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TYPES OF SINGULARITIES OF FEYNMAN AMPLITUDES

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We present a consistent analysis of the conditions under which the Feynman diagrams acquire singularities of various types—Landau and non-Landau (or singularities of the second type)—on the basis of an algebraic investigation of the extremal equations for the denominator of the corresponding Feynman integral. A complete transition from the internal 4-momenta q to loop 4-momenta k and external 4-momenta p is performed in the set of Landau equations, and an analysis is made of the changes in the structure of the Feynman diagrams in p, k notation (unlike the structure in q-notation). We consider a graphic procedure for the transformation of the p, k Feynman diagrams, which would establish the nature of the "contraction" of the diagrams corresponding to the investigated singularities of the second type. A parallel investigation is made of ordinary "contractions" of diagrams with account of their influence on the form of the surface of the singularities as compared with the corresponding surface of the main diagram. In conclusion, the general analysis of the types of singularities of Feynman amplitudes is illustrated by means of a concrete example of a 4-point diagram with a loop.

1. INTRODUCTION

THE analytic properties of the Feynman integral

$$\int \prod_i d^4k_i \prod_j \frac{d\alpha_j}{f^n} \delta(\Sigma\alpha_j - 1), \tag{1}$$

$$f = \sum_j \alpha_j (m_j^2 - q_j^2 - i\varepsilon), \tag{2}$$

where q_j are the internal and k_i the loop 4-momenta of the corresponding Feynman diagram with n internal lines characterized by masses m_j (the index j numbers the internal lines of the diagram and i its loops) are investigated, according to Landau [1], by analyzing the extremal properties of the quadratic form (2) with respect to the loop 4-momenta k_i, from which follow the equations [linear in q_j (and in a_j)]

$$\sum_{\langle i \rangle} \alpha_j q_j = 0 \tag{3}$$

for each contour k_i (the sum over $\langle i \rangle$ is a sum over the contours), and with respect to the parameters α_j, from which one establishes the fact that the 4-momenta q_j are located on the mass shell

$$q_j^2 = m_j^2. \tag{4}$$

The latter, in particular, indicates that the solutions of (3) depend on the masses m_j, and makes it possible by the same token to use the method of vector dual diagrams (see [1,2]) to find these solutions. The system of equations (3)—(4) is a necessary (but not sufficient) condition for the existence of singularities of the integral (1); the aggregate of the solutions of this system determines the position of all the possible singularities of the integral (1) and forms the surface of Landau singularities in the space of the invariant variables of this integral.

Recently, Cutkosky [3], Fairlie et al.[4], Fowler [5], Drummond [6], and the author [7] pointed out those singularities of the integral (1) which correspond to the solutions of (3) but are independent of the masses m_j of the internal lines. These singularities were called singularities of the second type (or non-Landau singularities). They appeared upon imposition of certain formal conditions that exclude the masses m_j from the extremal equations [4,5] (see also [7]). As was stated first by Fairlie et al.[4], singularities of the second type correspond to solutions of (3) which cannot be represented by dual diagrams, and in this sense cannot be obtained on the basis of the usual method of dual diagrams in accordance with (4). Subsequently, however, the same authors have shown [8] that in the analysis of singularities of the second type the condition (4) is replaced by a condition of the type

$$q_j^2 = m_j^2 + \lambda\varphi_j(\alpha), \tag{5}$$

where λ is a Lagrange multiplier and $\varphi_j(\alpha)$ is a known function of the parameters α. Because of this it is possible in principle, by suitable choice of λ, to construct dual diagrams also in the case of singularities of the second type (see [7]).

Nonetheless, the question of the existence of various possible types of singularities of Feynman amplitudes (see, for example, [9,10]), and in particular the regular appearance of singularities of the second type, still remains open. We therefore present in the present article a consistent analysis of the conditions for the occurrence of singularities of various types—Landau and non-Landau—for the integral (1) on the basis of an algebraic investigation of the extrema of the form (2) transformed to loop 4-momenta k_i and external 4-momenta p_j. Corresponding to the transformation $q \to p,k$ is the transition from ordinary Feynmann q-diagrams to p,k-diagrams, the specific properties of which make it possible to interpret graphically the character of the singularities of various types.

2. GENERAL METHOD

We consider the Feynman diagram corresponding to integral (1), with N external 4-momenta p_j, n internal 4-momenta q_j, and l loops with contour 4-momenta k_i, chosen from among the q_j in such a manner that the remaining $n - l$ internal 4-momenta can be uniquely expressed in terms of p and k. We number in succession the internal 4-momenta

$$q_1, q_2, \ldots, q_n (\equiv q_j;\ j = 1, 2, \ldots, n) \tag{6}$$

and separate the loop 4-momenta k_i:

$$q_1, q_2, \ldots, q_{n-l} (\equiv q_{j'};\ j' = 1, 2, \ldots, n-l), \tag{7$'$}$$

$$q_{n-l+1} \equiv k_1, \quad q_{n-l+2} \equiv k_2, \ldots q_n \equiv k_l (\equiv q_{j''} \equiv k_i;\ j'' = n - l + i, i = 1, 2, \ldots, l). \tag{7$''$}$$

