THE BETHE-SALPETER EQUATION AND ROLE OF "CENTRAL" INTERACTIONS

I. M. DREMIN, I. I. ROĬZEN, R. B. WHITE, and D. S. CHERNAVSKIĬ

P. N. Lebedev Physics Institute, Academy of Sciences, U.S.S.R.

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It is shown that the equation for the imaginary part of the amplitude and the equation for the total cross section follow from the Bethe-Salpeter equation. In the total cross section equation the integral term corresponds to the peripheral cross section and the free term to the central interaction cross section. A particular case of the equation is the multiperipheral model. The equation for partial waves $f_l(t)$ in the t channel makes it possible to use the complex-orbital-momentum apparatus to ascertain those singularities in the l-plane to which various assumptions regarding the nature of the inelastic processes correspond. A variant consistent with two-particle unitarity in which f_l possesses a fixed pole, a moving pole, and a moving cut is considered within the framework of the Bethe-Salpeter equation. The variant corresponds to the case of asymptotically constant central and peripheral interaction cross sections. It can also be applied for describing the production of "fireballs."

1. INTRODUCTION AND ANALYSIS OF THE BASIC EQUATIONS

In the theory of inelastic processes one should point out two limiting cases: 1) The theory of peripheral interactions, based on the assumption that the interaction in these processes can be described by the exchange of one virtual π meson. 2) The statistical theory of multiple production, which is applicable to the description of the central interaction and is based on the assumption that many particles are exchanged in the process of the interaction (the number of particles approaching infinity).

The experimental data, it seems to us, testify to the existence of both limiting cases of peripheral and central collisions. The cross sections for these processes can be measured independently of each other and for high energies they turn out to be constant. It also follows from the experimental data that among the peripheral collisions there is a class of processes in which the secondary π mesons are emitted in groups (of 6-10 mesons), which have been named fireballs. It is important to stress that the effective value of the "masses" of the fireballs $\mathfrak{M}$ does not depend on the energy of the colliding particles and appears constant ($\sim$3—5 GeV). The parameter $\mathfrak{M}$ = 3—5 GeV evidently plays an important role in the physics of high energies.

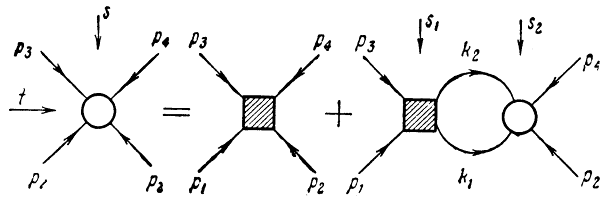
In the present and in following articles we will attempt to examine from a single point of view inelastic and elastic amplitudes for various processes, keeping in mind that both central and peripheral processes contribute asymptotically a constant term to the cross-section. A convenient method for this purpose is the utilization of the Bethe-Salpeter equation.

Using this equation we examine the following questions: 1) Derivation of an equation for the cross-section for inelastic processes from that for the amplitude for elastic scattering. 2) The inelastic processes associated with different types of kernels in the Bethe-Salpeter equation.[1)] 3) The manner in which the l-plane can be utilized to derive information concerning inelastic processes. 4) How the experimental data described above (and also other existing models for inelastic processes) can be utilized for the explanation of the properties of the amplitude for elastic scattering, and the consequences for elastic scattering that may arise from the experimental data concerning inelastic processes.

In the first part of this paper we consider only general theoretical questions. Concrete models and their characteristics are discussed in Sec. 2.

To begin with we examine the interaction of

[1)]Recently there appeared several articles which discuss the use of the Bethe-Salpeter equation in investigation of asymptotic properties of the scattering amplitude.[1-3] These papers do not, however, consider the properties of inelastic processes.



like spinless particles.[2)] We are interested only in states which in the t-channel possess the quantum numbers of the vacuum. (This means that in considering the partial waves in the t-channel only even partial wave amplitudes will have sense physically.)

We write the Bethe-Salpeter equation in the general form:

$$A(s,t,p_1^2,p_3^2)=A^c(s,t,p_1^2,p_3^2)$$
$$-\frac{i}{(2\pi)^4}\int A^c(s_1,t,p_1^2,p_3^2,k_1^2,k_2^2)$$
$$\times A(s_2,t,k_1^2,k_2^2)D(k_1^2)D(k_2^2)d^4k_1, \qquad (1)$$

A is the full amplitude (off the mass shell of two external momenta), A^C is the sum of all two-particle irreducible diagrams in the t-channel. $D(k_1^2)$ and $D(k_2^2)$ are propagators for particles with momenta k_1 and k_2. The remaining notation is clear from the graphical representation of the equation in the figure.

