INTERACTION OF LASER MODES

L. A. OSTROVSKIĬ

Radiophysics Institute, Gorkiĭ State University

Submitted to JETP editor July 31, 1964

J. Exptl. Theoret. Phys. (U.S.S.R.) 48, 1087-1096 (April, 1965)

A two-mode laser model is considered and abbreviated equations are derived. Monochromatic (single-mode) and two-frequency (two-mode) steady states are obtained, and their stability is investigated. Unsteady (transient) processes are studied qualitatively.

In the investigations of transient laser processes the presence of only one cavity mode has usually been assumed.[1-4] This assumption is usually not justified for optical systems where the cavity dimensions greatly exceed the wavelength of the light; for real laser processes it is more appropriate to consider several (at least two) modes. Of greatest importance are the nonlinear interaction (competition) of modes during transient processes and the conditions for the simultaneous generation of different modes. Only Haken and Sauermann,[5] apparently, have considered nonmonochromatic processes in a multimode laser.1) However, they were concerned exclusively with steady-state oscillations, mainly when the oscillation threshold was only slightly exceeded. (In one instance their evaluation of stability is incorrect.)

It was the aim of the present work to obtain a sufficiently complete picture of both steady-state and transient processes in our two-mode laser model. Our results are also applicable qualitatively to multimode lasers in general.

1. FUNDAMENTAL EQUATIONS

The equations of a two-level laser have been studied very thoroughly. In the case of two nonvanishing modes these equations are[3,7]

$$\ddot{q}_\lambda + \frac{2}{\tau_\lambda}\dot{q}_\lambda + \omega_\lambda{}^2 q_\lambda = -\mathbf{H}_\lambda\ddot{\mathbf{M}}, \qquad \lambda = 1, 2;$$

$$\dot{N} + \frac{N - N_0}{T_1} = \frac{2i}{\hbar}\sum_{\lambda=1,2} q_\lambda\{\rho_{12}(\mathbf{H}_\lambda\boldsymbol{\mu}_{21}) - \rho_{21}(\mathbf{H}_\lambda\boldsymbol{\mu}_{12})\};$$

$$\dot{\rho}_{12} - i\omega_0\rho_{12} + \frac{\rho_{12}}{T_2} = \frac{i}{\hbar}\sum_{\lambda=1,2} q_\lambda\{N(\mathbf{H}_\lambda\boldsymbol{\mu}_{12}) + \rho_{12}(\mathbf{H}_\lambda, \boldsymbol{\mu}_{11} - \boldsymbol{\mu}_{22})\};$$

$$\mathbf{M} = \mathrm{Sp}(\hat{\boldsymbol{\mu}}\hat{\rho}) = \boldsymbol{\mu}_{11}\rho_{11} + \boldsymbol{\mu}_{22}\rho_{22} + \boldsymbol{\mu}_{12}\rho_{21} + \boldsymbol{\mu}_{21}\rho_{12}. \qquad (1)$$

Here M is the magnetic or electric moment of the active molecules per unit volume, $\hat{\mu}$ is the matrix of the molecular dipole moment ($\mu_{12} = \mu_{21}^*$), $\hat{\rho}$ is the molecular density matrix ($\rho_{12} = \rho_{21}^*$), $N = \rho_{22} - \rho_{11}$ is the population difference of the energy levels, N_0 is a parameter depending on temperature and the pumping field, $\hbar\omega_0$ is the energy level separation, ω_λ are the frequencies of normal cavity modes in the absence of an active material, T_1 and T_2 are the relaxation times of the material, and τ_λ is the damping time of a cavity mode. The magnetic or electric field strength H is represented by

$$\mathbf{H} = \sum_{\lambda=1,2} q_\lambda(t)\mathbf{H}_\lambda(\mathbf{r}),$$

where $\mathbf{H}_\lambda$ are cavity resonator eigenfunctions normalized so that

$$\int_V H_\lambda{}^2 dv = 4\pi V'$$

(here V′ is the volume of active material in the cavity and V is the cavity volume).

