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RELATIVISTIC TWO-DIMENSIONAL MODEL OF A SELF-INTERACTING FERMION FIELD WITH NONVANISHING REST MASS

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A relativistic two-dimensional (one spatial and one temporal coordinate) model of a self-interacting fermion field with nonzero rest mass is investigated. The eigenfunctions of the energy operator and the scattering operator are determined in the unphysical space of the pseudoparticles. The s-matrix of elastic scattering of the physical particles is constructed with the aid of an improper canonical transformation.

1. INTRODUCTION

A few years ago, to study the contradictions which arise in relativistic quantum field theory, Thirring[1] proposed and investigated a two-dimensional model (one spatial, one temporal coordinate) of a self-interacting fermion field of zero rest mass. In the present paper we are considering a more general model, differing from Thirring's model in that the fermion field has nonzero rest mass. To investigate our model, we make use of the method essentially proposed by Thirring. This method consists of first introducing the unphysical space of pseudoparticles, connected with the space of the true particles by a canonical transformation. In this space we find the eigenfunctions of the energy operator, and the scattering operator. Then, carrying out a canonical transformation, we obtain the operator for elastic scattering of physical particles. This method seems to us also advantageous in the investigation of more meaningful models of field theory. Each of the two stages into which the problem is broken up, although nontrivial, nevertheless turns out to be much simpler than the initial problem as a whole.

Let us explain the idea of the paper in greater detail. We consider a Hamiltonian $H_u = H_0 + V$:

$$H_0 = -i\int dx\left\{\left[\psi_1^+(x)\frac{\partial\psi_1(x)}{\partial x} - \psi_2^+(x)\frac{\partial\psi_2(x)}{\partial x}\right] + m[\psi_1^+(x)\psi_2(x) - \psi_2^+(x)\psi_1(x)]\right\},$$

$$V = -2g\int dx\,dy\,\psi_1^+(x)\psi_2^+(y)u(x-y)\psi_1(x)\psi_2(y). \quad (1)$$

The operators $\psi_\alpha^+(x)$ and $\psi_\beta(y)$ satisfy the Fermi commutation relations

$$\{\psi_\alpha^+(x), \psi_\beta^+(y)\}_+ = \{\psi_\alpha(x), \psi_\beta(y)\}_+ = 0,$$
$$\{\psi_\alpha^+(x), \psi_\beta(y)\}_+ = \delta_{\alpha\beta}\delta(x-y).$$

The function $u(x)$ is a form factor. The Hamiltonian H, which describes the model in question is obtained from (1) by going to the limit as $U(x) \to \delta(x)$.

The operators $\psi_\alpha^+(x)$ and $\psi_\alpha(x)$ are expressed in terms of the operators of creation and annihilation $a^+(p)$, $b^+(p)$, $a(p)$, and $b(p)$ of the physical particles and antiparticles in accordance with the usual formulas

$$\psi_\alpha(x) \equiv \psi(\xi) = 2\pi^{-1/2}\int dp\, e^{ipx}[v_\alpha(+,p)a(p) + v_\alpha(-,p)b^+(-,p)],$$

$$\psi_\alpha^+(x) \equiv \psi^+(\xi) = 2\pi^{-1/2}\int dp\, e^{-ipx}[a^+(p)\bar{v}_\alpha(+,p) + b(-p)\bar{v}_\alpha(-,p)] \quad (2)$$

($\bar{v}_\alpha$ is the complex conjugate of v_α), in which the continuous spatial variable x and the discrete variable $\alpha = 1, 2$ are denoted for brevity by a single letter ξ, and $v_\alpha(+,p)$ and $v_\alpha(-,p)$ are positive and negative frequency solutions of the Dirac equation, i.e., they satisfy the equations

$$pv_1(\varepsilon,p) - imv_2(\varepsilon,p) = \omega(\varepsilon,p)v_1(\varepsilon,p),$$
$$imv_1(\varepsilon,p) - pv_2(\varepsilon,p) = \omega(\varepsilon,p)v_2(\varepsilon,p);$$
$$\omega(\varepsilon,p) = \varepsilon\sqrt{p^2+m^2} \equiv \varepsilon\omega(p) \quad (3)$$

(ϵ —the "energy-sign" variable, which assumes two values ± 1).

The operators $a^+(p)$, $a(p)$, $b(p)$, and $b^+(p)$ satisfy the same commutation relations as $\psi^+(\xi)$ and $\psi(\xi)$, and therefore (2) is a canonical transformation. This transformation, however, is improper. The latter means that there exists no unitary operator U which transforms a^+, a, b^+, b into ψ^+, ψ. The Hilbert space in which the operators $\psi^+(\xi)$ and $\psi(\xi)$ are creation and annihilation operators will be denoted by $\mathcal{H}_\psi$ and will be called the pseudoparticle space. The space of the physical particles will be denoted by $\mathcal{H}_{a,b}$. In this space, the operators $a^+(p)$, $b^+(p)$, $a(p)$, and $b(p)$ serve as creation and annihilation operators. The space $\mathcal{H}_\psi$ is characterized by the presence of a vector $|0)$ (of the pseudovacuum), satisfying the equation

$$\psi(\xi)|0) = 0. \tag{4}$$

In the space $\mathcal{H}_{a,b}$ there exists a vacuum vector $|0\rangle$ satisfying the conditions

$$a(p)|0\rangle = b(p)|0\rangle = 0. \tag{5}$$

It is important to emphasize that in $\mathcal{H}_\psi$ there exists no vector satisfying the conditions (5), and in $\mathcal{H}_{a,b}$ there exists no vector satisfying Eq. (4). This is connected with the fact that the canonical transformation (2) is improper.[1)]

The operators H and H_0, which are considered in $\mathcal{H}_\psi$, are self-adjoint, and there exists for them a scattering operator S_ψ, defined in the usual fashion. We shall calculate this operator explicitly, and express it as a normal form in ψ^+ and ψ, after which we shall make use of formulas (2) to express it as a normal form in a^+, b^+, a, and b. At this stage there arise infinite terms due to the improper character of the transformation (2). Discarding these, we obtain the true elastic-scattering operator.

We now proceed to execute the outlined plan.

