ELECTROMAGNETIC FIELD STATES WITH WELL-DEFINED FIELD STRENGTHS

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We consider states of the electromagnetic field with well-defined values of the field strength. We find the wave functions of such states in the occupation-number representation.

IT is well known that a free electromagnetic field can be described in quantum theory either by giving the photon occupation numbers or by giving the field strength.[1] We consider in the present paper the problem of finding states of the electromagnetic field in which the electrical or magnetic field strength has a well-defined value.

It follows from the commutation relations for the field operators that they commute in all points which can not be connected by a light signal. In the following we shall be interested in states in which the field strength is given in the whole of space at one particular time. In that case only the electrical and magnetic field operators taken at the same point of space do not commute. We can choose as a complete set the values of the electrical (or magnetic) field strength in all points in space. We can then clearly talk only about a probability distribution for the photon occupation numbers as in such states the occupation numbers themselves (and the field energy) do not have well-defined values.

We consider first states with a well-defined value of the electrical field.

1. The electrical field strength operator at time $t = 0$ has the form[2]

$$\hat{\mathbf{E}}(\mathbf{r}) = i\left(\frac{\hbar}{2V}\right)^{1/2}\sum_{\mathbf{k}\lambda}\omega^{1/2}\left(c_{\mathbf{k}\lambda}e^{i\mathbf{k}\mathbf{r}} - c_{\mathbf{k}\lambda}^{+}e^{-i\mathbf{k}\mathbf{r}}\right)\mathbf{e}_{\mathbf{k}\lambda}. \quad (1)$$

Here $\mathbf{k}$ is the wave vector, $\omega \equiv \omega_{\mathbf{k}} = ck$ the photon frequency, λ enumerates the two transverse polarizations, $\mathbf{e}_{\mathbf{k}\lambda}$ are the appropriate polarization unit vectors, V the normalization volume, leading to a discrete set of $\mathbf{k}$; $c_{\mathbf{k}\lambda}$ and $c_{\mathbf{k}\lambda}^{+}$ are the boson annihilation and creation operators for a photon in the state $\mathbf{k}\lambda$. Their action upon the eigenfunctions $\Phi\{N\} = \Phi_{\ldots N_{\mathbf{k}\lambda}\ldots}$ is defined by the following equations:

$$\begin{aligned} c_{\mathbf{k}\lambda}\Phi_{\ldots N_{\mathbf{k}\lambda}\ldots} &= N_{\mathbf{k}\lambda}^{1/2}\Phi_{\ldots N_{\mathbf{k}\lambda}-1\ldots}, \\ c_{\mathbf{k}\lambda}^{+}\Phi_{\ldots N_{\mathbf{k}\lambda}\ldots} &= (N_{\mathbf{k}\lambda}+1)^{1/2}\Phi_{\ldots N_{\mathbf{k}\lambda}+1\ldots}, \end{aligned} \quad (2)$$

where $N_{\mathbf{k}\lambda}$ is the number of photons in the state $\mathbf{k}\lambda$.

The eigenfunctions of the operator $\hat{\mathbf{E}}$ satisfy the equation

$$\hat{\mathbf{E}}(r)\Psi = \mathbf{E}(r)\Psi, \quad \Psi = \sum_{\{N\}} a_{\{N\}}\Phi_{\{N\}}. \quad (3)$$

2. We consider a state in which the electrical field has a well-defined value in all points in space at time $t = 0$. Such a state is possible, since the operators $\hat{\mathbf{E}}(\mathbf{r}_1)$ and $\hat{\mathbf{E}}(\mathbf{r}_2)$ commute with one another for all $\mathbf{r}_1$ and $\mathbf{r}_2$. The wave function of such a state must be the joint eigenfunction of the operators $\hat{\mathbf{E}}(\mathbf{r})$ in all points in space and must therefore be independent of the coordinates.