As in the majority of investigations it is here assumed for simplicity that the field amplitude of each mode in V′ is independent of the coordinates.2) In all real cases the nonlinear and relaxation terms in (1) are small, so that an averaging method can be applied, and the solution can be sought in the form

$$q_\lambda = Q_\lambda(t)e^{i\omega t} + Q_\lambda{}^*(t)e^{-i\omega t}, \quad \rho_{12} = \sigma(t)e^{i\omega t},$$

where ω is a constant frequency close to ω_0 and ω_λ (the differences between these two frequencies in the interesting cases are always small compared

1)Steady-state monochromatic oscillations in a two-mode maser were considered by Lugovoĭ in[6].

2)In[7] Yakubovich and the present author obtained equations similar to (2) for an arbitrary spatial field distribution without an expansion in terms of the modes.

with the frequencies themselves); Q_λ, σ, and N vary slowly compared with $e^{i\omega t}$.

Averaging over the period $2\pi/\omega$ in (1), we obtain a system of "abbreviated" equations of lower order than the original system:

$$\dot{Q}_\lambda + [\tau_\lambda^{-1} + i(\omega - \omega_\lambda)]\, Q_\lambda = -{}^1/_2\, i\omega\gamma_\lambda^* \sigma,$$

$$\dot{\sigma} + [T_2^{-1} + i(\omega - \omega_0)]\, \sigma = \frac{i}{\hbar} N \sum_{\lambda=1,2} \gamma_\lambda Q_\lambda,$$

$$\dot{N} + T_1^{-1}(N - N_0) = \frac{2i}{\hbar} \sum_{\lambda=1,2} (\sigma\gamma_\lambda^* Q_\lambda^* - \sigma^*\gamma_\lambda Q_\lambda). \quad (2)$$

We have here used the notation $\gamma_\lambda = H_\lambda \mu_{12}$.

Monochromatic laser oscillations correspond to the equilibrium of the system (2). Dropping all time derivatives, we can determine the possible frequencies ω of these oscillations and their respective amplitudes. It is an extremely complicated problem to derive any additional information from (2), e.g., to determine the steady state that is actually established for any given initial conditions, or to elucidate the possibility of establishing a nonmonochromatic periodic process. Thus, to arrive at a periodic two-frequency solution we must determine the limit cycle in a phase space of the system (2) with seven dimensions in the general case (because of the complexity of the equations). To be sure, the order of the system (2) can be lowered further due to the fact that the relaxation times in a laser satisfy the inequality

$$T_2 \ll \tau_\lambda \ll T_1. \quad (3)$$

(In practice, for a solid state laser $T_2 \sim 10^{-10}$–10^{-11} sec, $\tau_\lambda \sim 10^{-7}$–$10^{-8}$ sec, and $T_1 \sim 10^{-2}$–10^{-3} sec.) The smallness of T_2 enables us to distinguish in (2) the equations of "fast" (with the time constant T_2) and "slow" (with the time constant T_1) processes. It is easily shown that in a time of the order T_2 the fast processes bring the system into the region of slow movements.[6] For the latter (considered below) the derivative of $\dot{\sigma}$ can be neglected and σ can be expressed directly in terms of N and Q_λ. However, even after this has been done, there does not appear to be any fairly simple way of investigating unsteady oscillations.

Of more importance for the present problem is the assumption, also usually satisfied for lasers,[3)]

$$|\omega_1 - \omega_2| \gg \tau_\lambda^{-1}, \quad (4)$$

i.e., there is no overlapping of the frequency intervals in which the resonant excitations of different modes are possible.[4)] Each mode has its own frequency, and the characteristic time of transient processes is large compared with $1/(\omega_1 - \omega_2)$ (except, perhaps, for a process starting at a very high pumping level).

The foregoing permits a second averaging procedure, now in the system (2) itself, over a period corresponding to the frequency difference of the modes. Taking into account the fact that these frequencies ω_1' and ω_2' differ somewhat from ω_1 and ω_2 because of the active medium, and using the notation $\Delta_1(t) = \omega_1' - \omega_0$, $\Delta_2(t) = \omega_2' - \omega_0$, we shall solve (2) in the form[5)]

$$Q_\lambda = v_\lambda(t) \exp\left(i \int_0^t \Delta_\lambda dt\right), \qquad \sigma = \sum_{\lambda=1,2} \sigma_\lambda(t) \exp\left(i \int_0^t \Delta_\lambda dt\right). \quad (5)$$