[1)]We present a general definition. The canonical transformation

$$\psi(\xi) = \int d\tau[A(\xi|\tau)\varphi(\tau) + B(\xi|\tau)\varphi^+(\tau)],$$
$$\psi^+(\xi) = \int d\tau[\overline{B(\xi|\tau)}\varphi(\tau) + \overline{A(\xi|\tau)}\varphi^+(\tau)]$$

is called proper, if it has a generating operator U such that

$$\psi = U\varphi U^{-1}, \qquad \psi^+ = U\varphi^+ U^{-1}.$$

In the opposite case it is called improper. It is known[2,3] that in order for the transformation to be proper, it is necessary and sufficient that the following integral converge:

$$\int |B(\xi|\tau)|^2 d\xi d\tau < \infty.$$

In our case

$$\xi = (\alpha, x), \quad \tau = (\varepsilon, p), \quad \varphi(+, p) = a(p), \quad \varphi(-, p) = b(-p);$$
$$B(\alpha, x|+, p) = 0, \qquad B(\alpha, x|-, p) = (2\pi)^{-1/2} e^{ipx} v_\alpha(-, p);$$
$$\int |B(\xi|\tau)|^2 d\xi d\tau = \frac{1}{2\pi}\int |v_1(-, p)|^2 dx dp + \frac{1}{2\pi}\int |v_2(-, p)|^2 dx dp = \infty.$$

2. CONSTRUCTION OF EIGENFUNCTIONS OF THE HAMILTONIAN H IN PSEUDOPARTICLE SPACE

In the pseudoparticle space $\mathcal{H}_\psi$ introduced above, the operators H_u and H commute with the operator of the total number of pseudoparticles N:

$$N = \int dx[\psi_1^+(x)\psi_1(x) + \psi_2^+(x)\psi_2(x)].$$

It is therefore natural to seek the eigenvectors of these operators in the form

$$|F_{EN}) = \frac{1}{\sqrt{N!}}\int d\xi_1 \ldots d\xi_N F_{EN}(\xi_1, \ldots, \xi_N) \times \psi^+(\xi_1), \ldots, \psi^+(\xi_N)|0), \tag{6}$$

where the functions $F_{EN}(\xi_1, \ldots, \xi_N)$ are antisymmetrical. In (6), as everywhere below, the integration with respect to $d\xi$ implies summation over the discrete variables, $|0)$ is the vacuum vector in $\mathcal{H}_\psi$ [i.e., the vector satisfying (4)].

In order that the vector $|F_{EN})$ be an eigenvector for the operator H_u, it is necessary that the functions $F_{EN}(\xi_1, \ldots, \xi_N)$ satisfy the system of equations

$$-i\sum_{k=1}^{N}\Big\{\delta_{1,\alpha_k}\sum_{\beta_k=1}^{2}\Big[\delta_{1,\beta_k}\frac{\partial}{\partial x_k} + \delta_{2,\beta_k}m\Big] - \delta_{2,\alpha_k}\sum_{\beta_k=1}^{2}\Big[\delta_{2,\beta_k}\frac{\partial}{\partial x_k} + \delta_{1,\beta_k}m\Big]\Big\}$$
$$\times F_{EN}(\xi_1; \ldots; \xi_{k-1}; \beta_k, x_k; \xi_{k+1}; \ldots; \xi_N) + 2g\sum_{1\le k<j\le N} u(x_k - x_j)$$
$$\times F_{EN}(\xi_1, \ldots, \xi_N)[\delta_{1,\alpha_k}\delta_{2,\alpha_j} + \delta_{2,\alpha_k}\delta_{1,\alpha_j}] = EF_{EN}(\xi_1, \ldots, \xi_N). \tag{7}$$

Here α_k and $\beta_k = 1, 2$ and $\delta_{\alpha,\beta}$ is the Kronecker symbol.

In the case of interest to us $u(x) \to \delta(x)$, the functions $F_{EN}(\xi_1, \ldots, \xi_N)$ tend to the eigenfunctions $\Phi_{EN}(\xi_1, \ldots, \xi_N)$ of the Hamiltonian H. The functions Φ_{EN} satisfy when $x_1 \ne x_2 \ldots \ne x_N$ the free equations obtained from (7) with $g = 0$, and satisfy when $x_k = x_j$ the boundary conditions

$$\Phi_{EN}(\xi_1; \ldots; \xi_{k-1}; 1, x_k; \xi_{k+1}; \ldots; \xi_{j-1}; 2, x_j; \xi_{j+1}; \ldots; \xi_N)|_{x_k=x_j+0} = e^{ig}\Phi_{EN}(\xi_1; \ldots; \xi_{k-1}; 1, x_k; \xi_{k+1}; \ldots; \xi_{j-1}; 2, x_j; \xi_{j+1}; \ldots; \xi_N)|_{x_k=x_j-0} \tag{7a}$$

(see the Appendix).

We note that the free Hamiltonian has a complete system of eigenfunctions in the form

$$\Phi_{EN}{}^0(\xi_1,\ldots,\xi_N|\tau_1,\ldots,\tau_N)=\sum\pm u(\xi_{i_N}|\tau_1)\ldots u(\xi_{i_N}|\tau_N).$$

$$E=\sum_{k=1}^{N}\varepsilon_k\sqrt{p_k^2+m^2}, \quad (8)$$

where the sum is taken over the permutations of $\xi_1,\ldots,\xi_N$, the sign depends on the parity of the permutation,

$$u(\xi|\tau) = u(\alpha,x|\varepsilon,p) = (2\pi)^{-1/2}e^{ipx}v_\alpha(\varepsilon,p) \quad (9)$$

and $v_\alpha(\epsilon,p)$ is a solution of Eqs. (3) given by the formula

$$v_1(\varepsilon,p)=-i\varepsilon\left[\frac{\omega(p)+\varepsilon p}{2\omega(p)}\right]^{1/2},$$

$$v_2(\varepsilon,p)=\left[\frac{\omega(p)-\varepsilon p}{2\omega(p)}\right]^{1/2}. \quad (10)$$

We denote by $\theta(x_1>x_2>\ldots>x_N)$ a function equal to

$$\theta(x_1>x_2>\ldots>x_N)=\begin{cases}1 & \text{for } x_1\geqslant x_2\geqslant\ldots\geqslant x_N\\ 0 & \text{otherwise}\end{cases}$$

We now consider a function Φ_{EN} in the form

$$\Phi_{EN}=\Phi_{EN}(\xi_1,\ldots,\xi_N|\tau_1,\ldots,\tau_N)$$
$$=\sum\pm\theta(x_{k_1}>x_{k_2}>\ldots>x_{k_N})\sum\pm K(\tau_{j_1},\ldots,\tau_{j_N})$$
$$\times u(\xi_{k_1}|\tau_{j_1})\ldots u(\xi_{k_N}|\tau_{j_N}). \quad (11)$$

The outer sum is taken over permutations of x_k, and the inner over permutations of τ_j. In both cases the sign depends on the parity of the permutation. It is obvious that the functions (11) satisfy equation (7) with $g = 0$ when $x_k \neq x_j$, no matter what the function $K(\tau_1,\ldots,\tau_N)$ may be. In this case

$$E=\sum_{k=1}^{N}\varepsilon_k\sqrt{p_k^2+m^2}.$$

We now put

$$K(\tau_1,\ldots,\tau_N)=\prod_{1\leqslant k<j<N}K(\tau_k|\tau_j), \quad (12)$$

$$K(\tau_1|\tau_2) = e^{ig/2}v_1(\tau_1)v_2(\tau_2) + e^{-ig/2}v_2(\tau_1)v_1(\tau_2). \quad (13)$$

It turns out that with such a choice of the function $K(\tau_1,\ldots,\tau_N)$, the boundary conditions (7a) are satisfied. The corresponding calculations are somewhat cumbersome and will be omitted. Thus, formulas (9)—(13) determine the eigenfunctions of the total Hamiltonian.