If the total number of vertices of the diagram is denoted by v and accordingly the total number of internal vertices (not containing the external 4-momenta p) is denoted by $v - N$, then the $n - l$ non-loop 4-momenta $q_{j'}$ can be uniquely expressed in terms of the loop 4-momenta $q_{j''} \equiv k_i$ and the external 4-momenta p_j

$$q_{j'} = (\Sigma k)_{j'} + (\Sigma p)_{j'},\ j' = 1, 2, \ldots, n-l, \tag{8}$$

by solving with respect to q'_j the following system of v vector equations

$$\sum_{[j]} q = p_j, \quad j = 1, 2, \ldots, N,$$

$$\sum_{(j)} q = 0, \quad j = 1, 2, \ldots, (v - N). \tag{9}$$

The system (4) expresses the laws of conservation of 4-momenta in the external (sum over [j]) and internal (sum over (j)) vertices of the diagram; actually the system (9) contains only $v - 1$ independent equations, since one of the equations corresponds to the conservation law for the external 4-momenta p_j:

$$\sum_{j=1}^{N} p_j = 0. \tag{10}$$

We note here that if the external vertex contains several external 4-momenta $p_j^{(1)}, p_j^{(2)}, \ldots$ (j is the index of the vertex), then the vector equations (9) contain only the summary 4-momentum

$$p_j = p_j^{(1)} + p_j^{(2)} + \ldots,$$

as a result of which the number of external 4-momenta is always equal to the number of external vertices N.

The uniqueness of the solutions (8) of the system of equations (9) with respect to the $n - l$ non-loop 4-momenta $q_{j'}$ (7$'$) follows from the fact that the rank of the matrix of the coefficients of (9)—the incidence matrix I (see [11])—can be shown (see for example,[12]), to be equal precisely to

$$n - l = n - (n - v + 1) = v - 1.$$

This result is obvious in the case of single-loop diagrams ($l = 1$, $n = v$).

The positions of the singularities of the integral (1), characterizing the contribution of the Feynman diagram in question, is determined on the basis of the Landau equations (3) for the internal 4-momenta (loop and non-loop), written for all the contours of the diagram. Substituting in (3) the expressions for q_j from (7$'$), (7$''$), and (8), we obtain a system of equations of the form

$$\sum_{\langle i \rangle} \left[\alpha_{j'} \Big(\sum k \Big)_{j'} + \alpha_{j''} k_{j''} \right] = - \sum_{\langle i \rangle} \alpha_{j'} \Big(\sum p \Big)_{j'},$$

$$i = 1, 2, \ldots, l, \tag{11}$$

or, regrouping the terms in the left side of (11), we arrive at a system of inhomogeneous equations for the l loop 4-momenta k

$$\sum_{\langle i \rangle} \beta_{ij} k_j = - \sum_{\langle i \rangle} \alpha_{j'} \Big(\sum p \Big)_{j'}, \quad i = 1, 2, \ldots, l. \tag{12}$$

Here β_{ij} are linear combinations of α in the form of arithmetic sums of the α-parameters taken: either 1) along the i-th independent contour (coefficients β_{ii}, $j = i$), or 2) along the lines of tangency of the i-th and j-th independent contours (coefficients β_{ij}, $i \neq j$).

It is important to note that the determinant $\Delta(\beta)$ of the system (12) coincides with the determinant

$$C(\alpha) = \det(\beta_{ij}) \qquad (13)$$

of the $l \times l$ matrix (β_{ij}) consisting of the coefficients of the terms that are quadratic in k in the form (2) reduced to the momenta p and k [with the aid of (7′), (7″), and (8)]. This very same determinant is contained in the integral (1) written in the Chisholm-Nambu representation [13,7] (this representation is obtained by integrating in (1) with respect to k). The denominator of the Chisholm-Nambu representation can be obtained by direct substitution—without integrating with respect to k—of the solutions of (12) [$k_i^0(\alpha, p)$, $i = 1, 2, \ldots, l$], in the quadratic form (2):

$$f[k^0(\alpha, p)] = D(\alpha, p) / C(\alpha). \qquad (14)$$

The value thus obtained for the quadratic form at the extremal point (14) coincides, as is well known, with its free term in the canonical form.

Starting with the denominator of the integrand of (14), we can construct a system of extremal equations

$$\alpha_i \frac{\partial D(\alpha, p)}{\partial \alpha_i} = 0, \quad i = 1, 2, \ldots, n, \qquad (15)$$

which replace the Landau equations (3)—(4), but it is much more difficult to solve this system of equations algebraically.