The functions v_λ and σ_λ are real except for constant phase factors $\exp(i\varphi_\lambda)$ and vary slowly compared with

$$\exp\left[i \int_0^t (\Delta_1 - \Delta_2)\, dt\right].$$

The form of the function N(t) leads to a difficulty in averaging. According to (5) and the last equation in (2), we have

$$N = \tilde{N}(t) + \sum_{l=1}^{\infty} \{N_l(t)\, e^{il\delta t} + N_l^*(t)\, e^{-il\delta t}\}, \quad (5a)$$

where $\delta = \Delta_1 - \Delta_2$ (for definiteness we shall assume $\Delta_1 > 0$, $\delta > 0$); $\tilde{N}$ and N_l are slowly varying functions, and in the general case all terms of the series (5a) are comparable to $\tilde{N}$, so that they would be determined by solving an infinite system of equations. Then σ includes components of different combination frequencies; although the equations for the fields contain only the terms of (5) with the frequencies Δ_λ, the values of these terms depend on other components and all the quantities N_l. However, upon fulfillment of a condition to be introduced subsequently, each term of (5a) is smaller than the preceding term and we shall hereafter retain only the terms containing $\tilde{N}$ and N_1. Then in view of (3) and (4) ($\dot{N}_1 \ll \delta N_1$) we obtain

---

[3)]We note that (4) and subsequently (7) are also required for the correctness of the results given in[5], although this is not stated by Haken and Sauermann.

[4)]Thus, if in a plane cavity losses depend mainly on the mirror transmittance D, then also for neighboring longitudinal modes we have

$$|\omega_1 - \omega_2| / \tau_\lambda \sim (2\pi/D) \sim 10^2.$$

[5)]Since ω in (2) is arbitrary (the amplitudes are complex), we let $\omega = \omega_0$. It is possible, of course, to average directly over the period $2\pi/(\omega_1' - \omega_2')$ in (1).

from (2)

$$N_1 = \tilde{N}\frac{2iT_2(s_1+s_2^*)\gamma_1 v_1\gamma_2^* v_2^*/\hbar^2\delta}{1-2iT_2(s_2^*|\gamma_1 v_1|^2+s_1|\gamma_2 v_2|^2)/\hbar^2\delta};$$

$$s_\lambda = (1+iT_2\Delta_\lambda)^{-1}. \tag{6}$$

This shows clearly the condition for the smallness of the ratio $N_1/\tilde{N}$; it is easily shown that this condition also ensures the smallness of all the ratios N_{l+1}/N_l, i.e., the correctness of (6).

Substituting the values of γ_λ, we write the foregoing condition in the instructive form

$$W_\lambda W_{\lambda'} \ll (\hbar\delta)(\hbar T_2^{-1}), \quad \lambda, \lambda' = 1, 2, \tag{7}$$

where W_λ is the energy of the interaction between an active molecule and the field of the corresponding mode. For the longitudinal modes of a solid state laser ($\delta \sim 10^9$–10^{10} sec^{-1}, $H \lesssim 10^2$ cgs emu) the ratio of the left- and right-hand sides of (7) is of the order 10^{-2}–10^{-3}.

Substituting (5) and (6) into (2), averaging, and using (3), (4), and (7), we obtain the following equations for v_λ and $\tilde{N}$:

$$\dot{v}_1 = v_1\left\{\frac{\omega T_2 s_1|\gamma_1|^2}{2\hbar}\tilde{N}\left[1+\frac{2iT_2}{\hbar^2\delta}(s_1+s_2^*)|\gamma_2 v_2|^2\right] - [\tau_1^{-1}+i(\Delta_1-\Delta_{10})]\right\},$$

$$\dot{v}_2 = v_2\left\{\frac{\omega T_2 s_2|\gamma_2|^2}{2\hbar}\tilde{N}\left[1-\frac{2iT_2}{\hbar^2\delta}(s_1^*+s_2)|\gamma_1 v_1|^2\right] - [\tau_2^{-1}+i(\Delta_2-\Delta_{20})]\right\},$$

$$\dot{\tilde{N}} = T_1^{-1}N_0 - \tilde{N}\left\{T_1^{-1}+\frac{4T_2}{\hbar^2}\left[\operatorname{Re}s_1|\gamma_1 v_1|^2+\operatorname{Re}s_2|\gamma_2 v_2|^2 + \frac{4T_2}{\hbar^2\delta}(\operatorname{Re}s_1+\operatorname{Re}s_2)(\operatorname{Im}s_2-\operatorname{Im}s_1)|\gamma_1 v_1\gamma_2 v_2|^2\right]\right\}, \tag{8}$$

where $\Delta_{\lambda_0} = \omega_\lambda - \omega_0$. The terms in (8) that contain δ explicitly result from fluctuations of the population difference and are small when (7) is fulfilled.