We note that the function $K(\tau_1|\tau_2)$ does not depend on N. Below we shall frequently encounter sums over permutations, similar to (8) and (11). In order not to overcomplicate the indices, we shall agree to write such sums in the form $\sum_\xi f(\xi_1\ldots\xi_n|\tau_1\ldots\tau_n)$, and $\sum_\xi \pm f(\xi_1\ldots\xi_n|\tau_1\ldots\tau_n)$ if the sign depends on the parity of the permutation.

Formulas (9) and (13) determine the solution of the free equation with boundary conditions (7a) for arbitrary complex $p_1,\ldots,p_N$. However, the eigenfunctions of the Hamiltonian determine only that solution for which the eigenfunctions are bounded, and the corresponding energies are real. It is obvious that these conditions are satisfied by all real values of the variables $p_1,\ldots,p_N$. On going over to complex values of p_k, it is necessary to stipulate that the coefficients of the growing exponents vanish and that the energy E remain real. The functions obtained as a result of such a continuation into the complex domain describe states with complexes of bound pseudoparticles.

Let us consider by way of an example the third sector. The wave function of three independent pseudoparticles is according to (9)—(12)

$$\Phi_{E3}(\xi_1;\xi_2,\xi_3|\tau_1,\tau_2,\tau_3)=\sum_x\pm\theta(x_1>x_2>x_3)$$
$$\times\sum_\tau\pm K(\tau_1|\tau_2)K(\tau_1|\tau_3)K(\tau_2|\tau_3)v_{\alpha_1}(\tau_1)v_{\alpha_2}(\tau_2)v_{\alpha_3}(\tau_3)$$
$$\times e^{i(p_1x_1+p_2x_2+p_3x_3)} \quad (14)$$

We go over into the complex domain, putting

$$p_1=q_1(2),\quad \operatorname{Im}q_1(2)>0;\quad p_2=q_2(2),$$
$$\operatorname{Im}q_2(2)=-\operatorname{Im}q_1(2);\quad p_3=q_1(1),\quad \operatorname{Im}q_1(1)=0.$$

In the region $x_1 > x_2 > x_3$ the infinite terms in (14) are obtained from the terms that contain the following exponentials:

$$e^{i(p_2x_1+p_1x_2+p_3x_3)},\quad e^{i(p_2x_1+p_3x_2+p_1x_3)},\quad e^{i(p_3x_1+p_2x_2+p_1x_3)}.$$

From (12) and (14) it follows that each of the coefficients in front of the growing exponents will contain $K(\tau_2|\tau_1)$ as a factor. By using the explicit form of this function [see (13)], we find that if we assume p_3 to be an arbitrary real number,

$$p_1=\frac{1}{2}P(2)-i\frac{1}{2}E(2)\operatorname{ctg}\frac{g}{2}\equiv q_1(2),$$

$$p_2=\frac{1}{2}P(2)+i\frac{1}{2}E(2)\operatorname{ctg}\frac{g}{2}\equiv q_2(2),$$

$$E(2)=-\frac{g}{|g|}\left[P^2(2)+4m^2\sin^2\frac{g}{2}\right]^{1/2}, \quad (15)^*$$

then $K(\tau_2|\tau_1)$ vanishes, and we obtain the wave function of two bound and one unbound pseudoparticle[2)], where P(2) is the momentum of a com-

*ctg = cot.

[2)]Owing to the antisymmetry the function $\Phi_{E3}(\xi_1,\xi_2,\xi_3|\tau_1,\tau_2,\tau_3)$ is bounded not only in the region $x_1>x_2>x_3$, but in all other regions of the form $x_i>x_j>x_k$, that is, in all of space.

plex of two pseudoparticles and is an arbitrary real number, while E(2) is its energy.

To conclude this section, we note that the obtained eigenfunctions are very strongly reminiscent of the eigenfunctions of the simpler Hamiltonian

$$H = -\left(\frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}\right) + g \sum_{i<j} \delta(x_i - x_j).$$

This Hamiltonian describes a system of n nonrelativistic one-dimensional particles with zero spin and with point-like interaction. Its eigenfunctions were investigated in detail earlier.[4]

3. EIGENFUNCTIONS OF SCATTERING THEORY

The eigenfunctions of the Hamiltonian H obtained in the preceding section, are closely connected with the eigenfunctions $\Phi_E^{\pm}$ of scattering theory. In the case of scattering of N unbound pseudoparticles $[\tau = (\epsilon, p)]$

$$\Phi_{EN}^{+}(\xi_1, \ldots, \xi_N | \tau_1, \ldots, \tau_N) = \sum_{\tau} \theta(\varepsilon_N p_N > \varepsilon_{N-1} p_{N-1} > \ldots > \varepsilon_1 p_1) \times \frac{\Phi_{EN}(\xi_1, \ldots, \xi_N | \tau_1, \ldots, \tau_N)}{K(\tau_1, \ldots, \tau_N)} \quad (16)$$

(the sum is taken over all the permutations of $\tau_1, \ldots, \tau_N$) Φ_{EN}^{-} differs from Φ_{EN}^{+} in that $\theta(\epsilon_N p_N > \ldots > \epsilon_1 p_1)$ is replaced by $\theta(\epsilon_1 p_1 > \epsilon_2 p_2 > \ldots > \epsilon_N p_N)$.

