INELASTIC SMALL ANGLE NEUTRON SCATTERING IN FERROMAGNETS

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We consider the problem of small-angle inelastic scattering of slow neutrons in ferromagnets, in the case when the change in neutron energy during the scattering is comparable with the energy of the magnetic interaction of the atomic spins with one another and with the external magnetic field. It turns out that scattering with absorption of a spin wave occurs in a wider range of angles than scattering with the emission of a spin wave. In the range of angles where only scattering involving absorption takes place there should be in a number of cases a strong dependence of the cross section upon the neutron polarization and if the incident neutrons are unpolarized considerable polarization must appear as a result of scattering. We show that sufficiently slow neutrons can not be scattered at all with absorption or emission of a single spin wave. We also discuss some recent experiments on small angle scattering in ferromagnets.

SMALL-angle scattering of neutrons in ferromagnets has been studied theoretically in detail on the basis of the assumption of a quadratic spin wave dispersion law (see, e.g., the paper by the present author[1] and the review by Izyumov[2]). The following results were then obtained: the scattering occurs basically with the absorption or emission of a single wave; the scattering angle ϑ_m is limited by the magnitude $\vartheta_m = 1/\alpha$ where $\alpha = 2mA\hbar^{-2}$ (A is the constant in the spin wave dispersion law $\epsilon_k = Ak^2$, and m the neutron mass) and finally the change in neutron energy on scattering is of the order of E/α (E is the energy of the incident neutrons).

The assumption that the dispersion law is quadratic is clearly inapplicable if the quantity E/α becomes comparable with the energy of the magnetic interaction between the atomic spins with one another or with the external magnetic field H, i.e., comparable with either $4\pi\mu M_0$ or $2\mu H$, where M_0 is the saturation magnetization and μ the Bohr magneton. The exact expression for the spin wave energy is well known to be of the form (see, e.g., the survey by Akhiezer, Bar'yakhtar, and Kaganov[3])

$$\varepsilon_k = [(Ak^2 + 2\mu H)(Ak^2 + 2\mu H + 8\pi\mu M_0 \sin^2\vartheta_k)]^{1/2}. \quad (1)$$

Here H is the magnetic field inside the specimen and ϑ_k is the angle between the wavevector **k** and the magnetic field. Generally speaking, in (1) should also occur terms caused by the magnetic anisotropy. However, this anisotropy is small in cubic crystals and we neglect it. Moreover, if the crystal is magnetized along one of the axes of easy magnetization the anisotropy can be taken into account by adding to the field H the anisotropy field H_A.

Taking the exact dispersion law (1) into account leads to a number of interesting effects in neutron scattering. In particular, it turns out that the scattering involving the emission of a spin wave proceeds in a narrower range of angles than the scattering involving absorption, and as a result a strong dependence of the scattering cross section on the polarization of the incident neutrons arises and also when an unpolarized beam is scattered appreciable polarization appears. Moreover, for sufficiently low neutron energies it turns out that single-quantum scattering is not possible at all.

The expression for the scattering cross section can easily be obtained by the standard method[2] if we use the expressions connecting the atomic spin operators with the spin wave absorption and emission operators (see, e.g., [3]). Omitting the corresponding rather long, albeit simple, calculations we give at once the result

$$d\sigma_\pm / d\Omega dE' = {}^1\!/_2 N s r_0^2 \gamma^2 (n_q + {}^1\!/_2 \pm {}^1\!/_2)\{(u_q^2 + |v_q|^2) \times [1 + (\mathbf{em})^2] \pm 2(\mathbf{em})(\mathbf{eP}_0)\}\delta(E' - E \pm \varepsilon_q). \quad (2)$$