The separate terms in the sum (1) in different points in space with the same $\mathbf{k}$ and λ do not commute with one another. We see, however, easily that if we combine terms referring to the wave vectors $\mathbf{k}$ and $-\mathbf{k}$ the corresponding (Hermitean) operators $\hat{\mathbf{F}}_{\mathbf{k}\lambda}(\mathbf{r})$ turn out to be already commuting in different points (at the same time). This enables us to speak about a complete set of states with well-defined values of $\hat{\mathbf{F}}_{\mathbf{k}\lambda}$ and to classify the states of the field $\hat{\mathbf{E}}$ in terms of them.[1)] The wave function of a state with a well-defined value of the field $\mathbf{E}(\mathbf{r})$ is equal to the product of the eigenfunctions of the operators $\hat{\mathbf{F}}_{\mathbf{k}\lambda}$.

For the sake of convenience we assume that the polarization unit vectors $\mathbf{e}_{\mathbf{k}\lambda}$ and $\mathbf{e}_{-\mathbf{k}\lambda}$ are connected as follows:*

$$\mathbf{e}_{-k1} = \mathbf{e}_{k1},\ \mathbf{e}_{-k2} = -\mathbf{e}_{k2};\ \mathbf{e}_{k2} = [\mathbf{k}/k, \mathbf{e}_{k1}];$$

the operators $\hat{\mathbf{F}}_{\mathbf{k}\lambda}(\mathbf{r})$ are then equal to

[1)] We can also consider states with well-defined values of $F_{\mathbf{k}\lambda}$ for one set of indices $\mathbf{k}\lambda$ and with well-defined occupation numbers $N_{\mathbf{k}\lambda}$ for the other $\mathbf{k}\lambda$.

*$[\mathbf{k}/k, \mathbf{e}_{k1}] \equiv \mathbf{k}/k \times \mathbf{e}_{k1}$.

$$\hat{\mathbf{F}}_{\mathbf{k}\lambda}(\mathbf{r}) = i\left(\frac{\hbar\omega}{2V}\right)^{1/2}\{(c_{\mathbf{k}\lambda} \mp c^{+}_{-\mathbf{k}\lambda})\, e^{i\mathbf{kr}} \pm (c_{-\mathbf{k}\lambda} \mp c^{+}_{\mathbf{k}\lambda}) e^{-i\mathbf{kr}}\}\, \mathbf{e}_{\mathbf{k}\lambda} \quad (4)$$

($\lambda = 1, 2$; the upper sign corresponds to $\lambda = 1$). As a result we obtain a set of equations for the wave function $a\{N\}$:

$$(N_{\mathbf{k}\lambda}+1)^{1/2} a_{N_{\mathbf{k}\lambda}+1,\, N_{-\mathbf{k}\lambda}} \mp N^{1/2}_{-\mathbf{k}\lambda} a_{N_{\mathbf{k}\lambda},\, N_{-\mathbf{k}\lambda}-1} = E_\lambda a_{N_{\mathbf{k}\lambda},\, N_{-\mathbf{k}\lambda}},$$
$$(N_{-\mathbf{k}\lambda}+1)^{1/2} a_{N_{\mathbf{k}\lambda},\, N_{-\mathbf{k}\lambda}+1} \mp N^{1/2}_{\mathbf{k}\lambda} a_{N_{\mathbf{k}\lambda}-1,\, N_{-\mathbf{k}\lambda}} = \mp E^{*}_\lambda a_{N_{\mathbf{k}\lambda},\, N_{-\mathbf{k}\lambda}}. \quad (5)$$

In Eqs. (5) we have indicated explicitly only those pairs of arguments of the wave function $a\{N\}$ which change under the action of the appropriate operators $\hat{\mathbf{F}}(\mathbf{r})$. The quantities E_1 and E_2 are equal to

$$E_\lambda = -i(2V/\hbar\omega)^{1/2}\mathbf{E}_{\mathbf{k}}\mathbf{e}_{\mathbf{k}\lambda}, \quad (6)$$

where $\mathbf{E}_{\mathbf{k}}$ is the Fourier component in the expansion of the field eigenvalue,

$$\mathbf{E}(\mathbf{r}) = \sum_{\mathbf{k}} \mathbf{E}_{\mathbf{k}} e^{i\mathbf{kr}}.$$

It is clear from (5) that the equations for the two possible degrees of polarization are separate.