The scattering functions of n bound complexes of $\nu_1, \ldots, \nu_n$ pseudoparticles are residues at the poles arising in the analytic continuation of the functions (16) in the momenta. For example, the following functions describe the scattering of one pseudoparticle by a complex of two others, having positive energy

$$\Phi_{E3}^{+}(\xi_1, \xi_2, \xi_3 | P(2), \tau) = \theta(P(2) > \varepsilon p) \frac{\Phi_{E3}(\xi_1, \xi_2, \xi_3 | \varepsilon_1, q_1(2); \varepsilon_2, q_2(2); \tau)}{K(\tau | \varepsilon_1, q_1(2)) K(\tau | \varepsilon_2, q_2(2))} + \theta(\varepsilon p > P(2)) \frac{\Phi_{E3}(\xi_1, \xi_2, \xi_3 | \varepsilon_1, q_1(2); \varepsilon_2, q_2(2); \tau)}{K(\varepsilon_1, q_1(2) | \tau) K(\varepsilon_2, q_2(2) | \tau)},$$

where P(2) is an arbitrary real number and $q_1(2)$ and $q_2(2)$ are complex numbers, determined by formula (15) (we assume that $g < 0$).

A check on (16) is in the general case cumbersome. We therefore confine ourselves to an outline of the proof for elastic scattering at N = 3. We denote the eigenfunctions of scattering by $\Phi_{EN}^{\pm}$ and show that $\Phi_{EN}^{\pm} = \tilde{\Phi}_{EN}^{\pm}$. We recall that the functions $\tilde{\Phi}_{EN}^{\pm}$ are determined by means of a transition to the limit

$$\tilde{\Phi}_{EN}^{\pm} = \lim_{\sigma \to +0} \{\pm i\sigma [E - H \pm i\sigma]^{-1}\} \Phi_{EN}^{0},$$

where H is the total Hamiltonian, and Φ_{EN^0} is the eigenfunction of the free Hamiltonian. In our case Φ_{E3^0} is given by the formula [see (8) and (9)]

$$\Phi_{E'3}^{0}(\xi_1, \xi_2, \xi_3 | \tau_1', \tau_2', \tau_3') = \sum_{\xi} \pm v_{\alpha_1}(\tau_1') v_{\alpha_2}(\tau_2') v_{\alpha_3}(\tau_3') e^{i(p_1' x_1 + p_2' x_2 + p_3' x_3)},$$

$$E' = \omega(\tau_1') + \omega(\tau_2') + \omega(\tau_3') \equiv E(\tau_1', \tau_2', \tau_3').$$

Expanding $\Phi_{E'3}{}^{0}(\xi_1, \xi_2, \xi_3 | \tau_1', \tau_2', \tau_3')$ in terms of functions $\Phi_{E3^+}(\xi_1, \xi_2, \xi_3 | \tau_1, \tau_2, \tau_3)$, we obtain[3] for $\epsilon_3 p_3 > \epsilon_2 p_2 > \epsilon_1 p_2$

$$\tilde{\Phi}_{E'3}{}^{+}(\xi_1, \xi_2, \xi_3 | \tau_1', \tau_2', \tau_3') = \int d\tau_1 d\tau_2 d\tau_3 \Phi_{E3}{}^{+}(\xi_1, \xi_2, \xi_3 | \tau_1, \tau_2, \tau_3) \times \lim_{\sigma \to +0} \frac{i\sigma A(\tau_1, \tau_2, \tau_3 | \tau_1', \tau_2', \tau_3')}{E(\tau_1', \tau_2', \tau_3') - E(\tau_1, \tau_2, \tau_3) + i\sigma},$$

where

$$A(\tau_1, \tau_2, \tau_3 | \tau_1', \tau_2', \tau_3') = \int d\xi_1 d\xi_2 d\xi_3 \overline{\Phi_{E3}{}^{+}}(\xi_1, \xi_2, \xi_3 | \tau_1, \tau_2, \tau_3) \times \Phi_{E'3}{}^{0}(\xi_1, \xi_2, \xi_3 | \tau_1', \tau_2', \tau_3') = \sum \pm \frac{K(\tau_{j_1}, \tau_{j_2}, \tau_{j_3})}{K(\tau_1, \tau_2, \tau_3)} \times \sum \pm \sum_{\alpha_1 \alpha_2 \alpha_3} \prod_{t=1} \bar{v}_{\alpha_t}(\tau_{j_t}) v_{\alpha_t}(\tau_{m_t}) \times \frac{\delta(p_1' + p_2' + p_3' - p_1 - p_2 - p_3)}{[p_{m_1}' - p_{j_1} + i0][p_{m_3}' - p_{j_3} - i0]}.$$

The outer sum extends over the permutation of the indices j_k and m_s.

In order to verify that the functions $\tilde{\Phi}_{E3^+}$ and Φ_{E3^+} coincide, we must check the equality

$$\lim_{\sigma \to +0} i\sigma \frac{A(\tau_1, \tau_2, \tau_3 | \tau_1', \tau_2', \tau_3')}{[E(\tau_1', \tau_2', \tau_3') - E(\tau_1, \tau_2, \tau_3) + i\sigma]} = \sum_{\tau'} \pm \delta(\tau_1 - \tau_1') \delta(\tau_2 - \tau_2') \delta(\tau_3 - \tau_3').$$

The required relation follows directly from the fact that the functions $v_\alpha(\tau)$ are orthonormal

$$\sum_{\alpha=1}^{2} \bar{v}_\alpha(\varepsilon, p) v_\alpha(\varepsilon', p) = \delta_{\varepsilon\varepsilon'}$$

and from the following general relation

[3] A complete autonormal system is made up of the functions

$$\Phi_{E3}{}^{+}(\xi_1, \xi_2, \xi_3 | \tau_1, \tau_2, \tau_3), \quad \Phi_{E3}{}^{+}(\xi_1, \xi_2, \xi_3 | p_1 + p_2, \tau), \quad \Phi_{E3}{}^{+}(\xi_1, \xi_2, \xi_3 | p_1 + p_2 + p_3),$$

but the contribution to the expression for Φ_{E3+} will be given only by the retained terms of the expansion.

$$\lim_{\delta\to+0} \frac{i\sigma\delta(k_1+k_2+k_3-p_1-p_2-p_3)}{[f(k_1)+f(k_2)+f(k_3)-f(p_1)-f(p_2)-f(p_3)+i\sigma]}$$
$$\times \frac{1}{[k_{i_1}-p_{j_1}+i0][k_{i_3}-p_{j_3}-i0]}$$
$$=(2\pi)^2\theta(f'(p_{j_3})>f'(p_{j_2}))\theta(f'(p_{j_2})>f'(p_{j_1}))$$
$$\times\delta(k_{i_1}-p_{j_1})\delta(k_{i_2}-p_{j_2})\delta(k_{i_3}-p_{j_3}). \quad (17)$$

Here $f'(p) = df(p)/dp$.