Here N is the number of magnetic atoms in the scatterer, s their spin, r_0 the classical electron radius, γ the neutron magnetic moment in nuclear magnetons, $\mathbf{q} = \mathbf{p} - \mathbf{p}'$, $\mathbf{e} = \mathbf{q}/q$, **m** the direction of the magnetization in the scatterer, $\mathbf{P}_0$ the polarization vector of the incident neutrons,

$$u_q = \left(\frac{A_q + \varepsilon_q}{2\varepsilon_q}\right)^{1/2}, \quad v_q = -\frac{B_q}{|B_q|}\left(\frac{A_q - \varepsilon_q}{2\varepsilon_q}\right)^{1/2},$$

$$\varepsilon_q = (A_q^2 - |B_q|^2)^{1/2},$$

$$A_q = A_q^2 + 2\mu H + 4\pi\mu M_0 \sin^2\vartheta_q,$$

$$B_q = 4\pi\mu M_0 \sin^2\vartheta_q \exp(2i\varphi_q). \qquad (3)$$

In (2) and in the following the upper sign refers to the scattering involving emission and the lower sign to scattering with spin wave absorptions. In deriving (2) the magnetic formfactor, the Debye-Waller factor and the ratio p′/p were taken to be equal to unity since we are interested in small angle and small energy-transfer scattering.

The expression for the polarization of the scattered neutrons can also be obtained in the standard way and has the form

$$\mathbf{P}_\pm = \frac{\{2[\mathbf{P}_{0\perp} - \mathbf{e}_\perp(\mathbf{P}_0\mathbf{e}) + \mathbf{e}(\mathbf{em})(\mathbf{MP}_0)] - \mathbf{P}_0[1 + (\mathbf{em})^2]\}(u_q^2 + |v_q|^2) \mp 2\mathbf{e}(\mathbf{em})}{[1 + (\mathbf{em})^2](u_q^2 + |v_q|^2) \pm 2(\mathbf{P}_0\mathbf{e})(\mathbf{em})}; \quad \mathbf{a}_\perp = \mathbf{a} - (\mathbf{am})\mathbf{m}, \quad \mathbf{M} = \mathbf{m} - (\mathbf{em})\mathbf{e}. \qquad (4)$$

The next problem for us is the analysis of the angular distribution and the total polarization of the scattered neutrons, determined by the formula

$$\mathbf{P}(\vartheta, \varphi) = \frac{\sigma_+(\vartheta,\varphi)\mathbf{P}_+(\vartheta,\varphi) + \sigma_-(\vartheta,\varphi)\mathbf{P}_-(\vartheta,\varphi)}{\sigma_+(\vartheta,\varphi) + \sigma_-(\vartheta,\varphi)}, \qquad (5)$$

where φ is the azimuthal angle, which conveniently is reckoned from the projection of the magnetic field on the plane perpendicular to the incident beam; it is clear that if the field is parallel to the beam there is no dependence on φ.

The character of the angular distribution is mainly determined by the energy conservation law occurring in (2) which we can write in the form

$$(2 + x)x \pm \alpha[(x^2 + (1 + x)\vartheta^2 + a)(x^2 + (1 + x)\vartheta^2 + a + b\sin^2\vartheta_q)]^{1/2} = 0;$$

$$x = (p' - p)p^{-1}, \quad a = 2\mu H(\alpha E)^{-1}, \quad b = 8\pi\mu M_0(\alpha E)^{-1}, \qquad (6)$$

where $\sin^2\vartheta_q$ depends on the angle between the magnetic field and the incident beam and also on ϑ and φ.

We restrict our consideration to two cases: 1) the magnetic field is parallel to the incident beam and 2) the magnetic field is at right angles to the incident beam. Simple calculations lead to the following results:

$$1) \quad \sin^2\vartheta_q = \frac{(1 + x)\vartheta^2}{x^2 + (1 + x)\vartheta^2},$$

$$2) \quad \sin^2\vartheta_q = 1 - \frac{(1 + x)\vartheta^2}{x^2 + (1 + x)\vartheta^2}\cos^2\varphi. \qquad (7)$$

Using (7) we get from Eq. (6) an equation of sixth degree, the solution of which can not be obtained in general form. Our problem is thus to study qualitatively the behavior of its roots. To do this it is convenient to write (6) in the form

$$y(x, \vartheta^2) = z(x, \vartheta^2),$$

$$y(x, \vartheta^2) = [x^2 + (1 + x)\vartheta^2 + a]^2 - (2 + x)^2x^2/\alpha^2,$$

$$z(x, \vartheta^2) = -b[x^2 + (1 + x)\vartheta^2 + a]\sin^2\vartheta_q. \qquad (8)$$

