{k}{2k''},$$

$$k' = k + \frac{4}{3\sqrt{3\eta}}\frac{k^2u}{\omega_{pi}}, \qquad k'' = \frac{4}{3\sqrt{3\eta}}\frac{k^2u}{\omega_{pi}} \tag{22}$$

(in finding the solutions of (21) we have used the fact that $ku/\omega_{pi} \ll \eta^{1/2}$ is small). For a Maxwellian electron distribution function

$$V^{123}_{kk'k''} = 8\pi^4 \frac{2^{10}}{3^7}\left(\frac{e}{M}\right)^2 \frac{k''}{u\eta^3}. \tag{23}$$

We note that (9) and (18) can be obtained from the hydrodynamic equations for a three-component plasma under the assumption that the electron distribution is $n = n_0 \exp(e\varphi/T_e)$ in the electric field of the oscillations; however, in this case the quantity $V^{123}_{\mathbf{kk'k''}}$ is much smaller

$$V^{123}_{kk'k''} \sim 100\left(\frac{e}{M}\right)^2\left(\frac{ku}{\omega_{pi}\sqrt{\eta}}\right)^4\frac{k''}{u\eta}.$$

The equations (22) have two integrals:

$$N_1 - N_1^{(0)} = \xi(N_3 - N_3^{(0)}),$$

$$N_2 - N_2^{(0)} = -\xi(N_3 - N_3^{(0)});$$

$$\xi = 2k''/k \ll 1$$

(in order to conserve space we have omitted the subscripts k, k′, k″ on N_{k1}, $N_{k'2}$, and $N_{k''3}$). Substituting these integrals in the equation for N_3 we have

$$\dot{N}_3 = \frac{4}{3u\eta}V^{123}_{kk'k''}\left[N_3^2 + N_3\frac{N_1^{(0)} - N_2^{(0)} - 2\xi N_3^{(0)}}{2\xi} - \frac{(N_1^{(0)} - \xi N_3^{(0)})(N_2^{(0)} + \xi N_3^{(0)})}{2\xi}\right].$$

From the definition of $N_{k\alpha}$ and (19) for $\epsilon'_{\omega_\alpha}$ we assume that when $k > 0$, $N_1, N_3 > 0$, $N_2 < 0$; these relations are used to find the solution of the equation:

$$N_3 = N_3^{(0)}(A-1)\frac{\exp Bt}{1-\exp Bt};$$

$$A = \frac{N_1^{(0)} - N_2^{(0)}}{2\xi N_3^{(0)}}, \qquad B = \frac{4}{3u\eta}V^{123}_{kk'k''}N_3^{(0)}(A-1). \tag{24}$$

The solution in (24) holds for any initial condition with the exception of the narrow range of values of $N_1^{(0)}$, $N_2^{(0)}$, and $N_3^{(0)}$ that satisfy the inequality

$$|N_1^{(0)} - N_2^{(0)} - 2\xi N_3^{(0)}| \lesssim \{\xi|N_1^{(0)} - \xi N_3^{(0)}|\,|N_2^{(0)} + \xi N_3^{(0)}|\}^{1/2}.$$

Since $V^{123}_{\mathbf{kk'k''}} > 0$, $N_3^{(0)} > 0$, if the dependence on the initial conditions is neglected, the solution in (24) increases without limit.

The characteristic time for the development of this nonlinear instability τ can be estimated from the following:

$$\tau \sim \frac{1}{100}\frac{1}{\omega_{pi}}\left(\frac{ku}{\omega_{pi}\sqrt{\eta}}\right)^2\frac{\eta^{9/2}}{|A-1|}\frac{nT}{\mathscr{E}_3^{(0)}} \lesssim \frac{1}{100\omega_{pi}}\left(\frac{n'}{n}\right)^{3/2}\frac{nT}{\mathscr{E}_3^{(0)}}, \tag{25}$$

where $\mathscr{E}_3^{(0)} \sim \omega_3 N_3 k''$ is the plasmon energy per unit volume (N_3) in the range $-k'' < k < k''$. We note that the growth time for the nonlinear instability depends on $\mathscr{E}_3^{(0)}$ as $1/\mathscr{E}_3^{(0)}$ and not logarithmically, as in the case of the linear instability. It is obvious that this derivation giving the increase in the number of waves holds only so long as the perturbation theory holds.

The example given above contains all of the characteristic features of the general discussion in Sec. 2. The energy $\mathscr{E}_{k\alpha} = \omega_{k\alpha}\epsilon'_{\omega_\alpha}k^2|\varphi_{k\alpha}|^2/8\pi$ in the first and third branches is positive while the energy in the second branch is negative. Formally (22) has stationary solutions $N_3 = N_1N_2/(N_1 - N_2)$, but these solutions have no physical meaning (one is easily convinced of this by noting that $N_1, N_3 > 0$, $N_2 < 0$). Eventually the number of waves grows without limit.