In explicit form, the solutions $k_i^0(\alpha, p)$, $i = 1, 2, \ldots, l$ of the system (12) have the form of linear combinations of the external 4-momenta p_j:

$$k_i^0(\alpha, p) = \sum_j \gamma_{ij}(\alpha) p_j, \qquad (16)$$

where the coefficients $\gamma_{ij}(\alpha)$ are homogeneous functions of the α-parameters of zero order. Substitution of (16) in the Landau equations (4) leads, with allowance for (7′), (7″), and (8), to a system of n equations

$$(\Sigma k^0(\alpha, p))_{j'}^2 + 2(\Sigma p)_{j'}(\Sigma k^0(\alpha, p))_{j'} + (\Sigma p)_{j'}^2 = m_{j'}^2,$$
$$j' = 1, 2, \ldots, (n - l), \qquad (17')$$

$$(k^0_{i=j''-n+l}(\alpha, p))^2 = m_{j''}^2, \; j'' = n - l + i,$$
$$i = 1, 2, \ldots, l. \qquad (17'')$$

The elimination of n parameters α_j ($j = 1, 2, \ldots, n$) from the system of n equations (17′) and (17″) gives the connection between the invariants $(p_j p_i)$ made up of the external 4-momenta p_j and the masses m_j and M_j ($p_j^2 = M_j^2$) of the internal and external lines:

$$\Phi(m, M; (p_j p_i)) = 0. \qquad (18)$$

Equation (18) is the equation for the surface of the Landau singularities in the space of the invariant variables $(p_j p_i)$ for the amplitude of the investigated Feynman diagram [or the corresponding integral (1)]. On this surface are located all the singularities of the diagram in question. A method for explicitly determining the surfaces of the singularities (18) for arbitrary Feynman diagrams will be considered in the author's next paper.

3. p, k DIAGRAMS

The transition made above with the aid of (7′), (7″), and (8) from the q-formalism to the p, k-formalism is perfectly valid, if we do not use for the solution of the system of Landau equations (3)—(4) the graphic method of dual diagrams, in which the condition (4) for the 4-momenta q_j to be located on the mass shell is regarded as ancillary with respect to the linear-geometrical—conditions (3). After carrying out such a transition, equations (3) and (4) form together with (9) a single system of equations that are linear and quadratic in the 4-momenta. To solve this system, as shown above, it is necessary to perform the following operations: 1) express in a form such as (8) the non-loop 4-momenta $q_{j'}$ ($j' = 1, 2, \ldots, (n - l)$) in terms of loop 4-momenta k_i and the external 4-momenta p_j, after solving the system of linear equations (9), which is characterized by the determinant of an incidence matrix I of rank $n - l$, and go over from the q-equations (3) to the k-equations (12), introducing at the same time the coefficients β (the arithmetic sums of the parameters α); 2) express in a form such as (16) the loop 4-momenta k_i^0 ($i = 1, 2, \ldots, l$) in terms of the external 4-momenta p_j with the aid of the system of linear equations (12), characterized by the determinant $\Delta(\beta) \equiv C(\infty)$ of the matrix of β-coefficients of rank $\leq l$, and set up the coefficients γ (homogeneous combinations of the coefficients β or of the parameters α); 3) eliminate the coefficients γ with the aid of the system of quadratic equations (4), and also a series of supplementary conditions that establish the presence of a connection between some of the γ-coefficients (the method will be investigated in greater detail in the next paper).

Operation 1) effects in the system of Landau equation (3)—(4) a complete transition from the internal 4-momenta q_j to the loop 4-momenta k_i and the external 4-momenta p_j, and in this connection it should lead to a change in the structure of the Feynman diagram, in which the internal q_j lines are divided with the aid of (7′), (7″), and (8) into sets of $(\Sigma k)_j$ and $(\Sigma p)_j$ lines, characterized by the same parameter α_j. This structure-complicating graphical procedure does not change the number of contours or the degree of connectivity of the diagram, but causes the vertices of the diagram to become multiple for individual contours, that is, the lines of the contour pass through these vertices more than once. It is necessary to set in correspondence with the new structurally more complicated diagrams not the Landau contour equations in the form (3) but the same contour equations in the form (12), which contain only the momenta k and p.

Let us consider the procedure for the transition $q \rightarrow k$, using as an example the ordinary triangular diagram on Figure 1a. On going over to the momenta k and p the contour Landau equation is transformed in the following manner ($q_1 \equiv k$):

$$\alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 q_3 = 0 \rightarrow C(\alpha)k + \alpha_2 p_1 - \alpha_3 p_3 = 0, \quad (19)$$

where $C(\alpha) = \alpha_1 + \alpha_2 + \alpha_3$. The new equation already corresponds to the diagram of Fig. 1b. The singly-connected contour $(q_1 q_2 q_3)$ of diagram 1a is in Fig. 1b again a singly-connected contour $(\underline{k} p_1 p_3)$, where the bracket under the 4-momentum k denotes the presence of a closed k-contour in the diagram 1b, but all the vertices of 1b become double.

A characteristic feature of the diagrams in terms of the variables k and p and, in particular diagram 1b, is the presence of a "p-core" (solid lines), which is "overgrown" in the diagram by the loop 4-momenta k_i (dashed lines) and can be regarded as a standard "core" for all diagrams having the same number of external 4-momenta p_j. For example, the p-core for the 2-loop 3-point diagram on Fig. 2b will be the same as for the single-loop 3-point diagram in Fig. 1b.