The function $y(x, \vartheta^2)$ has the roots

$$x_{0j}^{(\pm)} = \frac{\mp(1 \pm a\vartheta^2/2) + j[\alpha^{-2} - \vartheta^2 - a(1 \pm \alpha^{-1})]^{1/2}}{\alpha \pm 1}, \qquad (9)$$

where $j = \pm 1$. It follows from this formula that $y(x, \vartheta^2) < 0$ if x lies in one of two intervals:

$$x_{0-}^{(+)} < x < x_{0+}^{(+)} < 0, \quad 0 < x_{0-}^{(-)} < x < x_{0+}^{(-)},$$

and this inequality can only occur if the $x_{0j}^{(\pm)}$ are real. Therefore, if $x < 0$, the function $y(x, \vartheta^2)$ can only be negative if

$$\vartheta_{0+}^2 = \alpha^{-2} - a(1 + 1/\alpha) > \vartheta^2 > 0,$$

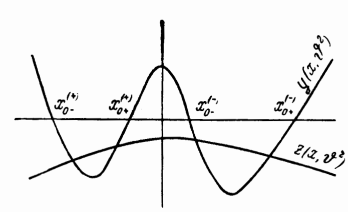
and when $x > 0$, if

$$\vartheta_{0-} = \alpha^{-2} - a(1 - 1/\alpha) > \vartheta^2 > 0.$$

At the same time $z(x, \vartheta^2) < 0$.[1)] The y- and z-curves can thus intersect only if x lies in one of the above mentioned intervals. The situation occurring here is schematically shown in the figure. Moreover, if $\vartheta_{0+}^2 < 0$, $y > 0$ for all $x < 0$, while if $\vartheta_{0-}^2 < 0$, $y > 0$ for all $x > 0$. Scattering is thus possible only if at least one of the quantities $\vartheta_\pm^2$ is positive.

The further discussion must be given separately for the cases 1) and 2).

Case 1). It follows from (7) and (8) that when



1)We are only interested in small scattering angles. One shows easily that if $\vartheta^2 \ll 1$, $z(x, \vartheta^2) < 0$ for all x.

ϑ^2 increases the curve $y(x, \vartheta^2)$ is raised and the curve $z(x, \vartheta^2)$ lowered, i.e., for $\vartheta^2 = 0$ the y-curve has its lowest position and the z-curve its highest position, coinciding with the x-axis. (Of course, with increasing ϑ^2 the shape of the y- and z-curves also changes but this is not important to us.)

As we mentioned already, when $\vartheta_{0+}^2 > 0$ the y-curve comes into the lower half-plane when $x < 0$ if $\vartheta^2 < \vartheta_{0+}^2$; for small ϑ^2 the z-curve lies close to the x-axis and therefore one can always find sufficiently small ϑ^2 for the y- and z-curves to intersect, provided $\vartheta_{0+}^2 > 0$. Bearing in mind the connection between a and the energy, we get from this inequality the condition that scattering with the emission of a spin wave takes place:

$$E > E_{a+} = 2\mu H(\alpha + 1). \tag{10}$$

Similarly, scattering involving the absorption of a spin wave is possible, provided

$$E > E_{a-} = 2\mu H(\alpha - 1). \tag{11}$$

When $x > 0$ the y- and z-curves intersect, generally speaking in two points. This follows from the fact that in the left half-plane the function y has one minimum while z monotonically decreases with decreasing x. However, these curves intersect only for sufficiently small ϑ^2. There exists thus a limiting angle $\vartheta_+ < \vartheta_{0+}$ such that when $\vartheta > \vartheta_+$ scattering involving emission is impossible. It is clear that when $\vartheta = \vartheta_+$ the y- and z-curves touch one another. Similarly, when $x > 0$ the y- and z-curves intersect also in two points and there exists an angle $\vartheta_- < \vartheta_{0-}$ such that for angles $\vartheta > \vartheta_-$ scattering involving absorption is impossible.

We show in Appendix I that $\vartheta_- > \vartheta_+$. It follows from (9) that the roots of the equation $y = z$ are of order $1/\alpha$. Moreover, it is clear that scattering is possible only if $a \lesssim \alpha^{-2}$. Assuming that b is a quantity of the same order of magnitude we reach the conclusion that $\vartheta_\pm \sim \alpha^{-1}$ and that the difference between them is of the order α^{-2}. However, for energies close to E_{a+} when ϑ_+ is anomalously small, ϑ_- may turn out to be of the order of $\alpha^{-3/2}$. The difference $\vartheta_- - \vartheta_+$ will then be of the same order of magnitude. This is connected with the fact that ϑ_-^2 and ϑ_+^2 differ from one another by terms of the order α^{-3} and when ϑ_+^2 tends to zero, at the same time the terms of order α^{-2} in ϑ_-^2 also tend to zero.