Equation (1) is the exact integral equation for the amplitude, valid in the s, t, and u channels. However, the only thing known about A^C with certainty is the fact that $\mathrm{Im}A_t^C(s,t)=0$ for $t<16$. This proves to be sufficient for the derivation of models of the type of the multiperipheral model [4]. We shall give A^C a definite form[3)] only upon going to specific models. Naturally, the choice of kernel is restricted by the fact that the amplitude must satisfy the unitarity condition in the s- and t-channels for arbitrary values of s and t. However, in the case of the selection of a concrete form for A^C satisfying the above requirements, it is possible to prove from Eq. (1) only that the amplitude is unitary in the interval $4<t<16$.

If we assume in Eq. (1) that A^C and A are independent of k_{10} and integrate over k_{10}, introducing the quantities $V_0=-A^C/(4\pi)^3$ and $T=-A/(4\pi)^3$, then we arrive at the equation used in the method of quasipotentials (see Eq. (1.4) of [15]). Going further to the limit $t\to 4$ we arrive at the non-relativistic Bethe-Salpeter equation in the form [6]

$$T(\mathbf{p},\mathbf{p}')=V(\mathbf{p},\mathbf{p}')-\frac{1}{(2\pi)^3}\int d^3qV(\mathbf{p},\mathbf{q})\frac{T(\mathbf{q},\mathbf{p}')}{\varepsilon_q-E-i\delta}. \qquad (2)$$

For the following it will be essential only that the kernel of the relativistic equation A^C goes over in that limit to the expression $-V(p,p')$, where V is the Fourier transform of the usual potential. Thus a positive A^C determines in the non-relativistic limit an attractive interaction, and a negative A^C a repulsive interaction.

If attraction takes place and A^C can be expressed in the dispersion form in s, then $A_{12}^C(s,0)>0$ if they are of fixed sign. Notice that in the case when A^C depends only on s, the related non-relativistic potential in the coordinate representation is local, and thus the behavior of A^C for increasing s defines the character of the potential as $r\to 0$ (for example, if $A^C\sim s$ then $V\sim r^{-5}$ [5]). Dependence on other variables (especially on $k_{1,2}^2$) implies a non-local character of the interaction in the non-relativistic limit.

We examine now Eq. (1) in the s-channel. Let us derive the equation for the imaginary part of the amplitude:

$$A_1(s,t)=A_1^c(s,t)-\frac{\mathrm{Re}}{(2\pi)^4}\int d^4k_1A^cAD(k_1^2)D(k_2^2). \qquad (3)$$

We write the dispersion relations in s_1 for A^C and A with $t<0$, substitute them in (3), and noting that $D(k_1^2)$ and $D(k_2^2)$ are non-singular in the s-channel, we find (for a detailed derivation of this equation, see Appendix I of [7])

$$A_1(s,t,p_1^2,p_3^2)=A_1^c(s,t,p_1^2,p_3^2)+\frac{1}{8\pi^4}\int d^4k_1\frac{A_1^c(s_1,t,p_1^2,p_3^2,k_1^2,k_2^2)A_1(s_2,t,k_1^2,k_2^2)+A_2^c(u_1,\ldots)A_2(u_2,\ldots)}{(k_1^2+1)(k_2^2+1)} \qquad (4)$$

This equation agrees with the equation of Amati and others,[4] except for the presence of the second term $(A_2^CA_2)$ in the integrand[4)]. If A^C is treated as described above, then Eq. (4) becomes the

[2)]For simplicity we take them to be neutral, pseudoscalar, with mass equal to 1.

[3)]The usual ladder approximation consists of choosing A^c as a simple pole in s.

[4)]In the case under consideration, that of interaction of neutral pseudoscalar particles, the integrals of the first and second term (4) are identical. Thus equation (4) differs from (4.2) of [4] by a factor of 2.

exact equation of field theory. Consequently, analytically continuing this equation into the region $4 \leq t \leq 16$, we can find the usual Mandelstam spectral function $\rho_{13}(s, t)$[8] if we calculate the discontinuity across the cut in that region.[5)]

Examining now Eq. (4) in the s-channel for $t = 0$ and using the optical theorem, we easily find the equation for the total cross sections

$$\sigma(s, p^2) = \sigma^c(s, p^2) + \frac{1}{8\pi^3}\int \frac{dk^2 s_1 ds_1 s_2 ds_2}{(k^2+1)^2}\sigma^c(s_1, k^2)\sigma(s_2, k^2)H(s_1, s_2, k^2, p^2),$$

$$H = s_1^{-1}s_2^{-1}[-su + (p^2+1)^2]^{-1/2}[(s_1+p^2+k^2)^2 - 4p^2k^2]^{1/2} \times [(s_2-1+k^2)^2 + 4k^2]^{1/2}[(s-1+p^2)^2+4p^2]^{-1/2},$$

$$p_1^2 = p_3^2 = p^2, \quad k_1^2 = k_2^2 = k^2. \tag{5}$$

Let us explain the physical meaning of the quantity σ^C. Recalling that A^C is two-particle-irreducible in the t-channel, we see that σ^C represents in the s-channel the sum of the cross sections of all processes which result from the exchange of more than one meson, and also from the result of interference of single-meson and many-meson diagrams. The question of the interference of these quantities was considered in [9,10], where it was found that these terms are negligibly small. We define as central collisions all processes proceeding through the exchange of more than one π meson. (This definition agrees with the definition of central interactions used in [11].) Then it follows from the foregoing [11] that σ^C, being the cross section for central collisions, is a positive quantity. Note that a positive σ^C, as can be seen above, corresponds to an attractive potential in the non-relativistic region in the t-channel.