We present a proof of (17) for $i_1 = j_1 = 1$, $i_2 = j_2 = 2$, and $i_3 = j_3 = 3$. We first make use of the fact that

$$\delta(k_1+k_2+k_3-p_1-p_2-p_3)=\frac{1}{2\pi i}$$
$$\times\left(\frac{1}{k_1+k_2+k_3-p_1-p_2-p_3-i0}\right.$$
$$\left.-\frac{1}{k_1+k_2+k_3-p_1-p_2-p_3+i0}\right).$$

Using this equation, we represent the right side of (17) in the form of two terms. We transform the first term, using the partial fraction expansion:

$$i\sigma/\{2\pi i[k_1+k_2+k_3-p_1-p_2-p_3-i0][k_1-p_1+i0][k_2-p_2-i0]\ [f(k_1)+f(k_2)+f(k_3)-f(p_1)-f(p_2)-f(p_3)+i\sigma]\}$$
$$=(i\sigma/\{2\pi i[(k_1-p_1)^2R_1+(k_2-p_2)^2R_2+(k_3-p_3)^2R_3+i\sigma]\})$$
$$\times\left(\frac{1}{k_2-p_2-i0}-\frac{f'(p_2)-f'(p_3)}{f(k_2)-f(p_2)-f'(p_3)(k_2-p_2)+(k_1-p_1)^2R_1+(k_3-p_3)^2R_3+i\sigma}\right)$$
$$\times\left(\frac{1}{k_1-p_1+i0}-\frac{f'(p_1)-f'(p_3)}{f(k_1)+f(k_2)-f(p_1)-f(p_2)-f'(p_3)(k_1+k_2-p_1-p_2)+(k_3-p_3)^2R_3+i\sigma}\right)$$
$$\times\left(\frac{1}{k_1+k_2+k_3-p_1-p_2-p_3-i0}-\frac{f'(p_3)}{f(k_1)+f(k_2)+f(k_3)-f(p_1)-f(p_2)-f(p_3)+i\sigma}\right)$$

R_i denotes here the expression

$$R_i=[f(k_i)-f(p_i)-f'(p_i)(k_i-p_i)]\,/\,(k_i-p_i)^2. \quad (18)$$

We note that the first factor in the limit as $\sigma\to 0$ differs from 0 only when

$$(k_1-p_1)^2R_1+(k_2-p_2)^2R_2+(k_3-p_3)^2R_3=0. \quad (19)$$

Let us find the limit of the second factor as $\sigma\to 0$. We note that

$$f(k_2)=f(p_2)+f'(p_2)(k_2-p_2)+(k_2-p_2)^2R_2.$$

Taking this identity into account, we transform the denominator of the second term:

$$f(k_2)-f(p_2)-f'(p_3)(k_2-p_2)+(k_1-p_1)^2R_1$$
$$+(k_3-p_3)^2R_3+i\sigma=(k_2-p_2)(f'(p_2)-f'(p_3))$$
$$+(k_1-p_1)^2R_1+(k_2-p_2)^2R_2+(k_3-p_3)^2R_3+i\sigma.$$

From this, taking (19) into account, we obtain the limit of the second factor as $\sigma\to 0$. This limit is equal to

$$2\pi i\delta(k_2-p_2)\theta(f'(p_2)>f'(p_3)). \quad (20)$$

Taking (19) and (20) into account, we obtain in analogous fashion the limit of the third factor. This limit is equal to

$$-2\pi i\delta(k_1-p_1)\theta(f'(p_3)>f'(p_1)). \quad (21)$$

Taking (19)–(21) into account, we find the limit of the last factor:

$$2\pi i\delta(k_3-p_3)\theta(f'(p_3)>0). \quad (22)$$

Thus, the limit of the first term as $\sigma\to 0$ is equal to

$$(2\pi)^2\theta(f'(p_3)>0)\theta(f'(p_2)>f'(p_3)$$
$$>f'(p_1))\delta(k_1-p_1)\delta(k_2-p_2)\delta(k_3-p_3). \quad (23)$$

In analogous fashion we find that the limit of the second term as $\sigma\to 0$ is equal to

$$(2\pi)^2\theta(0>f'(p_3))\theta(f'(p_2)>f'(p_3)>f'(p_1))\delta(k_1-p_1)$$
$$\times\delta(k_2-p_2)\delta(k_3-p_3). \quad (24)$$

Adding (23) and (24), we obtain the right side of (17).

4. SCATTERING AMPLITUDE FOR PSEUDOPARTICLES AND THEIR COMPLEXES

We proceed to construct the scattering amplitude

$$\int d\xi_1\ldots d\xi_N\overline{\Phi_{EN}{}^+}(\xi_1,\ldots,\xi_N|\tau_1,\ldots,\tau_N)$$
$$\times\ \Phi_{EN}{}^-(\xi_1,\ldots,\xi_N|\tau_1',\ldots\ \tau_M')$$
$$=(\tau_1,\ldots,\tau_N|S|\tau_1',\ldots,\tau_M'). \quad (25)$$

We consider the scattering of unbound pseudoparticles, that is, the so called elastic channel. In this case it follows from (16) that

$$(\tau_1,\ldots,\tau_N|S|\tau_1',\ldots,\tau_M')=\delta_{NM}S_N(\tau_1,\ldots,\tau_N)$$

$$\times \sum_{\tau'} \pm \delta(\tau_1 - \tau_1') \ldots \delta(\tau_N - \tau_N'), \tag{26a}$$

$$S_N(\tau_1, \ldots, \tau_N) = \sum_{\tau} \theta(\varepsilon_1 p_1 > \ldots > \varepsilon_N p_N) \times K(\tau_N, \ldots, \tau_1)/K(\tau_1, \ldots, \tau_N). \tag{26b}$$

Using the explicit expression (12) for $K(\tau_1, \ldots \tau_N)$ and taking into account (13) and (10), we obtain a final expression for the S-matrix of elastic scattering of N independent pseudoparticles by one another

$$S_N(\tau_1, \ldots, \tau_N) = \prod_{1 \leqslant m < n \leqslant N} S_2(\tau_m, \tau_n), \tag{26c}$$

where*

$$S_2(\tau_1, \tau_2) = \theta(\varepsilon_1 p_1 > \varepsilon_2 p_2) \frac{K(\tau_2|\tau_1)}{K(\tau_1|\tau_2)} + \theta(\varepsilon_2 p_2 > \varepsilon_1, p_1) \frac{K(\tau_1|\tau_2)}{K(\tau_2|\tau_1)}$$

$$= \theta(\varepsilon_1 p_1 > \varepsilon_2 p_2) \left[1 - i \operatorname{tg} \frac{g}{2} \frac{p_1 - p_2}{\omega(\tau_1) + \omega(\tau_2)}\right] \times \left[1 + i \operatorname{tg} \frac{g}{2} \frac{p_1 - p_2}{\omega(\tau_1) + \omega(\tau_2)}\right]^{-1} + \theta(\varepsilon_2, p_2 > \varepsilon_1 p_1)$$

$$\times \left[1 + i \operatorname{tg} \frac{g}{2} \frac{p_1 - p_2}{\omega(\tau_1) + \omega(\tau_2)}\right] \times \left[1 - i \operatorname{tg} \frac{g}{2} \frac{p_1 - p_2}{\omega(\tau_1) + \omega(\tau_2)}\right]^{-1} \tag{26d}$$

is the matrix element for the scattering of two pseudoparticles: $\omega(\tau) = \omega(\epsilon, p) = \epsilon(p^2 + m^2)^{1/2}$.