3. We consider the system of finite difference equations (5). To simplify the notation we write $N_{\mathbf{k}1} = n$; $N_{-\mathbf{k}1} = m$. We introduce the substitution

$$a_{n,\,m} = (-1)^m \left(\frac{m!}{n!}\right)^{1/2} E_1^{n-m}\, b_{n,\,m}, \quad (7)$$

which changes the first of Eqs. (5) into an equation with constant coefficients. The equations for the functions $b_{n,m}$ have the form

$$b_{n+1,\,m} + b_{n,\,m-1} = b_{n,\,m}; \quad (8)$$

$$(m+1)\, b_{n,\,m+1} + n b_{n-1,\,m} = \varepsilon b_{n,\,m}; \quad \varepsilon \equiv |E_1|^2. \quad (9)$$

As n and m are occupation numbers, the wave function $a_{n,m}$, and of course also the function $b_{n,m}$, must vanish whenever one of its indices is negative.

For $m = 0$, Eq. (8) gives, clearly, $b_{n,0} = b_{0,0}$ or, from (7) $b_{n,0} = a_{0,0}$ where the constant $a_{0,0}$ may depend on all occupation numbers except n and m. In the following we shall omit this constant which occurs everywhere as a factor.

For $n = 0$, Eq. (9) has the solution

$$b_{0,\,m} = \varepsilon^m / m! \quad (10)$$

The solution (10) can be considered to be a boundary condition for Eq. (8).

We can find the solution of Eq. (8) using a generating function $B(x, y)$ connected with $b_{n,m}$ through the equation

$$B(x,y) = \sum_{n,\,m=0}^{\infty} b_{n,\,m}\, x^n\, y^m. \quad (11)$$

We multiply both sides of Eq. (8) by $x^n y^m$ and sum over n and m from zero to infinity. Bearing in mind that $b_{n,-1} = 0$ we get the following equation for $B(x, y)$:

$$\frac{1}{x}\left[B(x,y) - \sum_{m=0}^{\infty} b_{0,m} y^m\right] + yB(x,y) = B(x,y).$$

Using the boundary condition (10) we find from this

$$B(x,\,y) = e^{\varepsilon y}/(1 - x + xy). \quad (12)$$

Expanding the expression obtained for $B(x, y)$ in a double power series in x and y, we find from (11) and (7) an explicit expression for the function $a_{n,m}$:

$$a_{n,\,m} = E_1^{n-m} \sum_{k=0}^{m} (-1)^k \frac{\varepsilon^k}{k!} \frac{(n!\, m!)^{1/2}}{(m-k)!\,(n-m+k)!} \qquad (n \geqslant m). \quad (13)$$

In this form the expression is valid for $n \geqq m$. When $n \leqq m$ the summation over k must start from $k = m - n$.

If we replace in that case the summation variable k by $k' = n - m + k$, we see that the solution satisfies the following symmetry condition

$$a_{m,\,n} = (-1)^{m-n} a^{*}_{n,\,m},$$

reflecting the symmetry of the set (5) under a replacement of $\mathbf{k}$ by $-\mathbf{k}$. It follows from this that the solution (13) of the first of Eqs. (5) with the boundary condition (10) also satisfies the second of Eqs. (5).