Equation (5) corresponds to the equation in the multiperipheral model [4] if one assumes that the central collision gives a non-vanishing term only for low energies, and reduces to the equation for the single-meson approximation [9] if one of the cross sections appearing in the integrand in this equation is regarded as that for central collisions. In this manner the properties of inelastic processes are related to those of the amplitude for elastic scattering. Specifically, the singularities of the elastic scattering amplitude in the l-plane may depend on the character of the inelastic processes.

We therefore turn now from Eq. (1) to the corresponding equation for the partial wave amplitude in the t-channel:

$$f_l(t, p_1^2, p_3^2) = f_l^c(t, p_1^2, p_3^2) - \frac{2i}{(2\pi)^3} \times \int_0^\infty |\mathbf{q}|^2 d|\mathbf{q}| \int_{-\infty}^{\infty} dq_0 \frac{f_l^c(t, p_1^2, p_3^2, k_1^2, k_2^2) f_l(t, k_1^2, k_2^2)}{(t/4 + q_0^2 - \mathbf{q}^2 - 1 - i\varepsilon)^2 - tq_0^2}, \tag{6}$$

$$\mathbf{q} = \mathbf{k}_1, \quad q_0 = (p_{10} + p_{30})/2 - k_{10} = \sqrt{t}/2 - k_{10}.$$

For the following it is convenient to introduce relativistically invariant integration variables having the physical meaning of the sum and difference of the virtualities of the exchanged particles, which in the center of mass system of the t-channel are related with q and q_0 by:

$$q = [t/4 - (r+2)/2 - v^2/4t]^{1/2}, \quad q_0 = v/2\sqrt{t}, \quad r = -k_1^2 - k_2^2 - 2, \quad v = k_2^2 - k_1^2. \tag{7}$$

Equation (6) in terms of these variables takes the form:

$$f_l(t, r_0, v_0) = f_l^c(t, r_0, v_0) - \frac{4i}{(4\pi)^3 t} \times \int dr dv \frac{[t(t-4) - 2tr + v^2]^{1/2} f_l^c(t, r_0, v_0, r, v) f_l(t, r, v)}{(r + i\varepsilon)^2 - v^2}, \tag{8}$$

where $r_0 = -p_1^2 - p_3^2 - 2$, $v_0 = -p_1^2 + p_3^2$, and the integration runs over the region $t(t-4) - 2tr + v^2 > 0$.

Equations (6) and (8) are valid for arbitrary values of t. Note that on going to the region $t < 4$ all zeros of the denominator of (6) are located in the second and fourth quadrants of the q_0 plane. The quantities f_l^c and f_l also may have singularities in the q_0 plane, but if they are of Feynman character they also lie in the second and fourth quadrants. In this case the contour of integration in q_0 can be rotated to the imaginary axis [12,13]. Substituting then $q_0 \to -iq_0$ we find:

$$f_l = f_l^c + \frac{2}{(2\pi)^3} \times \int_0^\infty q^2 dq \int_{-\infty}^{\infty} dq_0 \frac{f_l^c f_l}{(t/4 - q_0^2 - q^2 - 1 + i\varepsilon)^2 + tq_0^2} \tag{9}$$

or substituting $v \to -iv$

$$f_l = f_l^c + \frac{4}{(4\pi)^3 t}\int dr dv \frac{[t(t-4) - 2tr - v^2]^{1/2}}{(r+i\varepsilon)^2 + v^2} f_l^c f_l \tag{10}$$

with the region of integration $t(t-4) - 2tr - v^2 > 0$.

This equation can be analytically continued to the region $t > 4$. It is necessary, however, to keep in mind the singularities which deform the contour of integration with respect to q_0 at $t > 4$. This leads to the appearance of a cut at $f_l(t)$ in the t-plane when $t > 4$. The discontinuity across the

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[5)]See appendix II of the authors' paper [7].

cut can be calculated using the expression for the above equation. In the region $4 < t < 16$ we find

$$f_l - f_l^{II} = \frac{2i}{16\pi}\left(\frac{t-4}{t}\right)^{1/2} f_l f_l^{II}, \quad (11)$$

where f_l and f_l^{II} are the values of the partial-wave amplitude on the first and second sheets of the t-plane. On the mass shell for integer even values of l (11) reduces to the usual unitarity relation.