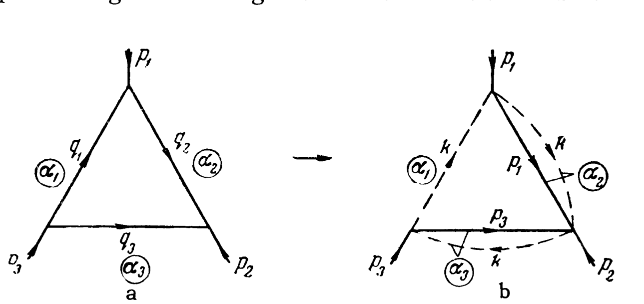


FIG. 1

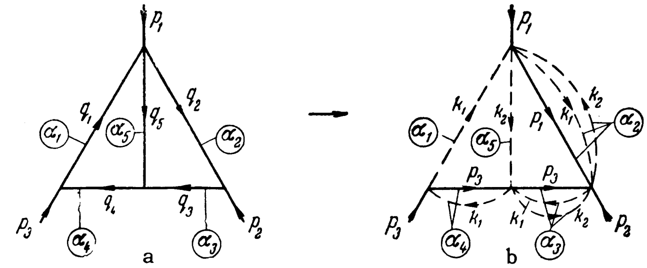


FIG. 2

In the case of the 2-loop diagram of figures 2a and 2b (and any multi-loop diagrams), a new problem arises, viz., operation 2), which is trivial in the case of single-loop diagrams [see (19)]. After transforming to the momenta k and p of the system of Landau contour equations ($q_1 \equiv k_1$, $q_5 \equiv k_2$)

$$\alpha_1 q_1 + \alpha_5 q_5 + \alpha_4 q_4 = 0, \qquad \alpha_2 q_2 - \alpha_5 q_5 + \alpha_3 q_3 = 0, \quad (20a)$$

with determinant $C(\alpha) = -\alpha_5(\alpha_2 + \alpha_3) - (\alpha_1 + \alpha_4)(\alpha_2 + \alpha_3 + \alpha_5)$, it is necessary to solve the obtained system

$$(\alpha_1 + \alpha_4)k_1 + \alpha_5 k_2 - \alpha_4 p_3 = 0,$$
$$(\alpha_2 + \alpha_3)k_1 - (\alpha_2 + \alpha_3 + \alpha_5)k_2 + \alpha_2 p_1 - \alpha_3 p_3 = 0, \quad (20b)$$

with respect to the loop 4-momenta k_1 and k_2. As a result we obtain a system of two contour equations with separated loop 4-momenta k_1 and k_2:

$$-C(\alpha)k_1 + \alpha_2\alpha_5 p_1 - [\alpha_3\alpha_5 + \alpha_4(\alpha_2 + \alpha_3 + \alpha_5)]p_3 = 0,$$
$$-C(\alpha)k_2 - \alpha_2(\alpha_1 + \alpha_4)p_1 + (\alpha_1\alpha_3 - \alpha_2\alpha_4)p_3 = 0, \quad (21)$$

where each of the equations is the analog of a single-contour equation (19) and should correspond to a definite single-loop diagram.

In fact, whereas in the case of a single-loop diagram 1b the k-lines are closed upon passage through the "elongated" loop $(\underline{k}p_1p_3)$, in the case of the two-loop diagram the k_1 and k_2 lines form closed contours only on the complete two-loop diagram of Fig. 2b, but not on its two separate "elongated" loops $(k_1k_1k_2p_3)$ and $(k_1\underline{k_1}k_2p_3p_1)$ on Fig. 3b, made up of loops $(q_1q_5q_4)$ and $(q_2q_5q_3)$ on Fig. 3a of the initial diagram 2a. We note here that, as in the case of diagrams 1a and b, the "elongation" of the loops in the $q \rightarrow k$ transition, $3a \rightarrow 3b$, leads because of the invariant number

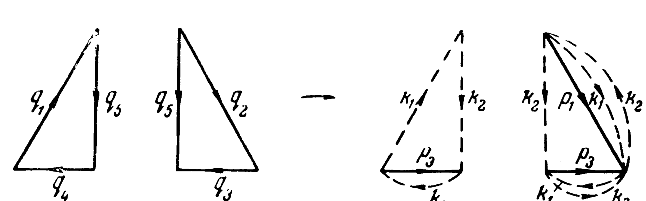
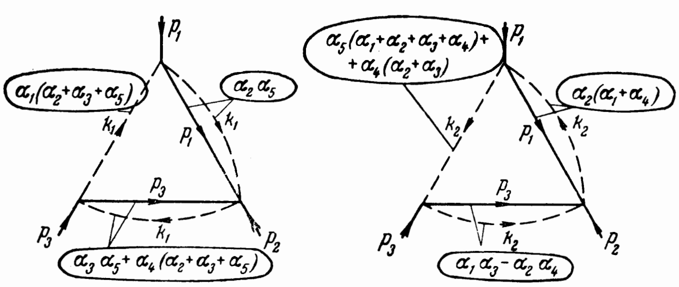