We note again that in the cases of practical interest we have $T_2^2\Delta_2^2 \ll 1$, i.e., the mode frequencies are close to the center of the emission line of the active material; then the s_λ are close to unity. This circumstance is unimportant for our subsequent discussion, but we shall employ it to simplify our notation: in the terms containing δ we assume

$$s_\lambda = 1 - iT_2\Delta_\lambda,$$

and in the remaining terms,

$$s_\lambda = 1 - T_2^2\Delta_\lambda^2 - iT_2\Delta_\lambda.$$

It is also convenient to use dimensionless terms in (8):

$$m_\lambda = 4T_1T_2|\gamma_\lambda v_\lambda|^2/\hbar^2, \quad n = \tilde{N}/N_0,$$

$$\alpha_\lambda = \omega_0 T_2|\gamma_\lambda|^2 N_0\tau_\lambda/2\hbar. \tag{9}$$

After separating out the real parts of (8), we obtain a system of equations for m_λ and n:

$$\tau_1\dot{m}_1 = 2m_1\{\alpha_1(1-T_2^2\Delta_1^2)n \times [1+T_2(\delta+2\Delta_1)m_2/2\delta T_1] - 1\},$$

$$\tau_2\dot{m}_2 = 2m_2\{\alpha_2(1-T_2^2\Delta_2^2)n \times [1+T_2(\delta-2\Delta_2)m_2/2\delta T_1] - 1\},$$

$$T_1\dot{n} = 1 - n\{1+m_1(1-T_2^2\Delta_1^2) + m_2(1-T_2^2\Delta_2^2) + 2T_2m_1m_2/T_1\}. \tag{10}$$

The imaginary parts of (8) give expressions for the frequency of each mode:

$$\Delta_1 = \frac{\Delta_{10}+2\alpha_1 nm_2/\delta T_1\tau_1}{1+2\alpha_1 T_2 n/\tau_1}, \quad \Delta_2 = \frac{\Delta_{20}-2\alpha_2 nm_1/\delta T_1\tau_2}{1+2\alpha_2 T_2 n/\tau_2}. \tag{11}$$

Consequently, each frequency depends on the amplitudes of both modes, and the expressions in (11) are, from a rigorous point of view, not explicit, since they contain $\delta = \Delta_1 - \Delta_2$. It will appear subsequently, to be sure, that the difference $\Delta_\lambda - \Delta_{\lambda_0}$ is usually small, but we cannot yet assume this. We note that even if $\Delta_{10} = \Delta_{20}$, i.e., degenerate modes are here interacting, this degeneracy is lifted ($\delta \neq 0$) because of the nonlinearity. In principle δ can become sufficiently large to satisfy (7); in this case (10) remains correct.

It is now our problem to investigate the third-order system (10) for the real functions m_λ and n, taking (11) into account. The advantage of (10) over (2) is clear from the fact that a two-frequency periodic state, if it occurs, does not correspond to a complicated limit cycle in (10), but, as in the single-frequency cases, to an equilibrium condition where both m_1 and m_2 are nonvanishing.

2. ANALYSIS OF MODE INTERACTION

We shall begin our investigation of laser processes by determining the equilibrium states of (10). It is easily seen that there are in the general case four such states, for which we give the respective values of m_λ and n (dropping small terms wherever they are not essential):