In the following it will be necessary also to examine the partial wave equation at the point $t = 0$. In the limit as $t \to 0$ the region of integration in the v-variable in (10) contracts to a point. With this the numerator and denominator of the kernel in the integral term tend to zero. The possibility of performing the transformation depends on the behavior of the functions f_l^c and f_l at the point $t = 0$. It is convenient here to introduce new functions:

$$\varphi_l^c = f_l^c(t-4-2r_1-v_1^2/4t)^{-l/2}(t-4-2r_2-v_2^2/4t)^{-l/2},$$

$$\varphi_l = f_l(t-4-2r_1-v_1^2/4t)^{-l/2}(t-4-2r_2-v_2^2/4t)^{-l/2}. \quad (12)$$

They satisfy the equations

$$\varphi_l = \varphi_l^c + \frac{4}{(4\pi)^3 t^{l+1}}\int dr dv \frac{[t(t-4)-2tr-v^2]^{l+1/2}}{r^2+v^2}\varphi_l^c\varphi_l. \quad (13)$$

Using the well known [4] expressions for f_l and f_l^c in terms of A_1 and A_1^c

$$f_l = \frac{2}{\pi}\int_{z_0}^{\infty} A_1(z, t, r, v) Q_l(z) dz,$$

$$f_l^c = \frac{2}{\pi}\int_{z_0}^{\infty} A_1^c(z, t, r, v) Q_l(z) dz \quad (14)$$

(where $Q_l(z)$ is a Legendre function of the second kind), and using the analytic properties of A_1 and A_1^c, we find that the functions φ_l and φ_l^c do not take on the values zero or infinity as $t - 4 - 2r - v^2/t \to 0$. In view of this it is possible to perform the integration over v as $t \to 0$. We then find

$$\varphi_l(r_0) = \varphi_l^c(r_0) + \frac{2^{l+3}\sqrt{\pi}\,\Gamma(l+3/2)}{(4\pi)^3\Gamma(l+2)} \times \int_{-\infty}^{-2} dr \frac{(-r-2)^{l+1}}{r^2}\varphi_l^c(r_0, r)\varphi_l(r). \quad (15)$$

By solving this inhomogeneous equation we can determine the form of the partial wave amplitude for $t = 0$.

Thus, (10) and (15) are equivalent to (1) and, like (4) and (5), permit in principle to ascertain the singularities of the functions f_l in the l-plane to which definite assumptions concerning the character of the central interactions correspond, especially assumptions concerning their total cross-sections.

2. PROPERTIES OF ELASTIC AND INELASTIC PROCESSES

In this section, using different models, we consider the following questions: a) The distribution in the square of the transferred momentum k^2 of the peripheral interactions in which several centers of emission of secondary pions (fireballs) occur; b) the distribution in the "masses" of these emission centers and its relation to the distribution in k^2; c) the dependence of the total number of fireballs on the energy of the colliding particles; d) the cross section of the central interaction at high energies. We pay particular attention here to those general properties of these distributions which are, as far as possible, independent of the particular assumptions of the model (namely, the asymptotic behavior of the distribution, the location of the maxima, and similar characteristics). A few of the models are in our opinion not realistic; their examination is of methodological importance in that they lead to models of more physical importance. We hope to carry out a more detailed investigation of such models in the future. We begin with the general characteristics of the process.

We are interested in the total cross section for processes in the s-channel. It is thus convenient to make use of the Bethe-Salpeter equation for the imaginary part of the amplitude for elastic processes at $t = 0$ in the form:

$$A_1(s, p^2) = A_1^c(s, p^2) + \frac{1}{8\pi^3(-su+(p^2+1)^2)^{1/2}}\int\frac{ds_1\,ds_2\,dk^2}{(k^2+1)^2} \times A_1^c(s_1, k^2, p^2)A_1(s_2, k^2). \quad (16)$$

In the l-plane it has the form (15). This is an integral equation whose character depends on the value of l. Further we will demand that the equation satisfy the Fredholm condition for $l > 1$.[6] Of particular interest will be the investigation of this equation for $l \to 1 + 0$. The peripheral interaction is described by the second term of (15) or (16).

In all of the models which we will consider,

[6]We stress that Eq. (16) or Eq. (15) for $l = 1 + 0$ has physical meaning in that the terms of the iteration series represent observable quantities. The Fredholm requirement is related to the existence of an iteration series.

there appears a singularity in the integral term of (16) in the form of a pole at $l = l_0(t)$, the position of which depends on t but not on the virtuality k^2. This is the so-called moving pole[7)], the appearance of which is related to the unitarity condition [14]. We assume that $l_0(0) = 1$ for $t = 0$. The function near the point $l = l_0$ can be written in the form

$$\varphi_l = R / (l - l_0) = R / (l - 1). \tag{17}$$

Substituting (17) in (15) and assuming that in general φ_l^c is regular near $l = l_0$ we easily find a homogeneous equation for R. However, since the contribution to the amplitude A_1 from the moving pole (A_1^P) is related to the residue of the function by

$$A_1^P \equiv \Psi(p^2) s \sigma^P (p^2 = -1) = 3\pi R(p^2) s, \tag{18}$$

where σ^P is the cross section of the peripheral interaction; σ^P = const, it will be more convenient to use not R, but a function Ψ which coincides with R apart from a coefficient:

$$\Psi(p^2) = \lambda \int_0^\infty K(p^2, k^2) \Psi(k^2) dk^2,$$

$$K(p^2, k^2) = \frac{(k^2)^2}{(k^2+1)^2} \varphi_l^c (t = 0, p^2, k^2), \quad \lambda = \frac{3}{16\pi^2}. \tag{19}$$

