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HIDDEN SYMMETRY OF THE KEPLER PROBLEM

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The hidden symmetry of the n-dimensional Kepler problem is considered. The treatment is based on the algebraic properties of the operators.

IN his famous paper of 1935,[1] Fock pointed out that the Hamiltonian for the hydrogen atom (Kepler problem) has a "hidden" symmetry: it is invariant under the four-dimensional rotation group (or the Lorentz group). Fock, treating momentum space as the stereographic projection of the unit hypersphere of a four-dimensional Euclidean space, reduced the Schrödinger equation in momentum space to an integral equation for four-dimensional spherical functions. Bargmann, starting from the solution of the hydrogen atom found by Pauli,[2] in a note concerning Fock's paper[3] showed that the symmetry group discovered by Fock is generated by the known integrals of motion for the problem: the angular momentum and the Laplace integral (Runge-Lenz vector).[1)]

Thus Fock first made clear that the so-called accidental degeneracy, which is not associated with any explicit geometrical symmetry (as, for example, invariance under spatial rotations) expresses a higher "hidden" symmetry of the Hamiltonian. Another example is the higher symmetry of the Hamiltonian for the n-dimensional harmonic oscillator—invariance under the group of n-dimensional unitary transformations, which was pointed out by Jauch and Hill,[5] and later, obviously independently of them, by Demkov (for $n = 3$)[6] and Baker.[7] The n-dimensional generalization of the Kepler problem was treated by Alliluyev [8] on the basis of Fock's method. He transformed the Schrödinger equation for the n-dimensional Kepler problem to an integral equation for the $(n + 1)$-dimensional spherical functions, and thus explained the symmetry of the problem under the $(n + 1)$-dimensional rotations.

We shall also consider the n-dimensional Kepler problem, but shall not choose a specific representation of the operators. Following Pauli, Bargmann, and Hulthen, we shall use only algebraic (commutation) properties of the operators. Because of the close analogy between quantum mechanical operators and classical quantities, such an approach to the problem enables us to explain how the "hidden" symmetry of the problem manifests itself within the framework of classical mechanics.

The Hamiltonian for the problem has the form

$$H = {}^1/_2 p_s p_s - (x_s x_s)^{-1/2}. \tag{1}$$

(We assume summation over repeated indices from 1 to n.) It is immediately clear that the group R_n of n-dimensional rotations is the symmetry group of this Hamiltonian. The infinitesimal generators of the group

$$L_{rs} = x_r p_s - p_s x_r \quad (r, s = 1, 2, \ldots, n), \tag{2}$$

which are the components of the angular momentum tensor, commute with H. (The total number of components is obviously $\binom{n}{2}$.)

The additional integrals of the motion are the components of the Lenz-Runge vector:

$$A_r = x_r (x_s x_s)^{-1/2} + {}^1/_2 (L_{sr} p_s + p_s L_{sr}) = x_r (x_s x_s)^{-1/2} + L_{sr} p_s - {}^1/_2 \hbar \sqrt{i} (n-1) p_r \quad (r = 1, 2, \ldots, n), \tag{3}$$

generalized to the case of n dimensions.

We shall restrict ourselves here to the subspace of Hilbert space in which H is negative definite (bound states). We define the hermitian vector operator

$$A_r' = (-2H)^{-1/2} A_r. \tag{4}$$

The components L_{sr} and A_r, or L_{sr} and A_r', constitute $\binom{n}{2} + n = \binom{n+1}{2}$ operators commuting with H. But it is better to use the A_r', since we can, from the components L_{sr} and A_r', using the formulas

$$D_{rs} = L_{rs} \quad (r, s = 1, 2, \ldots, n), \qquad D_{r\, n+1} = -D_{n+1\, r} = A_r' \quad (r = 1, 2, \ldots, n) \tag{5}$$

[1)]Cf. also the remark of Klein cited by Hulthén[4].

form one (n + 1)-dimensional antisymmetric tensor D_{rs}, whose components satisfy the commutation relations

$$[D_{rs}, D_{tu}] = i\hbar(\delta_{ru}D_{ts} + \delta_{st}D_{ur} + \delta_{rt}D_{su} + \delta_{su}D_{rt}) \quad (r, s, t, u = 1, 2, \ldots, n+1), \tag{6}$$

which coincide with the commutation relations of the Lie algebra of the group R_{n+1} of the (n + 1)-dimensional rotations.

Using formulas (1)—(5), we get

$$H^{-1} = -2[G(R_{n+1}) + {}^1/_4\hbar^2(n-1)^2], \tag{7}$$

where

$$G(R_{n+1}) = {}^1/_2 D_{rs}D_{rs} \tag{8}$$

(summation over r, s = 1, 2, ..., n + 1) is the Casimir operator for the group R_{n+1} in a suitable normalization. The eigenvalues of the operator G, as follows from (6), have the form $\hbar^2 N(N + n - 1)$, [9] where N is a nonnegative integer. Using this, we get from (7),

$$E_N = -1/2\hbar^2[N + {}^1/_2(n-1)]^2.$$

This formula coincides with Alliluyev's result, which he obtained by solving the Schrödinger equation for the eigenvalues of the energy, and which is a generalization of the familiar Balmer formula for the hydrogen atom and also of the two-dimensional and one-dimensional Balmer formulas found respectively by Jauch and Hill [5] and by Louck. [10]

The expression (7) for the energy can also be used in the classical limit $\hbar \to 0$. If, according to (5), we construct from the integrals L_{rs}, A'_r of the motion an (n + 1)-dimensional antisymmetric tensor, the (n + 1)-dimensional orthogonal transformations of this tensor will constitute the symmetry group of the Hamiltonian of the equations of motion.

One can similarly treat the case of positive energies. Then, in place of (4), we can define the hermitian vector operator A'_r by the formula

$$A_r' = (2H)^{-1/2}A_r.$$

The group generated by the operators L_{rs}, A'_r is now the (n + 1)-dimensional Lorentz group (with one timelike variable).

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