I. $m_1 = m_2 = 0, \quad n = 1;$

II. $m_1 = \alpha_1 - (1+T_2^2\Delta_1^2), \quad m_2 = 0,$

$n = (1+T_2^2\Delta_1^2)/\alpha_1;$

III. $m_1 = 0, \quad m_2 = \alpha_2 - (1+T_2^2\Delta_2^2),$

$n = (1+T_2^2\Delta_2^2)/\alpha_2;$

$$\text{IV. } 4m_1 = (\alpha - 1)(1 + 2\Delta_1/\delta) + (2T_1/T_2)[(\alpha_1/\alpha_2 - 1) + T_2^2(\Delta_2^2 - \Delta_1^2)],$$

$$4m_2 = (\alpha - 1)(1 - 2\Delta_2/\delta) - (2T_1/T_2) \times [(\alpha_1/\alpha_2 - 1) + T_2^2(\Delta_2^2 - \Delta_1^2)],$$

$$n \approx 1/\alpha_1 \approx 1/\alpha_2 = 1/\alpha. \tag{12}$$

A laser is self-excited when the "zero" equilibrium state I is unstable. The single-mode states II and III correspond to monochromatic oscillations in the first and second modes. Under real conditions α_λ is of the order $1-10^2$, with the field amplitude $H \lesssim 10^2$ cgs emu.

Both oscillatory modes are established simultaneously in the two-frequency state IV, which is possible only when the terms in square brackets are sufficiently small. Indeed, these are preceded by the factor $(2T_1/T_2) \sim 10^7-10^8$, and if the respective second terms in the equations of state IV in (12) exceed the first terms, then at least one of the m_λ cannot be positive. Specifically, the parameters α_1 and α_2 must be close to each other; this fact was utilized to obtain state IV in (12) ($\alpha_1 \approx \alpha_2 = \alpha$ can be assumed in the first terms). For example, in the case of symmetric modes ($\Delta_1 = -\Delta_2$) state IV exists upon fulfillment of the condition

$$\alpha - 1 > \frac{T_1}{T_2}\left|\frac{\alpha_1}{\alpha_2} - 1\right|. \tag{13}$$

For longitudinal cavity modes having identical spatial patterns[6] we have $|\alpha_1/\alpha_2 - 1| \sim 10^{-5}-10^{-7}$, and for $\alpha \sim 10^1-10^2$ the two sides of (13) are of comparable magnitude. For unsymmetric modes the difference $T_2^2(\Delta_2^2 - \Delta_1^2) \approx 10^{-4}-10^{-2}$ predominates in the last terms of state IV in (12); then two-mode steady-state oscillations are possible only at very high pumping levels. It is clear, in any event, that the condition for the existence of these oscillations is much more severe than the condition for the self-excitation of the two modes; this is accounted for by the emission from the medium that is stimulated by the field of one mode, thus preventing growth of the other mode.[7]

[6]Because of the difference in τ_λ we have $\alpha_1 \neq \alpha_2$, which results from frequency-dependent losses in the cavity mirrors or from diffractive losses ($|\alpha_1/\alpha_2 - 1| \sim \delta/\omega_0$).

[7]The practical conditions for the existence of a two-mode state appear to be somewhat more favorable than those that follow from IV in (12); this results from inhomogeneity of the active medium and of the field amplitude distribution in this medium.[5, 8] The mutual stimulated emission of the modes is then reduced. The establishment of a two-mode state is also aided by the splitting of a level of the active molecules.[9]

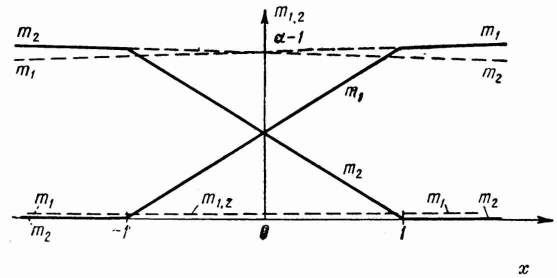

FIG. 1. Dependence of steady-state field amplitudes on the difference of parameter modes. Solid lines – stable states; dashed lines – unstable states.

Figure 1 shows the dependence of steady-state oscillation amplitudes on the parameter

$$x = (\alpha_1/\alpha_2 - 1)T_1/T_2(\alpha - 1)$$

(the "bifurcation diagram") for symmetric modes. We note that for $x = \pm 1$, before equilibrium state IV disappears it coincides with either II or III.