In an analogous manner we can rewrite Eq. (16), if we consider only the contribution from the moving pole:

$$\Psi(p^2)\sigma^P = \frac{1}{4\pi^3} \int ds_1 dk^2 \frac{(k^2)^2}{(k^2+1)^2} \times \frac{A_1^c(s_1, p^2, k^2)\Psi(k^2)\sigma^P}{\{s_1 + p^2 + k^2 + [(s_1 + p^2 + k^2)^2 - 4p^2k^2]^{1/2}\}^2}. \tag{20}$$

This equation corresponds to (19). The diagram explains the physical meaning of the quantities S_1 and k^2, S_1 being the square of the "mass" of the irreducible center of the diagram and k^2 the virtuality of the exchanged pion. It follows that the quantity

$$\frac{d\sigma^P}{ds_1} = \frac{\sigma^P}{4\pi^3 \Psi(p^2)} \int_0 dk^2 \frac{(k^2)^2}{(k^2+1)^2} \times \frac{A_1^c(s_1, p^2, k^2)\Psi(k^2)}{\{s_1 + p^2 + k^2 + [(s_1 + p^2 + k^2)^2 - 4p^2k^2]^{1/2}\}^2} \tag{21}$$

represents the distribution in the squares of the masses of the irreducible center of the diagram, and the quantity

$$\frac{d\sigma^P}{dk^2} = \frac{\sigma^P(k^2)^2}{4\pi^3(k^2+1)^2} \frac{\Psi(k^2)}{\Psi(p^2)} \int_4^s ds_1 \times \frac{A_1^c(s_1, p^2, k^2)}{\{s_1 + p^2 + k^2 + [(s_1 + p^2 + k^2)^2 - 4p^2k^2]^{1/2}\}^2} \tag{22}$$

is the distribution in k^2, the virtuality of the exchanged pion when the virtuality of the incoming pion is p^2. The properties of these distributions depend on the form of Ψ, and thus on the kernel K.

The Fredholm conditions impose some bounds on K. In the case of a symmetric kernel this consists of the requirement that $K(p^2, k^2)$, decrease faster than $(k^2)^{-1}$ for fixed p^2 and $k^2 \to \infty$. It is not difficult to convince oneself that $\Psi(k^2)$ behaves in the same fashion as $k^2 \to \infty$. It follows that as $k^2 \to \infty$ the distribution in virtualities decreases rapidly enough so that its integral converges.

The distribution $d\sigma/ds_1$ depends on the behavior of A_1^c. If $A_1^c(s_1) \sim s_1^\nu$ with $\nu < 1$ for $k^2 > 0$[8)], then $d\sigma/ds_1 < s_1^{-1}$ as $s_1 \to \infty$, and it follows that the integral of this distribution over s_1 also converges as $s_1 \to \infty$. Thus in both distributions the upper limit s can be replaced by infinity for large s.

On the other hand as $k^2 \to 0$ we have $d\sigma/dk^2 \to 0$, as follows directly from (22). Thus the distribution in k^2 has a maximum which is asymptotically independent of the energy of the colliding particles. For small values of s_1, near threshold ($s_1 = 4$), $A_1^c(s_1)$ is small. (Note $A_1^c(s_1) = 0$ for $s_1 = 4$.) Thus the function $d\sigma/ds_1$ has a maximum. Its position does not depend on the energy of the colliding particles.

For a more detailed description of peripheral inelastic processes it is necessary to consider iterations of Eqs. (15) and (16). The point is that an iteration of n-th order corresponds to a Feynman diagram for inelastic processes with n centers of emission of secondary particles. It is not difficult to verify that no finite iteration of (15) gives rise to a moving pole in the l-plane, but that this pole arises only through summation of the iteration series. If the function φ_l^c has in the l-plane a pole for $l = l_1$ (exactly this situation occurs as a rule, with $l_1 < l_0$), then the n-th iteration has a pole of the n-th order at the same point and is of the form

$$\varphi^{(n)} = \Phi^{(n)}(l, p^2) / (l - l_1)^n. \tag{23}$$

[7)]Note that the moving pole is the leading singularity of the entire amplitude for $k^2 > -1$.

[8)]We show below that the condition $\nu < 1$ for $k^2 > 0$ is necessary for the fulfillment of the Fredholm conditions of Eq. (15).