4. Differentiating the function $B(x, y)$ with respect to ϵ, x, and y and using the expansion (11) we obtain a differential equation for the function $b_{n,m}(\epsilon)$:

$$\varepsilon \frac{d^2 b_{n,\,m}}{d\varepsilon^2} + (n - m + 1 - \varepsilon)\frac{db_{n,\,m}}{d\varepsilon} + m b_{n,\,m} = 0. \quad (14)$$

The regular solution of this equation is the generalized Laguerre polynomial $L_m^{n-m}(\epsilon)$.[3] Comparing the explicit expression for the Laguerre polynomial with Eq. (13) for $a_{n,m}$ we find [2)]

$$a_{n,\,m} = \begin{cases} E_1^{n-m}\,(m!/n!)^{1/2} L_m^{n-m}(\varepsilon), & n \geqslant m, \\ (-E_1^{*})^{m-n}\,(n!/m!)^{1/2} L_n^{m-n}(\varepsilon), & n \leqslant m. \end{cases} \quad (15)$$

A similar comparison gives us for the function $b_{n,m}$:

$$b_{n,\,m} = (-1)^m L_{m^{n-m}}(\varepsilon).$$

It is clear from this that the function $B(x, y)$

[2)] Each of Eqs. (15) is valid whatever the sign of the difference $n - m$, but then the formula is not symmetrical in n and m.

given by (12) is the generating function for the generalized Laguerre polynomials with two integral indices.[3)]

We shall show that the functions $a_{n,m}$ corresponding to different eigenvalues of the field Fourier components are orthogonal. We denote by φ the argument of the complex amplitude E_1: $E_1 = \epsilon^{1/2}e^{i\varphi}$. For the normalization of the eigenfunctions we consider the sum

$$S \equiv \sum_{n,m} a^*_{n,m}(\varepsilon,\varphi)a_{n,m}(\varepsilon',\varphi').$$

We split the sum S into two parts, in one of which $n \geqq m$, while in the other $n < m$. Denoting these sums respectively by S_1 and S_2 we can write

$$S_1 = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{m!}{n!}e^{i(n-m)(\varphi'-\varphi)}(\varepsilon\varepsilon')^{(n-m)/2}L_m^{n-m}(\varepsilon)\,L_m^{n-m}(\varepsilon'),$$

$$S_2 = \sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty}\frac{n!}{m!}e^{-i(m-n)(\varphi'-\varphi)}(\varepsilon\varepsilon')^{(m-n)/2}$$

$$\times L_n^{m-n}(\varepsilon)L_n^{m-n}(\varepsilon').$$

Changing the order of summation in the sum S_1 and changing to a summation over the upper index of the Laguerre polynomials we get

$$S = \sum_{k=0}^{\infty}(\varepsilon\varepsilon')^{k/2}e^{ik(\varphi'-\varphi)}\sum_{m=0}^{\infty}\frac{m!}{(m+k)!}L_m^k(\varepsilon)L_m^k(\varepsilon')$$

$$+\sum_{k=1}^{\infty}(\varepsilon\varepsilon')^{k/2}e^{-ik(\varphi'-\varphi)}\sum_{n=0}^{\infty}\frac{n!}{(n+k)!}L_n^k(\varepsilon)L_n^k(\varepsilon').$$

The inner sum is equal to, owing to the orthogonality of the normalized Laguerre polynomials,

$$\sum_{m=0}^{\infty}\frac{m!}{(m+k)!}L_m^k(\varepsilon)L_m^k(\varepsilon') = (\varepsilon\varepsilon')^{-k/2}e^{\varepsilon}\delta(\varepsilon-\varepsilon').$$