We again consider the steady-state frequencies. The quantity $2\alpha_\lambda T_2 n/\tau_\lambda$ in (11) is usually small ($\sim 10^{-1}-10^{-3}$) even for $n = 1$, i.e., in state I. The value becomes even smaller in states II and III, when

$$n \approx 1/\alpha_\lambda, \quad 2\alpha_\lambda T_2 n/\tau_\lambda \approx T_2/\tau_\lambda \sim 10^{-3} - 10^{-4}.$$

For state IV we must also take into account the terms containing m_λ. However, these are of the order $10^2-10^4\ \text{sec}^{-1}$, and are thus small compared with δ. Therefore (in any event, when investigating steady-state oscillations and their stability) we can usually neglect frequency variations, assuming $\Delta_\lambda \approx \Delta_{\lambda_0}$ (see, also, footnote 8)

We shall now investigate the stability of the derived steady states. Linearizing (10) about an equilibrium state and assuming that the perturbations are proportional to $e^{\beta t}$, we in each case obtain three values of the characteristic root β. For the "zero" state I we have

$$\beta_1(\mathrm{I}) = 2\tau_1^{-1}[\alpha_1(1 + T_2^2\Delta_1^2)^{-1} - 1],$$

$$\beta_2(\mathrm{I}) = 2\tau_2^{-1}[\alpha_2(1 + T_2^2\Delta_2^2)^{-1} - 1], \quad \beta_3(\mathrm{I}) = -T_1^{-1}. \tag{14}$$

The conditions for the self-excitation of each mode follow herefrom:

$$\alpha_\lambda > (1 + T_2^2\Delta_\lambda^2).$$

For the single-frequency state II linearization gives the values

$$\beta_{1,2}(\mathrm{II}) = -\alpha_1/2T_1 \pm [\alpha_1^2/4T_1^2 - 2(\alpha_1 - 1)/T_1\tau_1]^{1/2},$$

$$\beta_3(\mathrm{II}) = \frac{T_2}{T_1\tau_2}\left\{(\alpha_1 - 1)\left(1 - \frac{2\Delta_2}{\delta}\right) - \frac{2T_1}{T_2}\left[\left(\frac{\alpha_1}{\alpha_2} - 1\right) + T_2^2(\Delta_2^2 - \Delta_1^2)\right]\right\}. \tag{15}$$

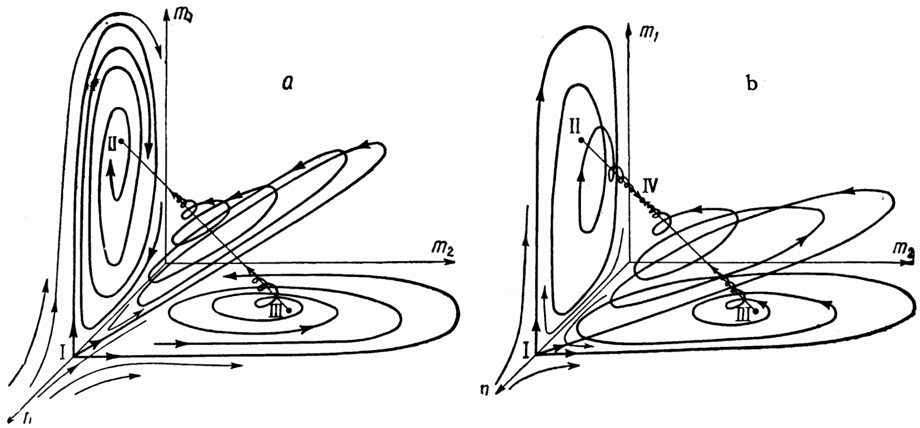

FIG. 2. Character of trajectories in a phase space for a two-mode laser, in the absence (a) and presence (b) of a two-frequency steady state.

It follows herefrom that the roots $\beta_{1,2}$ characterize the stability of a given mode with respect to field perturbations of the same mode (under ordinary conditions $\beta_{1,2}$ are complex, i.e., damped oscillations occur in the plane $m_2 = 0$); β_3 characterizes the stability of the first mode with respect to perturbations of the second mode (this process is aperiodic). If α_1 and α_2 are not very close, then

$$\beta_3 \approx 2\tau_2^{-1}(\alpha_2 - \alpha_1) / \alpha_2,$$

and the stability condition is $\alpha_1 > \alpha_2$. If $\alpha_1/\alpha_2 - 1$ is small, then the expression in curly brackets agrees with IV in (12) for $4m_2$ (to the degree of accuracy with which the expression has been written). Therefore if a two-frequency state exists, a single-frequency state is unstable.