It is easy to estimate the magnitude of the interaction between the plasmons and the background in the present example. The linear damping (growth) $\gamma_{k\alpha}$ (for a Maxwellian electron distribution) is

$$-\frac{\omega_{pi}^2}{k^2c_s^2}\sqrt{\frac{m}{M}}\frac{1}{\varepsilon_{\omega_\alpha}'}.$$

This quantity is a maximum on branches 1 and 2:

$$\gamma_{k1,2} \approx \mp\sqrt{m/M}\,\omega_{pi}(n'/n)^{1/6}. \tag{26}$$

Plasmons are absorbed in branch 1 and created in branch 2.

The absorption (emission) of plasmons by particles can also be related to nonlinear effects. An expression for the nonlinear damping (growth) $\gamma_{k\alpha}(N)$ is given in[5]:

$$\gamma_{k\alpha}(N) = 2\sum_{\beta\gamma}\int\int dk'\,dk''\delta(k'+k''-k)$$
$$\times \operatorname{Im} P\left\{\frac{1}{\varepsilon(\omega_{k\alpha}-\omega_{k'\beta}, k'')\,\varepsilon'_{\omega_\alpha}\varepsilon'_{\omega_\beta}}\left[\sum_{\mu}\frac{4\pi e_\mu^3}{m_\mu^2}\right.\right.$$
$$\left.\times\int dv\frac{\partial f_\mu/\partial v}{(\omega_{k\alpha}-kv)(\omega_{k'\beta}-k'v)(\omega_{k\alpha}-\omega_{k'\beta}-k''v)}\right]^2$$
$$\left.\times\; N_{k'\beta}\right\} \tag{27}$$

(the symbol ImP means that we are to take the imaginary part in the expressions in the curly brackets and that the integral over k″ is to be understood in the sense of the principal value). Since the ions are assumed to be cold we need only consider the plasmon interaction with the electrons. The largest contribution in the expression for γ_{k1} is given by plasmons $N_{k'2}$ of momentum k′, close to k, so that

$$\left|\frac{\omega_{k1}-\omega_{k'2}}{k-k'}\right| \sim \sqrt{\frac{T_e}{m}}$$

whence

$$\left|\frac{k''}{k}\right| = \left|\frac{k-k'}{k}\right| \lesssim \eta^{1/2}\frac{ku}{\omega_{pi}}\sqrt{\frac{m}{M}}.$$

We use this feature in computing $\gamma_{k1,2}(N)$: k′ is replaced by k and the integral over k″ is estimated by the mean-value theorem. For a Maxwellian function $f_e(v)$ we find

$$\gamma_{k1,2}(N) \sim \left(\frac{n'}{n}\right)^{1/2}\sqrt{\frac{m}{M}}\frac{k^2\omega_{pi}}{nMu}N^{(0)}_{2,1}. \tag{28}$$

Since $N_1 > 0$, and $N_2 < 0$, plasmons are absorbed in branch 1 and created in branch 2.

Similar calculations show that $|\gamma_{k3}(N)| \ll |\gamma_{k1,2}(N)|$.

The linear and nonlinear absorption (emission) of plasmons will have no effect on the results obtained above if $|\gamma_k(N)\tau|$, $|\gamma_k\tau| \lesssim 1$. The condition $|\gamma_k(N)\tau| < 1$ is always satisfied while the inequality $|\gamma_k\tau| < 1$ imposes a limitation on $\mathscr{E}_3^{(0)}$:

$$\mathscr{E}_3^{(0)} > \frac{nT}{100}\sqrt{\frac{m}{M}}\left(\frac{n'}{n}\right)^{5/4}.$$

If this condition is satisfied the plasma can be regarded as transparent as far as the plasmons are concerned. This limitation will not be imposed on $\mathscr{E}_3^{(0)}$ if a plateau is formed on the function $f_e(v)$ in the region of the point $v = c_s$: $\partial f_e/\partial v = 0$.

The general properties of quasi-particles with negative energy are reflected in (26) and (28). Any interaction of the quasi-particles with the background which leads to an increase in the background energy is accompanied by an increase in the number of quasi-particles with negative energy. This statement follows from the conservation of energy (cf. also[6,7]).

In conclusion we note that the effect described here can lead to anomalous diffusion even in a plasma that is stable in the linear approximation. This effect is of interest from the point of view of the conversion of the energy of ordered beam motion into heat.

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