Bearing in mind the equation

$$\sum_{k=-\infty}^{\infty}e^{ik\varphi} = 2\pi\delta(\varphi),$$

we get then the normalization condition in the form

$$\sum_{n,m} g^*_{n,m}(\varepsilon,\varphi)g_{n,m}(\varepsilon',\varphi') = \delta(\varepsilon-\varepsilon')\delta(\varphi-\varphi'), \quad (16)$$

where the functions

$$g_{n,m}(\varepsilon,\varphi) = \begin{cases} e^{-\varepsilon/2}(m!/2\pi n!)^{1/2}\varepsilon^{(n-m)/2}e^{i(n-m)\varphi}L_m^{n-m}(\varepsilon), & n \geqslant m, \\ (-1)^{m-n}e^{-\varepsilon/2}(n!/2\pi m!)^{1/2}\varepsilon^{(m-n)/2}e^{-i(m-n)\varphi}L_n^{m-n}(\varepsilon), & n \leqslant m \end{cases} \quad (17)$$

are orthonormalized both as regards the discrete arguments n and m, and as regards the continuous arguments ϵ and φ, i.e., they satisfy not only Eq. (16), but also the condition

$$\int_0^{2\pi}d\varphi\int_0^{\infty}d\varepsilon\, g_{nm}^*(\varepsilon,\varphi)g_{n'm'}(\varepsilon,\varphi) = \delta_{nn'}\delta_{mm'}. \quad (18)$$

The equations (5) corresponding to the same k and the second degree of polarization reduce to Eqs. (8) and (9) for the functions $b_{n,m}$ through the substitution

$$a_{n,m} = (m!/n!)^{1/2}E_2^{n-m}b_{n,m};\quad n = N_{k2},\quad m = N_{-k2}. \quad (19)$$

The parameter ϵ occurring in (9) is now equal to $\epsilon = |E_2|^2$. It is clear from a comparison of (19) and (7) that the wave function $g_{N_{k2},N_{-k2}}(\epsilon,\varphi)$ differs from $g_{N_{k1},N_{-k1}}(\epsilon,\varphi)$ [see (7)] by a factor $(-1)^{N_{-k2}}$ and the replacement of the indices $1 \to 2$.

The quantity $|g_{N_{k\lambda},N_{-k\lambda}}|^2$ gives us the probability distribution for the different photon occupation numbers $N_{k\lambda}$, $N_{-k\lambda}$ in states with a well-defined electrical field strength. In the particular case $N_{-k\lambda} = 0$ the distribution for $N_{k\lambda}$ is, apart from a factor $1/2\pi$ the same as a Poisson distribution where the quantity ϵ plays the part of the average value $\bar{N}_{k\lambda}$. We note that when the emission processes which create the field are statistically independent, the photon distribution is also described by a Poisson distribution.[4] However, in our case the Poisson distribution for $N_{k\lambda}$ occurs only under the condition $N_{-k\lambda} = 0$.

5. We can find in complete analogy the wave function of the state in which the magnetic field strength has a well-defined value in the whole of space. If we use for the magnetic field an expansion of the operator $\hat{\mathbf{H}}(\mathbf{r})$ similar to Eq. (1):

$$\hat{\mathbf{H}}(\mathbf{r}) = i\left(\frac{\hbar}{2V}\right)^{1/2}\sum_{\mathbf{k}\lambda}\omega^{1/2}(c_{\mathbf{k}\lambda}e^{i\mathbf{kr}} - c_{\mathbf{k}\lambda}^+e^{-i\mathbf{kr}})[\mathbf{n}\mathbf{e}_{\mathbf{k}\lambda}]$$

($\mathbf{n} = \mathbf{k}/k$) one sees easily that then for the same connection between the polarization unit vectors the operators $\hat{\mathbf{F}}_{\mathbf{k}\lambda}(\mathbf{r})$ will correspond to operators $\hat{\mathbf{G}}_{\mathbf{k}\lambda}$:

$$\hat{\mathbf{G}}_{\mathbf{k}\lambda}(\mathbf{r}) = i(\hbar\omega/2V)^{1/2}\{(c_{\mathbf{k}\lambda} \pm c_{-\mathbf{k}\lambda}^+)e^{i\mathbf{kr}}$$

$$\mp(c_{-\mathbf{k}\lambda} \pm c_{\mathbf{k}\lambda}^+)e^{-i\mathbf{kr}}\}[\mathbf{n}\mathbf{e}_{\mathbf{k}\lambda}]. \quad (20)$$

[3)] Apparently, the expression (12) for the generating function of the Laguerre polynomials with two integral indices is not known in the literature.