FIG. 3

of vertices to multiple vertices in the loops on Fig. 3b (the "elongated" loop "folds up", as it were, between the specified number of vertices). With such a separate consideration of the loops on Fig. 3b, the p-core of the diagram is likewise broken and the conservation laws in the broken vertices disappear, corresponding to a mixed character (with respect to the k_1 and k_2 momenta) of the system of equation (20b), which violates the structure of the diagram and the integrity of the k_i contours ($i = 1, 2$). However, both the integrity of the p-core and the closed nature of the k_i contours ($i = 1, 2$) are restored when solving the system of equations (20b) with respect to k_1 and k_2, and by the same token on going over to two single-contour equations (21) with separated loop k_1 and k_2 momenta. The two-loop diagram 2b breaks up into the two single-loop diagrams shown in Fig. 4, with the conservation laws satisfied at the vertices and with the k_1-contour closed in the loop $k_1p_1p_3$ (Fig. 4a) and the k_2 contour closed in the loop $k_2p_1p_3$ (Fig. 4b).

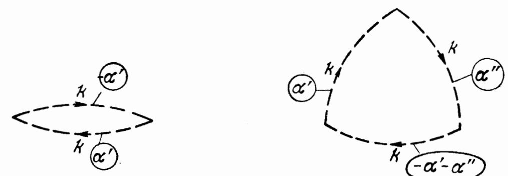


a b

FIG. 4

Graphically, the procedure of going over from the two loops in Fig. 3b to the two single-loop diagrams of Figs. 4a and b (which make up in the aggregate the 2-loop diagram of Fig. 2b) can be carried out in the following manner.

It is always possible to insert (or remove) between two or several vertices of the diagram, without violating the conservation laws, a closed two-sided or multi-lateral contour (see Fig. 5), the sides of which are identified with one of the loop 4-momenta k, to which are assigned parameters α or homogeneous combinations of these parameters (α', α'', etc.), with a vanishing sum (on going around the contour). In particular, it is always possible to insert between two vertices of the diagram an arbitrary k-line lacked by the contour, ascribing to this line a parameter $\alpha' = 0$, and then exclude the corresponding k-contour. Thus, in any diagram loop it is possible to "draw" the k-lines along the loop in the required direction and obtain for these lines com-



a b

FIG. 5.

binations of parameters α of appropriate type. For example, we can make the combinations of the parameters α identical for the k and p momenta that join identical vertices. Further, the loops can be multiplied by different combinations of parameters α and then "added" (or "subtracted") by making congruent identical k lines with equalized combinations of α-parameters; the congruent k-lines are then eliminated (with account of the circuiting directions along these lines) and the "added" loops can form (with annihilation of one of the loops) single-loop diagrams as component elements of some multi-loop diagram.

We note that the lines of the p-core of the diagram can also be "drawn along" or excluded to obtain new variants of the p-core, corresponding to a different initial choice of the loop-k-momenta. However, it is more advantageous to establish a single p-core for all the N-point diagrams with fixed N, to choose the loop k-momenta in accord with this core, and then effect the required balancing of the conservation laws at the vertices of the loops only by "drawing along" of the k-lines without touching the p-core.

With the aid of the described graphic procedure, the loops of any multi-loop diagram can be transformed into corresponding single-loop diagrams corresponding to single-contour Landau equations [of the type (19) or (21)]. This is precisely the way used to transform the two loops of Fig. 3b into the two single-loop diagrams of Figs. 4a and b. We emphasize here that homogeneous combinations of the parameters α near the external 4-momenta p_j, $j = 1, 3$ (and the corresponding adjacent k-momenta) on the single-loop diagrams of Figs. 4a and b are the sought γ-coefficients, which appear as a result of operation 2); the same γ coefficients precede the p-momenta in the corresponding Eq. (21).

In the elimination of the γ coefficients—operation 3)—with the aid of the quadratic Landau equations (4) [or (17′)—(17″)] and some additional connections between the coefficients, one must not lose sight of the fact that the eliminated γ-combinations of the parameters α can assume arbitrary values (in connection with the nonvanishing or the vanishing of the parameters α contained in them or in their combinations). However, in spite of this fact (if we disregard the case when one or several γ coefficients vanish), the general form of (18) for the Landau-singularity surface will remain unchanged.

The case of vanishing of some of the γ coefficients corresponds to a particular (degenerate) form of the Landau-singularity surface. On the other hand, inasmuch as the vanishing of one of the parameters α, for example the parameter α_j, corresponds to the operation of "contraction" the diagram, that is, crossing out the j-th internal line and coalescence of the corresponding vertices, some types of diagram contraction, not connected with the vanishing of the coefficients γ, will not influence the form of the Landau surface. In this case the Landau surface of the contracted diagram will be identical with the Landau surface of the main diagram (with a possible loss of dependence on one of the mass parameters $m_{j'}$); in the opposite case, when some of the coefficients γ vanish, the Landau surface of the contracted diagram will be part of the Landau surface of the main diagram.