We now consider how the cross section depends on the polarization of the incident neutrons. We shall assume that the neutron polarization vector $\mathbf{P}_0$ is parallel or antiparallel to the field, i.e., that

$$\mathbf{P}_0 = P_0\mathbf{m}, \quad -1 \leqslant P_0 \leqslant 1.$$

Otherwise, $\mathbf{P}_0$ will rotate around the field and the experimental study of polarization effects is made much more difficult. We restrict ourselves to the case of sufficiently small $\vartheta_\pm$ when $\sin^2\vartheta_q \ll 1$. The quantities $(\mathbf{e}\cdot\mathbf{m})^2 = \cos^2\vartheta_q$ and $u_q^2 + |v_q|^2$ occurring in (2) are then close to unity and thus

$$d\sigma_\pm / d\Omega \sim 2(1 \pm P_0).$$

The scattering cross section involving emission is thus increased when $P_0 > 0$, while the scattering cross section involving spin wave absorption decreases. In particular, using completely polarized neutrons we can experimentally study those two cross sections separately. Moreover, it is clear that the strongest polarization dependence will be in the range of angles $\vartheta_+ < \vartheta < \vartheta_-$.

Let now the incident neutrons be unpolarized. It follows from (4) that in the range of angles $\vartheta_+ < \vartheta < \vartheta_-$ where there is no cross section with emission the scattered neutrons will become strongly polarized and the component of their polarization along the magnetic field will be proportional to $\cos^2\vartheta_q$ so that for small ϑ_q the polarization $\mathbf{P} \approx \mathbf{m}$. In the region of angles $\vartheta < \vartheta_+$ the polarization of the scattered neutrons turns out to be proportional to the difference in the scattering cross sections with emission and with absorption of spin waves.

The above-mentioned effects can clearly be observed if the spread in the energy of the incident neutrons as to order of magnitude does not exceed E/α.

Case 2). In this case

$$z(x, \vartheta^2) = -b[x^2 + (1+x)\vartheta^2 + a]\left[1 - \frac{\vartheta^2(1+x)\cos^2\varphi}{x^2 + (1+x)\vartheta^2}\right]. \tag{12}$$

Let the y- and z-curves intersect for some ϑ^2. When ϑ^2 increases (in the case where $\vartheta^2 > \vartheta_{0+}^2$ for $x < 0$ and in the case when $\vartheta^2 > \vartheta_{0-}^2$ for $x > 0$) the y-curve will turn out to be completely above the x-axis while the z-curve for all ϑ^2 lies in the lower half-plane. It is thus clear that for sufficiently large ϑ^2 the y- and z-curves cease to intersect, i.e., in this case scattering is only possible in a limited range of angles ϑ. The function $z(x, \vartheta^2)$ both when $x > 0$ and when $x < 0$ is a monotonic function of x and there are thus again, generally speaking, up to two intersections both in the left and in the right half-plane. When $\cos^2\varphi = 1$, both curves move upwards when ϑ^2 increases. In principle there is thus a possibility that the y- and z-curves do not intersect when $\vartheta^2 = 0$, while with increasing ϑ^2 the z-curve overtakes the y-curve and intersection occurs.

One can show, however, that this possibility is not realized.

Thus, scattering is only possible, if the equation

$$y(x, 0) = z(x, 0)$$

has real roots (this equation is independent of φ). Neglecting x in the expressions for y and z compared to unity, we get the following condition for the reality of the roots:

$$2/\alpha^2 > a + b/2,$$
$$(2/\alpha^2 - a - b/2)^2 > a(a+b). \qquad (13)$$

Bearing in mind the connection between a and b and the energy we get from (13) the following condition for the occurrence of scattering:

$$E > E_b = \alpha\mu H[1 + 2\pi M_0/H + (1 + 4\pi M_0/H)^{1/2}]. \qquad (14)$$

A little later we give a more exact condition for the limiting energy when it turns out that, as in case 1), $E_{b+} > E_{b-}$. In Appendix I we show that also in that case $\vartheta_+ < \vartheta_-$ i.e., there is again a range of angles where only scattering involving spin wave absorption occurs. All we have said earlier about the order of magnitude of the angles $\vartheta_\pm$ and their difference is now also valid. We note further that the $\vartheta_\pm$ depend on φ.