Comparison of (4) with (20) shows that the eigenfunction of the operator $\hat{G}_{k1}(\mathbf{r})$ is the same as the eigenfunction of $\hat{F}_{k2}(\mathbf{r})$ in which the argument E_2 must be replaced by

$$H_1 = -i(2V/\hbar\omega)^{1/2}(\mathbf{H}_k[\mathbf{n}\mathbf{e}_{k1}]).$$

The eigenfunction of the operators $\hat{G}_{k2}(\mathbf{r})$ also is exactly the same as the eigenfunction of $\hat{F}_{k1}$ with E_1 replaced by

$$H_2 = -i(2V/\hbar\omega)^{1/2}(\mathbf{H}_k[\mathbf{n}\mathbf{e}_{k2}]).$$

6. If the quantity $F_{k\lambda}$ has a well-defined value in the whole of space at a time $t_0 \neq 0$, the corresponding eigenfunction is equal to

$$a_{n,m}(t_0) = a_{n,m}\exp\{i(n+m)\omega t_0\},$$

where $a_{n,m}$ is determined by Eqs. (15) and (19). The evolution of a state with time is described, according to the Schrödinger equation for a free electromagnetic field, by the formula

$$a_{n,m}(t_0, t) = a_{n,m}(t_0)\exp\{-i(n+m)\omega(t-t_0)\}.$$

It is clear from this formula that $a_{n,m}(t_0, t)$ will be the eigenfunction of the operator $\hat{F}_{k\lambda}(\mathbf{r}, t_0)$ at a time

$$t_l = t_0 + 2\pi l/\omega,$$

where l is an arbitrary integer. At other times $F_{k\lambda}$ does not have a well-defined value.

We can construct from the functions $a_{n,m}(t_0)$ wave functions of states in which different Fourier components of the field have well-defined values at different times. These wave functions have the form

$$a_{\{N\}}(t) = \prod_{k\lambda} a_{N_{k\lambda}, N_{-k\lambda}}(t_{0k\lambda}, t).$$

7. Bohr and Rosenfeld[1,4] have shown that a field strength averaged over a space-time region around the point $(\mathbf{r}_0, t_0)$ has a physical meaning. Assuming for the sake of simplicity that this region is a four-dimensional parallelepiped with edge lengths $2L_x$, $2L_y$, $2L_z$, and $2T$, we get for the averaged operator $\bar{\hat{F}}_{k\lambda}$ the following expression

$$\bar{\hat{F}}_{k\lambda} = \alpha\hat{F}_{k\lambda}(\mathbf{r}_0, t_0),$$

$$\alpha = \frac{\sin k_x L_x}{k_x L_x}\,\frac{\sin k_y L_y}{k_y L_y}\,\frac{\sin k_z L_z}{k_z L_z}\,\frac{\sin \omega T}{\omega T}.$$

It is clear that through the averaging the operator $\hat{F}_{k\lambda}(\mathbf{r}_0, t_0)$ is multiplied by a constant factor. Hence it follows that the eigenfunction of the averaged operator is the same as the function $a_{n,m}$ found above and that the eigenvalue is equal to

$$\overline{F_{k\lambda}(\mathbf{r})} = \alpha F_{k\lambda}(\mathbf{r}_0).$$

8. One can easily find the wave function of a state in which the electrical field has a well-defined value only in one, fixed point of space. The required function can in that case be found as the common eigenfunction of the operators

$$\hat{E}_{k\lambda}(\mathbf{r}) = i(\hbar\omega/2V)^{1/2}(c_{k\lambda}e^{i\mathbf{k}\mathbf{r}} - c_{k\lambda}^{+}e^{-i\mathbf{k}\mathbf{r}}),$$