4. TYPES OF SINGULARITIES

We now proceed to analyze the types of singularities of the Feynman integral (1) on the basis of various characteristics of the solutions $k_i^0(\alpha, p)$ of the inhomogeneous linear equations (12).

1. The case $C(\alpha) \equiv \Delta(\beta) \equiv 0$ is the usually investigated general case, when the singularities of the Feynman diagram depend on the masses of the internal lines, and the geometric method of dual diagrams can be used for the solution of the system of Landau equations (3)—(4) (in this case the algebraic solution of the system (3)—(4) usually becomes much more cumbersome on going over to more complicated Feynman diagrams). Taking (16) into account, the solutions of (12) can be written in this case in the form

$$k_i^0(\alpha, p) = -\frac{C_i(\alpha, p)}{C(\alpha)} = \frac{1}{C(\alpha)} \sum_m \gamma_{im}{}^\alpha p_m,$$

$$i = 1, 2, \ldots, l, \tag{22}$$

where $C^i(\alpha, p) \equiv \Delta_i(\beta, p)$ is the determinant obtained from $C(\alpha)$ by replacing the i-th column, made up of the coefficients β_{mi} $(m = 1, 2, \ldots, l)$ with unknown k_i in (12), by the column made up of free terms

$$\sum_{\langle m \rangle} \alpha_{j'} \Big(\sum p \Big)_{j'} \quad (m = 1, 2, \ldots, l);$$

$\gamma_{im}^\alpha = \gamma_{im}(\alpha) C(\alpha)$ are linear combinations of the coefficients of expansion of the determinant $C_i(\alpha, p)$ with respect to the column of the free terms, which are homogeneous functions of the parameters α of the same degree l [equal to the degree of the homogeneous function $C(\alpha)$].

Substitution of the solutions (22) into Eqs. (17′)—(17″) leads to a system of quadratic equations for $\gamma_{im}(\alpha)$, the solution of which gives the sought Landau-singularity surface (18). The graphic formalism developed for p, k diagrams and described in Sec. 3 is perfectly valid in our case.

2. The case $C(\alpha) \equiv \Delta(\beta) = 0$ corresponds to a possible appearance in the Feynman diagram of singularities of the second type, whose positions are independent of the masses of the internal lines m_j, and which are located on the section of the singularity surface defined only by the external 4-momenta p_j. Indeed, the system of inhomogeneous equations (12), the determinant of which is now assumed equal to 0, has a solution only when the compatability conditions

$$C_i(\alpha, p) = 0, \qquad i = 1, 2, \ldots, l \tag{23}$$

are satisfied, where $C_i(\alpha, p)$ are the determinants defined in (22). Equations (23) can be rewritten in the form of conditions for a linear dependence of p-momenta [see (22)]

$$\sum_m \gamma_{im}{}^\alpha p_m = 0, \qquad i = 1, 2, \ldots, l, \tag{24}$$

which in turn are equivalent to the vanishing of the Gram determinant for the external 4-momenta

$$\det(p_j p_i) = 0. \tag{25}$$

Equations (25) are equations for second-type singularity surfaces of the Feynman diagrams, and they can be treated, taking into account the properties of Gram determinants, as conditions for the location of the external 4-momenta p_j in a space with a smaller number of dimensions, that is, the singularities of the second type depend on the dimensionality of the space. In particular, for N-point diagrams with $N \geq 6$, conditions (25) for the existence of surfaces of singularities of the second type are always satisfied in 4-dimensional space, since these conditions (in the form of Gram determinants of fifth order and higher) are identical with the corresponding geometrical conditions of the diagrams (see [14]) connected with the 4-dimensionality of space. All the conditions with Gram determinants of higher order reduce here to conditions with Gram determinants of fifth order, which are equations of the surfaces of singularities of the second type for contracted diagrams. Thus, when $N \geq 6$ the surfaces of the singularities of the second type always exist, and the leading singularities of the main and of the suitably contracted diagrams should be located on these surfaces.

On going over to the case $C(\alpha) = 0$, the character of the single-contour equations (19) and (21) changes radically, as does the structure of the p, k diagrams. All the single-loop equations are transformed into the conditions of the type (24) for the linear dependence of the external 4-momenta p_m, which corresponds to elimination of the k-contours from the diagrams 1b and 2b (or 4a and 4b). As a result, we obtain on both triangle diagrams (both single-loop and two-loop) a new type of contracted diagram, as shown in Fig. 6,

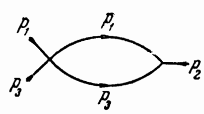


FIG. 6

which we also obtained by a different graphic method in an earlier paper [7]. The obtained contracted diagram is the p-core of triangle diagrams 1b and 2b (or of any other 3-point diagrams) in which the vertices separated by loop k-lines have been made congruent. The linear Landau equation (3) for the diagram of Fig. 6, "contracted" in this manner, is equivalent to the conditions (24) and (25) for the existence of surfaces of singularities of the second type. A similar procedure of going over to the case $C(\alpha) = 0$, using as an example the single-loop 4-point diagram, is shown in Figs. 7a, b, c ($q_3 \equiv k$).