With the aid of (26) it is easy to prove the unitarity relation

$$\int (\tau_1', \ldots, \tau_N'|S|\tau_1'', \ldots, \tau_N'') \times (\tau_1', \ldots, \tau_N'|S|\tau_1, \ldots, \tau_N) d\tau_1' \ldots d\tau_N' = 1,$$

which denotes the absence of transitions between states with and without bound complexes during scattering. This circumstance enables us to consider the elastic channel separately from the remaining channels, a fact of which we shall make use later.

In inelastic channels, which include bound complexes, the scattering amplitude can be obtained by substituting in (25) the corresponding scattering functions. It turns out here that the number of complexes of a given type is conserved during scattering, and the corresponding matrix elements of the scattering operator can be obtained as residues in an appropriate analytic continuation of the matrix elements (26). For example, in the third sector scattering of two bound pseudoparticles and one unbound one is described by the following amplitude ($g < 0$):

$$S_3(P(2), \tau) = S_3(\varepsilon_1, q_1(2);\ \varepsilon_2, q_2(2); \tau)$$

$$= \theta(P(2) > \varepsilon p) \frac{K(\tau|\varepsilon_1, q_1(2)) K(\tau|\varepsilon_2, q_2(2))}{K(\varepsilon_1, q_1(2)|\tau) K(\varepsilon_2, q_2(2)|\tau)} + \theta(\varepsilon p > P(2)) \frac{K(\varepsilon_1, q_1(2)|\tau) K(\varepsilon_2, q_2(2)|\tau)}{K(\tau|\varepsilon_1, q_1(2)) K(\tau|\varepsilon_2, q_2(2))}.$$

5. SCATTERING MATRIX OF PHYSICAL PARTICLES

In the preceding section we found in the unphysical space $\mathcal{H}_\psi$ all the matrix elements of the scattering operator S_ψ. We now carry out a transition to the space of the physical states $\mathcal{H}_{a,b}$, constructing with the aid of the improper canonical transformation (2) the elastic scattering operator for true particles. This transition will be realized in two stages: we first express the elastic part of S_ψ in terms of the creation and annihilation operators of pseudoparticles $\varphi^+(\tau)$ and $\varphi(\tau)$:

$$\varphi(\tau) = \frac{1}{\sqrt{2\pi}} \int dx \sum_{\alpha=1}^{2} e^{-ipx} \bar{v}_\alpha(\tau) \psi_\alpha(x).$$

We then carry out transformation (2) on the obtained operator; this transformation is written in terms of the operators $\varphi^+(\tau)$ and $\varphi(\tau)$ in the form

$$\varphi(+, p) = a(p), \quad \varphi(-, p) = b^+(-p);$$
$$\varphi^+(+, p) = a^+(p), \quad \varphi^+(-, p) = b(-p). \tag{2a}$$

In order to express S_ψ in terms of the creation and annihilation operators, we write down the operators with the aid of functionals. Let A be an arbitrary operator in normal form

$$A = \sum \int K_{mn}(\tau_1, \ldots, \tau_m|\sigma_1, \ldots, \sigma_n) \varphi^+(\tau_1) \ldots \varphi^+(\tau_m) \varphi(\sigma_1) \ldots \varphi(\sigma_n) d\tau\, d\sigma,$$

where φ^+ and φ are the creation and annihilation operators satisfying the Fermi commutation relations. We set in correspondence with this operator the functional

$$B(\alpha^*, \alpha) = \sum \int K_{mn}(\tau_1, \ldots, \tau_m|\sigma_1, \ldots, \sigma_n) \alpha^*(\tau_1) \ldots \alpha^*(\tau_m) \alpha(\sigma_1) \ldots \alpha(\sigma_n) d\tau d\sigma, \tag{27}$$

where α^* and α are functions with anticommuting values:

$$\{\alpha^*(\tau), \alpha^*(\sigma)\}_+ = \{\alpha^*(\tau), \alpha(\sigma)\}_+ = \{\alpha(\tau), \alpha(\sigma)\}_+ = 0.$$

It is obvious that knowing the functional $B(\alpha^*, \alpha)$ we can reconstitute the operator A.

*tg = tan.

Further, let $(\tau_1, \ldots, \tau_m \mid A_{mn} \mid \sigma_1, \ldots, \sigma_n)$ be the matrix elements of the same operator A in the Fock representation: if

$$\begin{pmatrix}\Phi_0 \\ \Phi_1(\sigma_1) \\ \cdots\end{pmatrix} \text{and} \begin{pmatrix}\Psi_0 \\ \Psi'(\sigma_1) \\ \cdots\end{pmatrix}$$

are Fock columns corresponding to the vectors Φ and $\Psi = A\Phi$ respectively, then

$$\Psi_m(\tau_1, \ldots, \tau_m) = \sum_n \int (\tau_1, \ldots, \tau_m | A_{mn} | \sigma_1, \ldots, \sigma_n) \Phi_n(\sigma_1, \ldots, \sigma_n) d\sigma.$$

We set in correspondence with the matrix $\| A_{mn} \|$ the functional

$$\tilde{B}(\alpha^*, \alpha) = \sum \frac{1}{\sqrt{m!n!}} \int (\tau_1, \ldots, \tau_m | A_{mn} | \sigma_1, \ldots, \sigma_n) \times \alpha^*(\tau_1) \ldots \alpha^*(\tau_m)\, \alpha(\sigma_1) \ldots \alpha(\sigma_n) d\tau d\sigma. \quad (28)$$