We now consider how the cross section depends on the polarization of the incident neutrons. We limit ourselves merely to two cases: $\cos^2\varphi = 1$ and $\cos^2\varphi = 0$, i.e., to scattering in a plane containing the magnetic field and in a plane at right angles to the field. As before, we can then assume that the incident neutrons are polarized parallel or antiparallel to the field. Let $\cos^2\varphi = 1$. Then

$$(\mathbf{em})^2 = \cos^2\vartheta_q \approx \vartheta^2/(x^2 + \vartheta^2)$$

and, according to (2), the cross section is proportional to

$$(1 + \cos^2\vartheta_q)(u_q^2 + |v_q|^2) \pm 2P_0\cos^2\vartheta_q.$$

The cross section depends thus strongly on the polarization at large scattering angles ($\vartheta \sim \vartheta_\pm$) and this dependence disappears in the range of small angles ($\vartheta \ll \vartheta_\pm$). In particular, changing the sign of the polarization may strongly change the scattering cross section in the range of angles $\vartheta_+ < \vartheta < \vartheta_-$. Moreover, according to (4), when an unpolarized beam is scattered in the range of angles $\vartheta_+ < \vartheta < \vartheta_-$ an appreciable polarization may arise, and the component of the polarization along the magnetic field is proportional to

$$\cos^2\vartheta_q[(1 + \cos^2\vartheta_q)(u_q^2 + |v_q|^2)]^{-1}.$$

For angles less than ϑ_+ the polarization due to the scattering must strongly decrease, as follows from (4) and (5). Of course, one can also in this case observe these effects only if the spread in energy of the incident neutrons does not exceed E/α in order of magnitude.

Recently, Drabkin et al.[4] have studied experimentally the scattering of neutrons in a plane, containing the magnetic field. In their experiments, the scatterer was an iron single crystal, the magnetic field was 26 kG and the neutron energy was in the interval 110—130° K corresponding to wavelengths from 2.7 to 2.9 Å. They observed a relatively large scattering up to angles of about 25′. A very strong dependence of the cross sections on the polarization was observed in a range of angles from about 18—20 to 25′. In the same interval of angles an unpolarized beam became strongly polarized (of the order of 30%) on being scattered. If we assume that the observed effects are caused by scattering involving absorption or emission of a single spin wave,[2)] we can conclude about the magnitude of α. Indeed, substituting into (14) the above-mentioned value of the magnetic field and $M_0 = 1.7$ kG (the magnetization of iron at room temperature) we get $E > 4.8$ [°K] and thus, assuming that $E \approx 120$° K, we get $\alpha < 25$. This estimate is also in agreement with the range of angles in which strong polarization effects are observed.

Lowde and co-workers[6,7] have also studied small angle scattering in ferromagnetics. In their papers they describe experiments on the scattering of very fast, non-monochromatic neutrons ($\lambda > 1.1$ Å). By reducing the experimental data and assuming a quadratic dispersion law, they found $\alpha \approx 130$. Since high-energy neutrons were used in these experiments, an account of the corrections necessitated by the magnetic interactions in the reduction of experimental data can not strongly change the magnitude of α.

We must note that the value $\alpha \approx 130$ agrees well with data on the exchange integral obtained by other means. Thus, a study of the $T^{3/2}$ law for the spontaneous magnetization in pure iron leads according to Argyle et al.[8] to $\alpha \approx 130$ to 140. The data given in that paper on the exchange integral, which were obtained by means of ferromagnetic resonance, agree within experimental errors with such a value of α. In the same paper a value was given for the exchange integral for

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2)Processes involving two or more spin waves have a negligible cross section because of the smallness of the phase volume for the additional spin waves. This was shown by the present author[5] for the case of a quadratic dispersion law.

iron with 3% silicon impurities. In that case the exchange integral (and thus also α) turns out to be about 10% less than in pure iron.