We note here that when $C(\alpha) = 0$ the diagram will be characterized also by surfaces of singularities that depend on the parameters m_j, besides the surfaces of singularities of the second type, defined by (25). The equations of the former surfaces are those solutions of (12), which correspond to nonvanishing minors of the determinant $C(\alpha)$.

In conclusion let us analyze the types of singu-

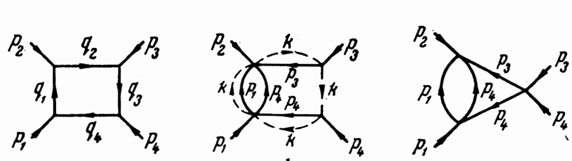


FIG. 7

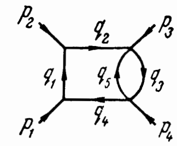


FIG. 8

larities, using the Feynman diagrams shown in Fig. 8 as a concrete example. The Landau equations of this diagram

$$\alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 q_3 + \alpha_4 q_4 = 0, \ \alpha_3 q_3 + \alpha_5 q_5 = 0,$$
$$q_i^2 = m_i^2 \quad (i = 1, 2, \ldots, 5) \tag{26}$$

assume on going over to k, p-notation the form

$$(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) k_1 - (\alpha_1 + \alpha_2 + \alpha_4) k_2 = P(\alpha),$$
$$\alpha_3 k_1 + \alpha_5 k_2 = 0; \tag{27'}$$
$$m_1^2 = t + k^2 + 2(p_1 + p_4) k, \ m_2^2 = M_3^2 + k^2 - 2p_3 k,$$
$$m_3^2 = k_1^2, \ m_4^2 = M_4^2 + k^2 + 2p_4 k, \ m_5^2 = k_2^2, \tag{27''}$$

where $P(\alpha) = -\alpha_1(p_1 + p_4) + \alpha_2 p_3 - \alpha_4 p_4$, and $k = k_1 - k_2 \equiv q_3 - q_5$ is the momentum of the loop side $(k_1 k_2) \equiv (q_3 q_5)$.

If in lieu of two loop 4-momenta k_1 and k_2 we consider the 4-momenta of the loop side $k = k_1 - k_2$, then the diagram of Fig. 8 in k, p-notation will have the form of the single-loop 4-point diagram on Fig. 7b.

In the case when

$$C(\alpha) = (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\alpha_5 + (\alpha_1 + \alpha_2 + \alpha_4)\alpha_3 \neq 0$$

the solutions of the system of linear equations (27′) take the form

$$k_1^0 = \frac{\alpha_5}{C(\alpha)} P(\alpha), \quad k_2^0 = -\frac{\alpha_3}{C(\alpha)} P(\alpha),$$
$$k^0 = k_1^0 - k_2^0 = \frac{\alpha_3 + \alpha_5}{C(\alpha)} P(\alpha). \tag{28}$$

Introducing γ-combinations of the parameters α:

$$\gamma_i = \alpha_i(\alpha_3 + \alpha_5) / C(\alpha), \quad i = 1, 2, 4 \tag{29}$$

and using the relation $(k_1 k_2) = \pm m_3 m_5$, which follows from the second equation of (27′), we obtain a system of equations for the elimination of the γ coefficients

$$2(p_1 + p_4) k^0 = m_1^2 - (m_3 \pm m_5)^2 - t, \tag{30}$$
$$2p_3 k^0 = -m_2^2 + (m_3 \pm m_5)^2 + M_3^2,$$
$$2p_4 k^0 = m_4^2 - (m_3 \pm m_5)^2 - M_4^2,$$
$$k^{0^2} = (m_3 \pm m_5)^2;$$
$$k^0 = -\gamma_1(p_1 + p_4) + \gamma_3 p_3 - \gamma_4 p_4. \tag{30'}$$

By determining three coefficients γ from the first three equations of (30) which are linear in γ, and by substituting them into the fourth equation of

(30) we obtain, taking (30′) into account, the sought equation for the Landau surface of the diagram for the case $C(\alpha) \neq 0$.

Let us consider different cases of "contraction" of the diagram in Fig. 8 under the condition $C(\alpha) \neq 0$. When $\alpha_3 = 0$, the 4-momenta $k_2^0 = 0$, but all the coefficients γ differ from 0 and $k^{0 2} = k_1^{0 2} = m_3^2$, that is, the form of the singularity surface will be the same as in the case of the main diagram. However, for the singularity surface of the "contracted" diagram (along one of the lines—α_3—of the loop side) the dependence on the parameter m_5 disappears. Analogously, when $\alpha_5 = 0$, the singularity surface of the "contracted" diagram will not contain the parameter m_3. When $\alpha_1 = 0$ we have $\gamma_1 = 0$ and the diagram is overdefined (there is one extra equation for γ_i); if we do not use in this case the first relation of (30) for γ_i, then the singularity surface will not depend on m_1. A similar situation takes place for $\alpha_2 = 0$ (m_2 is missing) and for $\alpha_4 = 0$ (m_4 is missing).