We separate from the iteration series those terms which upon summation contribute the pole at $l = l_0(0) = 1$ (denote them by $\varphi_{pol}^{(n)}$). We write (17) in the form

$$\varphi_l = \frac{R(l)}{l-1} = \varphi_{pol} + \frac{R(l)-R(1)}{l-1}. \qquad (17')$$

The last term does not have a pole at $l = 1$ if the function $R(l)$ is smooth and differentiable at this point. The first term has, of course, the same structure as (23):

$$\varphi_{pol} = \sum_n \varphi_{pol}^{(n)}, \quad \varphi_{pol}^{(n)} = \Phi_{pol}^{n}(1, p^2)/(l-l_1)^n. \qquad (23')$$

On the other hand, the function φ_{pol} may be expanded in a Laurent series

$$\varphi_{pol} = \frac{R(1)}{l-l_0} = R(1)\sum_n \frac{1}{l-l_1}\frac{(l_0-l_1)^n}{(l-l_1)^n}.$$

Comparing the n-th term of this expansion with that of (23′) and using (18), we find

$$\Phi_{pol}^{(n)} = R(1)(1-l_1)^{n-1}$$

$$= (1-l_1)^{n-1}\Psi(p^2)\frac{\sigma^P(p^2=-1)}{3\pi}. \qquad (24)$$

Thus the quantity $\Phi_{pol}^{(n)}$ depends on the virtuality p^2 in a manner independent of n. This is a consequence of the fact that the position of the moving pole depends only on t. The corresponding term in A_1 has the form

$$A_{1pol} = \frac{\Phi^{(n)}}{s^{l_1}(n-1)!}(\ln s)^{n-1} = \Psi(p^2)\frac{(l_0-l_1)\ln s}{s^{l_1}(n-1)!}. \qquad (25)$$

Using expression (25) it is possible to find in which energy region (that is, for what s) the maximum contribution is made by the formation of n emission centers.

$$\ln s = (n-1)/(1-l_1). \qquad (26)$$

From this it follows that with increasing s the number of centers grows logarithmically.

Note, that the terms $A_1^{(n)}$ of the iteration series of Eq. (16) are of course different from $A_{pol}^{(n)}$. For small s, when the conservation of energy in the s channel limits the iteration series to a small number of terms, the difference may be appreciable, even as a result of the singularities of φ_l located to the left of l_1. If, besides this, $\varphi_l^{(n)}$ has singularities located to the right of l_1 (but, as stipulated, to the left of $l_0 = 1$), then as $s \to \infty$ these are the ones which determine the behavior of $A_1^{(n)}$. (We will examine this situation for $n > 1$ below.) However for sufficiently large n, and for s chosen so that $A_{1pol}^{(n)}$ is a maximum [see (26)], the term $A_{1pol}^{(n)}$ will dominate $A_1^{(n)}$. This is due to the fact that the sum of $\varphi l_{pol}^{(n)}$ has a pole located to the right of all the singularities of the integral terms.

In the calculation of the angular distribution, the most important characteristic is $\overline{\gamma_0}$, the Lorentz factor relating two neighboring centers. Using the exposition of [9] we easily find that

$$\overline{\gamma_0} = \cosh\frac{1}{l_0(0)-l_1} \qquad (27)$$

i.e., $\overline{\gamma_0}$ is completely defined by the relative positions of the poles in φ_l^c and φ_l.

Thus, peripheral processes, described in the framework of the Bethe-Salpeter equation, posess the following properties. Secondary-particle emission centers are formed. The number of centers n grows slowly (logarithmically) with increasing energy s. For sufficiently large values of s and n, which are related by condition (26), the distribution with respect to the square of the masses s_i of these centers has a maximum and decreases quickly as $s_i \to \infty$. The position of the maximum does not depend on the energy of the incoming particles s nor on the number of centers n, if n and s satisfy (26).

The distribution with respect to the virtuality of the exchanged pion posesses these same properties. These properties are those of fireballs. We stress again that the formation of fireballs arises within the framework of the considered theory through the following two requirements: a) the Fredholm condition must be satisfied by (15) and (16), and b) the moving pole at $t = 0$ must be located at $l_0(0) = 1$. The latter is equivalent to the requirement that the cross-section for peripheral processes be asymtotically constant.

We examine now several models for inelastic processes.

A. The model with $\sigma^c \sim S^{\nu-1}$ ($\nu < 1$). We consider the case when A_1^c has the form

$$A_1^c(s, p^2) = \Psi^c(s/s_0)^\nu \quad \text{for} \quad s \geqslant s_0, \qquad (28)$$

where ν is a constant, $\nu < 1$. In this case the cross-section for central interactions decreases with increasing energy as a power of the energy, $\sigma^c \sim S^{\nu-1}$. In our opinion this is a non-realistic model (as noted above, the cross-section for central interactions is apparently asymptotically constant), but can be of methodological interest since its analysis can facilitate further research.