It turns out that a simple connection exists between the functionals $\tilde{B}(\alpha^*, \alpha)$ and $B(\alpha^*, \alpha)$ [5]:

$$B(\alpha^*, \alpha) = \tilde{B}(\alpha^*, \alpha) \exp\left(-\int \alpha^*(\tau) \alpha(\tau) d\tau\right). \quad (29)$$

Formula (29) enables us, if we know the matrix elements of the operator A to find its normal form and, conversely, knowing the normal form it enables us to find the matrix elements. We apply formula (29) to our case. Formulas (26a)—(26d) determine the matrix elements of the elastic scattering operator. Starting from these formulas, we obtain the matrix elements of the logarithm of this operator:

$$(\tau_1, \ldots, \tau_N | \ln S_{NM} | \tau_1', \ldots, \tau_M') = \delta_{NM} \sum_{i<j} \ln S_2(\tau_i, \tau_j) \delta(\tau_1 - \tau_1') \ldots \delta(\tau_N - \tau_N').$$

Going over to functionals, we find, using the anticommutativity of $\alpha^*(\tau)$ and $\alpha(\tau)$, the functional $\tilde{B}$ corresponding to the matrix of the logarithm of S_ψ:

$$\tilde{B} = \frac{1}{2} \int \ln S_2(\tau_1, \tau_2) \alpha^*(\tau_1) \alpha^*(\tau_2) \alpha(\tau_1) \alpha(\tau_2) d\tau_1 d\tau_2 \times \exp\left\{\int \alpha^*(\tau) \alpha(\tau) d\tau\right\}$$

Using formula (29), we find that the functional B corresponding to the normal form of the logarithm S_ψ is equal to

$$B = \frac{1}{2} \int \ln S_2(\tau_1, \tau_2) \alpha^*(\tau_1) \alpha^*(\tau_2) \alpha(\tau_1) \alpha(\tau_2) d\tau_1 d\tau_2.$$

Consequently

$$S_\psi = \exp\left[\frac{1}{2} \int \ln S_2(\tau_1, \tau_2) \varphi^+(\tau_1) \varphi^+(\tau_2) \varphi(\tau_1) \varphi(\tau_2) d\tau_1 d\tau_2\right]. \quad (30)$$

Substituting (2a) in (30), we now carry out a canonical transformation. The argument of the exponential, obtained as a result of such a substitution, we reduce to a form that is normal with respect to the operators a^+, b^+, a, and b. This gives rise to several infinite terms of the type const $\cdot\, \delta(0)$. Discarding these terms, we obtain the operator for the scattering of physical particles. In order to write it in compact form, let us consider in place of the operators $a(p)$ and $b(p)$ the operators $c(\tau)$ where, as usual, $\tau = (\epsilon, p)$,

$$c(+, p) = a(p), \quad c(-, p) = b(p).$$

The scattering operator is written in terms of $c(\tau)$ in the form

$$S = \exp\left[\frac{1}{2} \int c^+(\tau_1) c^+(\tau_2) \mathscr{L}(\tau_1, \tau_2) c(\tau_1) c(\tau_2) d\tau_1 d\tau_2\right];$$

$$\mathscr{L}(\varepsilon_1, p_1; \varepsilon_2, p_2) = \varepsilon_1 \varepsilon_2 \ln\left\{\left[1 - i \operatorname{tg} \frac{g}{2} \left(\frac{|p_1 - p_2|}{\omega(p_1) + \omega(p_2)}\right)^{\varepsilon_1 \varepsilon_2}\right] \times \left[1 + i \operatorname{tg} \frac{g}{2} \left(\frac{|p_1 - p_2|}{\omega(p_1) + \omega(p_2)}\right)^{\varepsilon_1 \varepsilon_2}\right]^{-1}\right\}, \quad (31)$$

where the variable $\epsilon_i = +, -$ now has the meaning of the sign of the charge, which distinguishes the particle from the antiparticle.

Let us show that the operator given by formula (31) is the sum of the usual series of perturbation theory. In fact, we introduce temporarily momentum cut-off, and consider the ordinary perturbation theory series for the scattering operator:

$$S = \sum i^n \int\int_{-\infty < t_1 < \ldots < t_n < \infty} :V(t_1)::V(t_2): \ldots :V(t_n): dt_1 \ldots dt_n,$$

$$:V(t): = e^{it:H_0:} :V: e^{-it:H_0:}, \quad (32)$$

where :V: and $:H_0:$ are the operators obtained from the interaction and free-field operators [see (1)] by transposing a^+, b^+, a, b into normal order, and discarding the infinite terms.

It is obvious that the same result can be obtained by replacing in the series (32) :V: by V, and $:H_0:$ by H_0, and then reducing the entire series (32) to a normal form in a^+, b^+, a, b and discarding the infinite terms. This is precisely the method used here: the series obtained from (32) by replacing :V: and $:H_0:$ by V and H_0 is the perturbation-theory series for the S-matrix in pseudoparticle space.

The infinities that have been discarded on going from V and H to :V: and $:H_0:$ have the specific form $c(p)\,\delta(0)$, where the function $c(p)$ (p is the momentum) can depend also on the cut-off parameter. It is important that no terms occur that are finite for a finite cut-off parameter and tend to infinity with the latter. Because of this, the connection between the S-matrix in pseudoparticle space and the true S-matrix is retained when the cut-off parameter goes to infinity. Thus, the absence of infinities in formula (31) offers evidence that all the infinite Feynman diagrams in our model cancel out and there are no infinite renormalizations.

We note in conclusion that, as can be seen from (31), the operator for the scattering of physical particles has the same structure as the operator for pseudoparticle scattering[4)]. Therefore, reversing the arguments that lead to (30), we obtain, starting from (31), the matrix elements of the true scattering operator.

6. CONCLUSION

The method used in this paper for solving field-theoretical problems has two attractive features. First, it enables us to find the exact elastic-scattering S-matrix, the derivation of which with the aid of perturbation theory, even in the simplest model considered here, entails extraordinary computational difficulties. Second, in the investigation of our model this method discloses clearly the difficulties in principle, the resolution of which will apparently contribute to the understanding of more realistic models of quantum-field theory. Let us stop to discuss them in greater detail.