A more interesting case is that of "mixed" singularities of the diagram [combination of singularities of the second type along the loop $(q_3 q_5)$ and ordinary singularities along the loop $(q_1 q_2 q_3 q_4)$], when the singularity surface should not depend on the masses m_3 and m_5. This case takes place when $\alpha_3 + \alpha_5 = 0$ and $k^0 = 0$, when all the coefficients γ vanish and the singularity surface is determined from the equations

$$t = m_1^2,\ M_3^2 = m_2^2,\ M_4^2 = m_4^2.$$

We note here that any pair of α_i (with the exception of α_3 and α_5) can simultaneously vanish, thus leading to "double contraction". The case $\alpha_3 = \alpha_5 = 0$ corresponds already to the condition $C(\alpha) = 0$.

When $C(\alpha) = 0$, the compatibility conditions for the system of linear equations (27′) are written in the form

$$C_1(\alpha, p) = \alpha_5 P(\alpha) = 0,\quad C_2(\alpha, p) = -\alpha_3 P(\alpha) = 0, \tag{31}$$

from which, generally speaking, follows the equation for the singularity surface of the second type

$$P(\alpha) = -\alpha_1(p_1 + p_4) + \alpha_2 p_3 - \alpha_4 p_4 = 0, \tag{32}$$

which is more conveniently written in the form of a condition for the vanishing of the Gram determinant:

$$\det(p_j p_i) = 0,\quad j, i = 1, 3, 4. \tag{33}$$

The "contracted" diagram corresponding to this equation of the singularity surface is of the form shown in Fig. 7c. The second solution of the system (31)

$$\alpha_3 = \alpha_5 = 0 \tag{34}$$

also corresponds to the surface of singularities of the second type, for when (34) is satisfied the diagram becomes single-loop, and with the aid of the first equation of (27′) the loop 4-momenta k_1, k_2, or k are expressed in terms of the external 4-momenta p_1, p_3, and p_4.

This case also corresponds to the "contracted" diagram of Fig. 7c. Solving the system (27′) with $C(\alpha) = 0$, but, for example, with

$$C_{11}(\alpha) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \neq 0,$$

we can express one of the loop 4-momenta k_1 in terms of a second free 4-momentum k_2:

$$k_1^0 = -\frac{C_{12}}{C_{11}} k_2 + \frac{P(\alpha)}{C_{11}},\quad C_{12} = -(\alpha_1 + \alpha_2 + \alpha_4). \tag{35}$$

The free 4-momentum k_2 is eliminated from the system of the first three equations of (27′) with the aid of the compatibility condition (32); this leads to the equation

$$\alpha_1(m_1^2 - t - k^2) + \alpha_2(m_2^2 - M_3^2 - k^2) + \alpha_4(m_4^2 - M_4^2 - k^2) = 0. \tag{36}$$

Simultaneous solution of this equation with any two others from the system of three scalar equations obtained from the vector equation (32) (by taking the scalar products with the vectors $p_1 + p_4$, p_3 and p_4) gives the equation for the surface of singularities that do not belong to the second type.

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[1] L. D. Landau, JETP 37, 62 (1959), Soviet Phys. JETP 10, 45 (1960); Nucl. Phys. 13, 81 (1959).

[2] L. B. Okun and A. P. Rudik, Nucl. Phys. 14, 261 (1960).

[3] R. Cutcosky, J. Math. Phys. 1, 429 (1960).

[4] Fairlie, Landhoff, Nuttall, and Polkinghorne, ibid. 3, 594 (1962).

[5] M. Fowler, ibid. 3, 936 (1062); Nuovo cimento 27, 952 (1963).

[6] I. T. Drummond, Nuovo cimento 29, 720 (1963).

[7] V. E. Asribekov, Nucl. Phys. 56, 161 (1964).

[8] Fairlie, Landhoff, Nuttall, and Polkinghorne, Phys. Lett. 3, 55 (1962).

[9] T. Kinoshita, J. Math. Phys. 3, 650 (1962).

[10] N. Nakanishi, ibid. 4, 1539 (1963).

[11] Logunov, Todorov, and Chernikov, Godishnik

Sofiĭsk. un-t. fiz.-mat. fak-t 1960-1961 (1962) (Annual of the Sofia University, Physical-Mathematical Faculty), Book 2, **55**, 117; H. Poincare, Proc. Lond. Math. Soc. **32**, 277 (1901).
[12] I. S. Zlatev and A. V. Nikolov, Godishnik Sofiĭsk. un-t. fiz.-mat. fak-t 1961-1962 (1963), Book 2, **56**, 51.
[13] R. Chisholm, Proc. Cambridge Phil. Soc. **48**, 400 (1952).
[14] V. E. Asribekov, JETP **43**, 1826 (1962), Soviet Phys. JETP **16**, 1289 (1963).

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