Using (12) and the asymptotic form of the function $Q_l(z)$ for $z \gg 1$ we find for φ_l^c [9)]

[9)]Here and below we assume that the region of importance for A_1^c is that of large but possibly bounded values of s, in which the asymptotic form already holds, and neglect the region near threshold, i.e., $4 \lesssim S \lesssim 0.5$ (GeV)^2.

$$\varphi_l^c = \Psi^c \frac{\Gamma(l+1)}{\sqrt{\pi}\Gamma(l+3/2)2^{2l}} F(l+1,\ l-\nu,\ l-\nu+1;\ z) \times \frac{1}{l-\nu}, \tag{29}$$

where $z = -(p^2 + k^2)/s_0$ and F is a hypergeometric function. Equation (19) takes the form

$$\Psi(p^2) = \frac{1}{8\pi^3}\int \frac{(k^2)^2}{(k^2+1)^2}\frac{\Psi^c}{1-\nu} \times F\left(2,\ 1-\nu,\ 2-\nu;\ -\frac{p^2+k^2}{s_0}\right)\Psi(k^2)\,dk^2. \tag{30}$$

If $\nu < 0$, then Eq. (30) is of Fredholm type and can be solved by usual methods. The case $\nu \leq -1$ contains the multiperipheral model of Amati et al.[4] If $A^c(s) \sim \delta(s - s_0)$ the function φ_l^c simplifies and the equation for $\psi(p^2)$ becomes Eq. (4.6) of [4].

We note also that the iterations of Eq. (15) have in the l-plane multiple poles for $l = -1$. Expressed in terms of s these iterations have terms of the type $A^{(n)} \sim (\kappa \ln s)^n/sn!$ examined in [4].

When $0 \leq \nu < 1$, in order to satisfy the Fredholm character of Eq. (30) it is necessary to require in addition that Ψ^c be a decreasing function of the virtuality. When $\nu = l_0 = 1$ such an assumption is no longer sufficient, since the coefficient of the integral term in (30) becomes infinite. This is related to the impossibility of there being a multiplicative dependence of the total cross sections on s and k^2 in this case[10)].

Note that the quantity $\overline{\gamma_0}$ defined in Eq. (27), is about $\gamma_0 \approx 1.15$ when $\nu \leq -1$, which is somewhat lower than the experimental results[11]. For $\nu = 1/2$ we have $\gamma_0 \sim 3$-4, which is closer to the experimental results.

B. The model with σ^c = constant. We now examine models in which the cross section for the central interaction (on the mass shell) is asymptotically constant. This means that as $s \to \infty$

$$A_1^c(s, t, p^2 = k^2 = -1) = (s/s_0)^\nu \Psi^c(t), \tag{31}$$

where $\nu = 1$ for $k_{1,2}^2 = p_{1,3}^2 = -1$, and s_0 is a parameter of dimension energy squared. From the above it follows that the mode (31) (with $\nu = 1$) can not be retained off the mass shell for space-like k^2, and that for the fulfillment of the Fredholm condition it is necessary that ν be smaller than unity as $k_{1,2}^2, p_{1,3}^2 \to \infty$. From this it follows that ν is a decreasing function of its arguments $(k_{1,2}^2;\ p_{1,3}^2)$.

10)This is because it proved impossible to satisfy the equations of Berestetskii and Pomeranchuk[14] by specifying asymptotically constant cross sections.

Expression (31) corresponds to $\varphi_l^c(k_{1,2}^2;\ p_{1,3}^2)$ given by Eq. (29), which on the mass shell (for $k_{1,2}^2 = p_{1,3}^2 = -1$) has in the l-plane a pole at $l = 1$, but off the mass shell the position of the singularity changes. This is already a more difficult variant of the Bethe-Salpeter equation.

In distinction to the above considerations, we will now examine the analytic properties of the elastic scattering amplitude in the l-plane in the case of t different from zero. It is convenient to use Eq. (13) for this purpose.

As will be shown below, the solution in this case posesses, besides a moving pole and a fixed pole (associated with the central interaction), a moving branch point in the l-plane.

In the moving pole method, it followed from the analyticity of the partial wave amplitudes that the existence of a fixed pole contradicted the unitarity condition. In our case this statement does not hold true. In the first place, it was shown above that the solution of the Bethe-Salpeter equation satisfies the unitarity condition automatically. Second, in our case the function possesses moving branch points.[11)]

Assume that in expression (31) the power ν depends on the virtualities: $\nu(k_{1,2}^2;\ p_{1,3}^2)$[12)]. We examine the case when ν is bounded below. If $\nu_0 \geq 0$, then it is necessary for the satisfaction of the Fredholm condition to introduce an additional factor $\Psi^c(k^2)$ which decreases with increasing virtuality. (If $\nu_0 < 0$ the Fredholm condition is automatically satisfied.)

For the following, the essential points concerning the singularities of φ_l^c are [see (29)]: 1. There is a pole at $l = \nu(r_0, v_0, r, v)$, the position of which depends on the virtualities, but not on the parameter t. This we refer to as the fixed pole. 2. There is a pole for $l = -1$, with position depending on no variables; this pole arises from the pole in the Γ-function in (29), or what is the same, from the related pole of $Q_l(z)$[13)].