1. The theory is based on the Hamiltonian $H_{a,b}$ obtained from the Hamiltonian H by means of the canonical transformation (2) with subsequent reduction to normal form and discarding of the infinities. The Hamiltonian H, like the free Hamiltonian H_0, is a self-adjoint operator in the space of the pseudoparticles $\mathcal{H}_\psi$. Because of this, H has a complete system of eigenvectors[5)] and there exists a scattering operator which can be determined by any of the universally known methods, which in this case are equivalent. We made use of the fact that

$$(i|S|f) = (\Phi^-(i), \Phi^+(f)), \tag{33}$$

where $\Phi^-(i)$ and $\Phi^+(f)$ are the scattering-theory eigenfunctions of the operator H. We could use with equal success the formula

$$S = U_+U_-^*, \quad U_+ = \lim_{t\to+\infty} e^{itH_0}e^{-itH}, \quad U_- = \lim_{t\to-\infty} e^{itH_0}e^{-itH} \tag{34}$$

or the perturbation theory series, or the formula

$$\Psi_{out} = S\Psi_{in}S^{-1}, \tag{35}$$

where Ψ_{in} and Ψ_{out} are the solutions of the Heisenberg equations with known asymptotic conditions as $t \to \mp\infty$. (The latter method determines the operator S accurate to an arbitrary phase factor.) If we were able to carry out the calculations based on any of these methods, we would obtain an answer coinciding with ours.

The situation is different with the Hamiltonian $H_{a,b}$ in the space of the physical particles. $H_{a,b}$ is not an operator in the Hilbert space $\mathcal{H}_{a,b}$. This means that there is no vector $\Phi \neq 0$ in the state space such that $H_{a,b}\Phi$ is also an element of Hilbert space[6)]. Since $H_{a,b}$ is not an operator, it is meaningless to speak of its spectrum and eigenvectors (even generalized). Therefore a determination of the S-matrix based on formula (33) is meaningless. Further, there exists no operator e^{itH}, so that it is meaningless to have a determination based on formula (34).

We are left with perturbation theory and with formula (35). Our paper can be regarded as a summation of a perturbation theory series, carried out by a roundabout way. It would be of interest to investigate formula (35). It is shown in the paper by one of the authors[7] that in the case when m = 0 the operators Ψ_{in} and Ψ_{out} exist, and this formula can be used in spite of the fact that $H_{a,b}$ has in this case, like in the case when $m \neq 0$, no operator meaning.

2. Closely related to questions touched upon in the preceding item is also the question of inelastic scattering. In the space $\mathcal{H}_\psi$ we can give two independent determinations of the inelastic scattering amplitude, one is formula (19) and the other by means of an analytic continuation of the elastic part of the S-matrix. In the space $\mathcal{H}_{a,b}$ we have constructed the operator of elastic scattering of physical particles, and we can construct by means of an analytic continuation the amplitude of inelastic scattering. Now, however, we have no other independent determination, since $H_{a,b}$ has no discrete-spectrum eigenfunctions even after the total momentum is separated.

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4)In particular, there are no transitions between states with different numbers of particles. The latter circumstance is a consequence of the fact that the scattering operator in pseudoparticle space is a multiplication operator in the p-representation [see (26)].

5)More accurately—generalized eigenvectors, since the operator H has a continuous spectrum.

6)This circumstance is the basis of the known Haag theorem.[6]

APPENDIX

DERIVATION OF BOUNDARY CONDITIONS

The Hamiltonian H in pseudoparticle space, corresponding to the case of a pointlike interaction, was defined in the introduction as the limit of the operator H_u as $u(x) \to \delta(x)$. We must first refine the meaning of $\lim H_u$ as $u(x) \to \delta(x)$. To this end we specify in explicit form the limiting operator H, i.e., we first describe the set of functions forming its domain of definition, and then indicate the formula in accordance with which it acts.

We denote by $H_u^{(N)}$ the value of the operator H_u in the subspace N of pseudoparticles, and by $H^{(N)}$ the value of H in this subspace.

1. Let D_u the domain of definition of the operator $H_u^{(N)}$, $F_u \in D_u$, and let there exist the limits

$$\Phi = \lim_{u\to\delta} F_u \in L_2, \qquad \lim_{u\to\delta} H_u F_u \in L_2.$$

The domain of definition of the operator $H^{(N)}$ is made up of the functions

$$\Phi = \lim_{u\to\delta} F_u.$$

2. The formula according to which H operates consists in the following:

$$H^{(N)}\Phi = \lim_{u\to\delta} H_u{}^{(N)} F_u.$$

We denote by $L_2^{(N)}$ the space of all the functions (not necessarily skew-symmetrical) with summable square. The operator $H_u^{(N)}$ extends in natural fashion over all of the space $L_2^{(N)}$ 7). It is obvious that when $x_1 \neq x_2 \neq \ldots \neq x_N$ the operator $H^{(N)}$ acts like the free operator $H_0^{(N)}$. We now consider the function $F_{N,u}(\xi_1, \ldots, \xi_N)$, which differs from zero only in the vicinity of $x_i = x_j$. $H_u^{(N)}$ acts on such a function like an operator with separable variables:

$$H_u{}^{(N)} F_{Nu}(\xi_1, \ldots, \xi_N) = H_0{}^{(N)} F_{Nu}(\xi_1, \ldots, \xi_N) + 2gu(x_i - x_j) \times F_{Nu}(\xi_1, \ldots, \xi_N)[\delta_{1\alpha_i}\delta_{2\alpha_j} + \delta_{2\alpha_i}\delta_{1\alpha_j}],$$

and therefore $F_{Nu}(\xi_1, \ldots, \xi_N)$ can be represented

$$F_{Nu}(\xi_1, \ldots, \xi_N) = \int d\tau_1 \ldots d\tau_N F_{2u}(\xi_i, \xi_j | \tau_i, \tau_j) u(\xi_1|\tau_1) \ldots u(\xi_{i-1}|\tau_{i-1}) u(\xi_{i+1}|\tau_{i+1}) \ldots u(\xi_{j-1}|\tau_{j-1}) u(\xi_{j+1}|\tau_{j+1}) \ldots u(\xi_N|\tau_N) A(\tau_1 \ldots \tau_N), \tag{36}$$

where $F_{2u}(\xi_i, \xi_j | \tau_i, \tau_j)$ are the eigenfunctions of the operator H_u in $L_2^{(2)}$.

We choose the form factor $u(x)$ in the form of a rectangular well. In this case we can calculate explicitly the eigenfunctions of the operator $H_u^{(2)}$ and their limits as $u(x) \to \delta(x)$, which are the eigenfunctions of $H^{(2)}$. From the explicit form of these functions it follows that they satisfy the boundary condition (7a). Taking this into account, we obtain, starting from (36) that the function $\Phi = \lim F_{Nu}$ as $u \to \delta$ satisfies the necessary boundary conditions. It follows from this in obvious fashion that the eigenfunctions of $H^{(N)}$ satisfy the same boundary conditions (7a).

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7)The operator $H_u^{(N)}$ is extended over $L_2^{(N)}$ with the aid of the left-hand side of (7).

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