Symmetric expressions are obtained for state III, which is stable only if $m_1 < 0$ in (12) for state IV, i.e., in the absence of a two-frequency state. It is easily seen that m_1 and m_2 cannot be negative at the same time; therefore at least one of the states II and III is unstable.

Finally, for state IV we obtain

$$\beta_{12}(\mathrm{IV}) = -\frac{\alpha}{2T_1} \pm \left[\frac{\alpha^2}{4T_1^2} - \frac{2}{T_1}\left(\frac{m_1}{\tau_1} + \frac{m_2}{\tau_2}\right)\right]^{1/2},$$

$$\beta_3(\mathrm{IV}) = -\frac{4\alpha T_2 m_1 m_2}{T_1(m_1\tau_2 + m_2\tau_1)}, \qquad (16)$$

where m_λ is given by the formulas of (12.IV); therefore a two-frequency state, if it exists, is always stable. For $\tau_1 \approx \tau_2$ the expressions for $\beta_{1,2}$ in (15) and (16) are identical, i.e., perturbations of a certain type in the vicinity of the point for IV also undergo oscillatory damping (Fig. 2b).

Thus, depending on the parameters of the active material and the cavity, there are two possible steady states for a laser. First, when the parameters of the modes are very close both modes exist in the equilibrium state, and the combined amplitude of the two different frequencies oscillates with the frequency δ; single-mode equilibrium states are then unstable.[8)] Second, if the parameters of the modes differ widely, the laser emits monochromatic light at the frequency of the mode for which m_λ of IV in (12) is negative.

It should be noted that when the mode parameters are not very close, if

$$|\alpha_1 / \alpha_2 - 1| \gg (\alpha_\lambda - 1) T_2 / T_1,$$

the small terms in (10) arising from fluctuations of the population difference can be dropped. Then, eliminating n from the first two equations of (10) and integrating, we obtain

$$\frac{m_1}{m_2^{\alpha_1\tau_2/\alpha_2\tau_1}} = \mathrm{const}\cdot\exp\left\{\frac{2t}{T_1}\left[\frac{\alpha_1(1+T_2^2\Delta_2^2)}{\alpha_2(1+T_2^2\Delta_1^2)} - 1\right]\right\}. \qquad (17)$$

(This expression is valid even when the terms $T_2\Delta_\lambda^2$ are not small.) It thus becomes clear that in a time of the order

$$\tau_{\mathrm{tr}} = \tau_1\left[\frac{\alpha_1(1+T_2^2\Delta_2^2)}{\alpha_2(1+T_2^2\Delta_1^2)} - 1\right]^{-1}$$

practically a single-mode state is established in the system. The existence of a steady two-frequency state in our model results from fluctuations of the population difference; this is easily seen from the derivation of IV in (12).

A more detailed quantitative analysis of laser

[8)]When there is a significant explicit dependence of the frequencies in (11) on m_λ, the situation can differ. Indeed, the previously identical conditions for the existence of state IV and the instability of II and III are then different because they contain different frequencies. Then in some interval of the parameters stable steady states may be altogether absent; for bounded oscillations this indicates the existence of periodic (or at least quasi-periodic) undamped fluctuations of the mode amplitudes. In the present case this interval is very narrow, but for close frequencies of the modes this "phase" instability is interesting as one of the possible causes of self-modulated laser emission.

oscillations requires integration of the system (10). A sufficiently complete qualitative picture of the processes for any initial conditions can, however, be obtained by plotting a family of trajectories in the phase space of the system (10), i.e., in the space of the variables m_1, m_2, and n. Figure 2 shows the structure of this phase space in the two basic cases, the existence and absence of a two-frequency steady state. We note that for $m_1 \equiv 0$ or $m_2 \equiv 0$, the equations of (10) describe the previously studied single-mode state, i.e., the pattern of trajectories in the respective planes is known.[9)]