The reason for the divergence in the results of the above-mentioned experimental papers is so far not clear. The problem requires further experimental study.

We now consider the scattering in a plane at right angles to the field. In that case the vector **m** is perpendicular to the vector **e** and therefore the cross section is independent of the polarization. Moreover, polarization cannot occur when an unpolarized beam is scattered, and if the incident neutrons are polarized along the field one sees easily by using (4) that on scattering the polarization vector turns around the beam over 180°, i.e., $\mathbf{P} = -\mathbf{P}_0$. Furthermore, since $\cos^2\varphi = 0$, $\sin^2\vartheta_q = 1$. Therefore, the equation

$$y(x, \vartheta^2) = z(x, \vartheta^2)$$

becomes a fourth-degree equation. It is solved in Appendix II. According to Eq. (II.7) we have for the limiting angles

$$\vartheta_\pm^2 = \frac{1}{\alpha^2} + \frac{\alpha^2 b^2}{16} - \left(a + \frac{b}{2}\right) \mp \left(a + \frac{b}{2} - \frac{b^2\alpha^2}{8}\right)\left(\frac{1}{\alpha^2} - \frac{b^2\alpha^2}{16}\right)^{1/2}. \qquad (15)$$

It is at once clear that $\vartheta_+^2 < \vartheta_-^2$. In this formula the first three terms are quantities of the order of α^{-2} and the last one of the order α^{-3} so that $\vartheta_\pm \sim \alpha^{-1}$ and $\vartheta_- - \vartheta_+ \sim \alpha^{-2}$; however, if the terms of order α^{-2} in (15) practically completely cancel one another, the difference $\vartheta_- - \vartheta_+$ may turn out to be of order $\alpha^{-3/2}$. Such a situation arises clearly for energies close to the limiting one. The limiting energies $E_{b\pm}$ are determined from the condition $\vartheta_\pm^2 = 0$ and are easily found by the method of successive approximations:

$$E_{b\pm} = E_b\left\{1 \pm \frac{\alpha E_b}{2\mu H}\,\frac{[\alpha^{-2} - (2\pi\mu M_0/E_b)^2]^{3/2}}{(1 + 4\pi M_0/H)^{1/2}}\right\}. \qquad (16)$$

As one should expect, the difference

$$E_{b+} - E_{b-} \sim E_b\alpha^{-1}.$$

We bear in mind that we have shown above that the limiting energy is independent of φ.

Although we know $x_j^{(\pm)}$ in the case considered, the equations for $d\sigma_\pm/d\Omega$ are practically unmanageable. However, in the case when in Eq. (II.7) of Appendix II we can neglect the correction terms, the cross section has the comparatively simple form:

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma_+}{d\Omega} + \frac{d\sigma_-}{d\Omega} = Nr_0^2\gamma^2 s\left[a + \frac{b}{2} - \frac{\alpha^2 b^2}{8} + \vartheta^2\right] \times\left[\frac{1}{\alpha^2} - a - \frac{b}{2} + \frac{\alpha^2 b^2}{16} - \vartheta^2\right]^{-1/2} \times\left[\left(a + \frac{b}{2} + \vartheta^2\right)^2 - \frac{1}{4}b^2\right]^{-1}\frac{T}{\alpha^2 E}. \qquad (17)$$

In deriving (17) we assumed that $E \ll T\alpha$. It is clear that if that condition is satisfied the cross section is proportional to the temperature also in the general case.

The radicand in (17) contains the quantity $\vartheta_0^2 - \vartheta^2$, where ϑ_0 is the limiting scattering angle so that $d\sigma/d\Omega$, as $\vartheta \to \vartheta_0$, becomes infinite as $(\vartheta_0 - \vartheta)^{-1/2}$; the total cross section remains then, of course, finite. The appearance of this infinity is caused by the fact that as $\vartheta \to \vartheta_0$ two roots in the argument of the δ-function in (2) approach one another and therefore the result of the integration of (2) over E' turns out to be inversely proportional to the distance between these roots, i.e., $(\vartheta_0 - \vartheta)^{1/2}$.

A similar case arises, clearly also in the general case where we can show that the character of the singularity is retained, i.e., $d\sigma_\pm/d\Omega \sim (\vartheta_\pm - \vartheta)^{-1/2}$. This must be borne in mind for numerical calculations.