$$\hat{\mathbf{E}}(\mathbf{r}) = \sum_{k\lambda}\hat{E}_{k\lambda}(r)\,\mathbf{e}_{k\lambda}, \qquad (21)$$

which commute in one point $\mathbf{r}$. This problem is analogous to the problem of finding the eigenfunction of the oscillator coordinate operator in the energy representation. It is well known[5] that the latter is the complex conjugate of the energy eigenfunction in the coordinate representation and is equal to

$$a_n(E_{k\lambda}, \mathbf{r}) = \frac{(-i)^n e^{-in\mathbf{k}\mathbf{r}}}{\pi^{1/4}(2^n n!)^{1/2}} H_n\left(\left(\frac{V}{\hbar\omega}\right)^{1/2} E_{k\lambda}\right)\exp\left\{-\frac{V}{2\hbar\omega}E_{k\lambda}^2\right\},$$

where $E_{k\lambda}$ is the eigenvalue of the operator $\hat{E}_{k\lambda}(\mathbf{r})$ and H_n a Hermite polynomial.

Since the eigenvalue of the field $\mathbf{E}$ is a sum of the eigenvalues $E_{k\lambda}\mathbf{e}_{k\lambda}$ we have, as should be the case, an infinite degeneracy since for a given $\mathbf{E}$ different values of $E_{k\lambda}$ are possible. The degeneracy is connected with the fact that the field in one point is, clearly, not a complete set and does not determine the value of the field in other points in space. The degeneracy can be removed, for instance, by requiring that the field has a well-defined value in all points in space. This corresponds to expanding the solution (15) in terms of Hermite polynomials:

$$a_{n,m}(F_{k1+}, F_{k1-}) = \int_{-\infty} dE_{k1}\, a_n(E_{k1}, \mathbf{r})\, a_m(F_{k1}(\mathbf{r}) - E_{k1}, \mathbf{r})\, K(E_{k1}, \mathbf{r}). \qquad (22)$$

Here

$$F_{k1}(\mathbf{r}) = F_{k1+}e^{i\mathbf{k}\mathbf{r}} + F_{k1-}e^{-i\mathbf{k}\mathbf{r}}$$

is the eigenvalue of the operator $\hat{F}_{k1}(\mathbf{r})\,\mathbf{e}_{k1}$. It is connected, according to what we have said above, with the eigenvalues E_{k1} and E_{-k1} through the formula $F_{k1}(\mathbf{r}) = E_{k1} + E_{-k1}$ ($\mathbf{r}$—fixed point).

The right-hand side of the equation for an arbitrary kernel K is an eigenfunction of the operator $\hat{F}_{k1}(\mathbf{r})\,\mathbf{e}_{k1}$ in the point $\mathbf{r}$ corresponding to the eigenvalue $F_{k1}(\mathbf{r})$ which is degenerate with respect to the magnitude of E_{k1}. If we require that this function should be the common eigenfunction

of the operators $\hat{F}_{\mathbf{k}1}(\mathbf{r})$ in all points in space, i.e., that Eq. (22) be satisfied, $K(E_{\mathbf{k}1}, \mathbf{r})$ turns out to be equal to

$$K(E_{\mathbf{k}1}, \mathbf{r}) = \left(\frac{\hbar\omega}{\pi V}\right)^{1/2} \exp\left\{-\frac{V}{\hbar\omega}[E_{\mathbf{k}1} - F_{\mathbf{k}1}+e^{i\mathbf{k}\mathbf{r}}]^2\right\}.$$

This follows from the integral relation between the Laguerre and Hermite polynomials (see [3] p. 852, Eq. 377).

In conclusion we note that the solutions obtained in the foregoing are valid for any real Bose field, for instance, for phonons or spin waves. The concrete expressions will differ only in differences in notation.

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