In the phase space the trajectories approach either a single-mode equilibrium state (Fig. 2a) or a two-mode state (Fig. 2b), spiraling down in the case of close modes to a straight line $m_1 + m_2 = \alpha - 1$. (In the limit when $\alpha_1 = \alpha_2$, $\tau_1 = \tau_2$, and the δ terms are unimportant, this entire straight line would be the geometric locus of equilibrium positions.) The period of the oscillations τ_{osc} (the duration of a "spike") and their damping time τ_d are estimated from the values of $\beta_{1,2}$ in (15) and (16): $\tau_{osc} \sim 10^{-6}$—10^{-7} sec and $\tau_d \sim 10^{-4}$ sec. The time required for the system to reach an equilibrium state corresponds either to τ_2 or to the time τ_T during which the representative point moves along the straight line $m_1 + m_2 = \alpha - 1$. The latter time is determined by the value of β_3, or from (17) for modes that are not too close. Usually $\tau_T \sim 10^{-3}$—10^{-6} sec, while for "symmetric" modes its value is $\sim 10^{-2}$ sec. Since most frequently $\tau_T > \tau_d$, the representative point, while remaining in the plane $m_1/m_2 = \text{const}$, first spirals into the aforementioned straight line and then moves along the latter to the equilibrium state. The trajectory pattern in this plane qualitatively resembles that for single-mode processes in the planes $m_1 = 0$ and $m_2 = 0$. Consequently, each "spike" contains the fields of both modes, even if a single-mode state is subsequently established.[10)] The opposite case $\tau_T < \tau_d$ is also possible, of course; practically a single mode then remains even before the transient process is terminated.

The foregoing analysis shows the essential influence of fluctuating processes on the relationship of the mode amplitudes and therefore on the frequency spectrum of the generated oscillations.[11)] Since at the start of a transient oscillatory process (Fig. 2) the phase trajectories pass close to the zero-equilibrium point I,[4] small fluctuations of the field can throw the representative point from one trajectory to another with an entirely different relation between m_1 and m_2. On the other hand, unavoidable small variations of the system parameters α_λ and τ_λ, such as can result from variations of temperature or pumping power, can before termination of the transient process qualitatively alter the structure of the phase space (by the ap-appearance or disappearance of a two-frequency state), or produce instability of a steady mode accompanied by the growth of the other mode etc. As a result the power and frequency spectrum of laser emission can vary randomly. These are probably among the causes of irregularity in the emission from most solid state lasers.

If more than two modes participate in the interaction the laser problem becomes correspondingly more complicated. However, as already mentioned, the foregoing results would still be applicable qualitatively. Specifically, for modes of not very close frequencies (when the population fluctuations at all difference frequencies can be neglected), Eq. (17) is valid for any pair of modes. It follows directly that for an arbitrary number of modes in the present model monochromatic oscillation is established at the frequency of the mode corresponding to the largest value of $\alpha_\lambda/(1 + T_2^2\Delta_\lambda^2)$.

The author wishes to thank A. V. Gaponov and G. I. Freĭdman for discussions of the results.

9)By virtue of $T_1 \gg \tau_\lambda$ [see Eq. (3)] these trajectories can be divided into "fast" and "slow" regions.[4]

10)The corresponding field amplitude fluctuations with a period δ within a spike have been observed experimentally.[10, 11]

11)Fluctuations in a single-mode model have been considered by Bespalov and Gaponov.[4]

[1] H. Statz and G. de Mars in Quantum Electronics, edited by C. H. Townes, Columbia Univ. Press, New York, 1960.

[2] D. M. Sinnett, J. Appl. Phys. **33**, 1578 (1962).

[3] V. N. Genkin and Ya. I. Khanin, Izv. VUZov, Radiofizika **5**, 423 (1962).

[4] V. I. Bespalov and A. V. Gaponov, Izv. VUZov, Radiofizika (in press).

[5] H. Haken and H. Sauermann, Z. Physik **173**, 261 (1963).

[6] V. N. Lugovoĭ, Radiotekhnika i élektronika (Radio Engineering and Electronics) **6**, 1701 (1961).

[7] L. A. Ostrovskiĭ and E. I. Yakubovich, JETP **46**, 963 (1964), Soviet Phys. JETP **19**, 656 (1964).

[8] T. I. Kuznetsova and S. G. Rautian, FTT **5**, 2105 (1963), Soviet Phys. Solid State **5**, 1535 (1964).

[9] G. I. Freĭdman, Izv. VUZov, Radiofizika (in press).

[10] A. J. de Maria and R. Gagosz, Appl. Optics **2**, 807 (1963).

[11] V. V. Korobkin and A. M. Leontovich, JETP **44**, 1847 (1963), Soviet Phys. JETP **17**, 1242 (1963).

Translated by I. Emin
159