Investigation of Eq. (13) with φ_l^c defined through (29) and ν depending on r_0, v_0 and r, v allows one to find several general properties of

11)A mention of such a possibility is contained in the paper of Gribov[15].

12)Here and below we consider ν independent of t. Questions concerning the physical meaning of (31) will be considered in future papers.

13)Generally speaking, one cannot rule out the variant in which a fixed pole of such character is located to the right of $l = 1$, i.e., at $l = l_2$, where $l_2 > -1$. The arguments given above imply only the limitation $l_2 = 1$.

the solution which are independent of the exact form of the function $\nu(r_0, v_0, r, v)$. First, the solution $\varphi_l(t, r_0, v_0)$ possesses in the l-plane a moving branch point. Its appearance is connected with zeros of the term $l - \nu$ in the denominator of expression (29). Actually, the prescence of the pole in $\varphi_l(t, r, v)$ for $l = \nu(r, v, 0, 0)$ leads to the appearance of a cut which should be drawn through the points[14];

$$\nu(0, 0, -\infty, 0) < l < \nu(0, 0, t/2 - 2, 0), \qquad (32)$$

and the pole in $f_l^c(t, r, v, r_0, v_0)$ leads to a cut at

$$\nu(r, v, -\infty, 0) < l < \nu(r, v, t/2 - 2, 0). \qquad (33)$$

The presence of the branch points defined through (33) leads after integration over r and v to two cuts at

$$\nu(-\infty, 0, t/2 - 2, 0) < l < \nu(t/2 - 2, 0, t/2 - 2, 0), \qquad (34)$$

$$\nu(-\infty, 0, -\infty, 0) < l < \nu(t/2 - 2, 0, -\infty, 0), \qquad (35)$$

which, however, appear as continuations of each other, because $\varphi_l^c(r, v, r_0, v_0)$ is symmetric in r, r_0 and v, v_0, so that there is simply one cut

$$\nu(-\infty, 0, -\infty, 0) < l < \nu(t/2 - 2, 0, t/2 - 2, 0). \qquad (36)$$

In this manner the function $\varphi_l(t, r_0, v_0)$ has in the general case three cuts in the l-plane with branch points whose positions depend on t. On the mass shell these cuts join into one cut.[15] We note also that for t = 4 the moving branch point moves past the fixed pole. [This is clear from (32)–(36)].

Let us examine in the framework of this model the question of iterations and of the character of the inelastic processes. Iterations of Eq. (13) have the following singularities. A fixed (in t) pole at $l = \nu$ is found only in the first iteration $\varphi^{(1)} = \varphi^c$. The subsequent iterations have moving (in t) branch points and fixed higher order poles at $l = 1$. It can be shown that the sum of these branch points cannot lead to a moving pole; the sum of the higher-order poles leads to the moving pole. Thus, the character of the iterations of Eq. (13) as $l \to l_0(t)$, and therefore the character of the peripheral processes contributing a non-vanishing term to the cross section, is exactly the same as in the models investigated above.

The peripheral interaction has a fireball character. For the evaluation of the number of fireballs, their distributions according to mass and transferred momentum, and an estimate of γ_0 we can use Eqs. (21), (22), and (27).

We note that besides the moving pole contribution, there is also a contribution from the branch points to the peripheral interaction cross-section. These terms, however, vanish asymptotically, in this model, since for t = 0 the branch is located to the left of the point $l = 1$. In a lower energy region these terms could be noticeable.

So far we have investigated only the interaction of identical particles (π mesons) but the apparatus of the Bethe-Salpeter equation can be generalized without difficulty to treat the interactions of different particles, for example $\pi\pi$, πN, and NN. Ignoring spin and isospin structure we may write:

$$\varphi_{\pi\pi} = \varphi_{\pi\pi}{}^c + \int K' \varphi_{\pi\pi}{}^c \varphi_{\pi\pi} dr\, dv,$$

$$\varphi_{\pi N} = \varphi_{\pi N}{}^c + \int K' \varphi_{\pi\pi}{}^c \varphi_{\pi N} dr\, dv,$$

$$\varphi_{NN} = \varphi_{NN}{}^c + \int K' \varphi_{\pi N}{}^c \varphi_{\pi N} dr\, dv. \qquad (37)$$

These functions satisfy the unitarity condition, and therefore the trajectories of moving poles in the l-plane coincide (i.e., there is one universal moving pole). We recall that the moving pole is contained in the integral terms of Eq. (37).

In the investigated model in the case of the interaction of real particles the central interaction introduces a non-vanishing term in the cross section, given by the poles in the first terms of (37); the residues of these poles are in general arbitrary. From this it follows that the total amplitudes of related processes do not necessarily have the same behavior. The situation in this case is analogous to that investigated by Feĭnberg and one of the authors of [11], and the main factor is the relative contribution of the central and peripheral processes to the cross-section for inelastic processes.

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