In conclusion the author expresses his gratitude to G. M. Drabkin, E. I. Zabidarov, Ya. A. Kaeman, A. I. Okorokov, and V. A. Trunov for a large number of interesting discussions without which this paper could not have been written, and also to A. Klochikhin and A. D. Piliya, with whom he has discussed a number of problems touched upon in the paper.

APPENDIX I

The aim of this appendix is to prove that $\vartheta_+ < \vartheta_-$.

Case 1). Let $\vartheta^2 = \vartheta_+^2$. This means that the y- and z-curves touch at $x = x_1 < 0$. It is necessary to explain why these curves intersect for $x > 0$ or not. If they intersect, $\vartheta_-^2 > \vartheta_+^2$, since with increasing ϑ^2 the y-curve goes up and the z-curve down. We split the y- and z-functions in parts even in x (y_1 and z_1) and parts odd in x (y_2 and z_2). Assuming that $x \ll 1$, we get

$$y_1 = (x^2 + \vartheta^2 + a)^2 - 4x^2/\alpha^2,$$

$$z_1 = -b[1 + a/(x^2 + \vartheta^2)]\vartheta^2;$$

$$y_2 = x\eta = 2x\vartheta^2(x^2 + \vartheta^2 + a) - 4x^3/\alpha^2,$$

$$z_2 = x\xi = -bx\vartheta^2[1 + ax^2/(x^2 + \vartheta^2)^2].$$

In the point of contact

$$y(x_1, \vartheta_+^2) = z(x_2, \vartheta_+^2).$$

It is clear that when $x > 0$ the curves intersect if

$$y(|x_1|, \vartheta_+^2) < z(|x_1|, \vartheta_+^2).$$

This follows from the fact that when $x > 0$ the y-curve has one minimum while the z-curve is monotonic. Since $x \ll 1$, the above inequality has the form $\eta < \xi$ under the additional conditions

$$y_1 = z_1, \qquad dy_1 / dx = dz_1 / dx$$

(the last of these equations is due to the fact that the curves touch). The additional conditions are of an approximate nature but as y_2, $z_2 \ll y_1$, z_1 this is not important.

We can thus show that

$$2\vartheta^2(x^2+\vartheta^2+a) - 4x^2/\alpha^2 < -b\vartheta^2[1 + ax^2/(x^2+\vartheta^2)^2] \quad \text{(I.1)}$$

under the conditions

$$(x^2+a+\vartheta^2)^2 - 4x^2/\alpha^2 = -b[1 + a/(x^2+\vartheta^2)]\vartheta^2, \quad \text{(I.2)}$$

$$2(x^2+a+\vartheta^2) - 4/\alpha^2 = ab\vartheta^2/(x^2+\vartheta^2)^2. \quad \text{(I.3)}$$

Using condition (I.2) to eliminate b from (I.1) and (I.3) we have:

$$-(x^2+a+\vartheta^2)^2(x^4+x^2a-\vartheta^4) < 4x^2a\vartheta^2/\alpha^2, \quad \text{(I.1}')$$

$$(x^2+\vartheta^2+a)^2(x^2+\vartheta^2+a/2) - 2(x^2+\vartheta^2+a)(x^2+\vartheta^2)/\alpha^2 = 2x^2a/\alpha^2. \quad \text{(I.3}')$$

Substituting then $2x^2a/\alpha^2$ from (I.3′) into (I.1′) we obtain the inequality

$$4\vartheta^2/\alpha^2 < (x^2+\vartheta^2+a)^2. \quad \text{(I.4)}$$

However, according to (I.3)

$$x^2+\vartheta^2+a = 2/\alpha^2 + \Delta \quad (\Delta > 0),$$

so that (I.4) is certainly satisfied if $\vartheta^2 < 1/\alpha^2$, but we know that

$$\vartheta^2 \leqslant 1/\alpha^2 - a(1 \pm 1/\alpha).$$

Case 2). By complete analogy with the previous subsection we obtain the following conditions which must be satisfied for the intersection of the y- and z-curves for $x > 0$, if for $x < 0$ these curves touch one another:

$$2(\vartheta^2+x^2+a)\vartheta^2 - \frac{4x^2}{\alpha^2} < \frac{b\vartheta^2}{(x^2+\vartheta^2)^2}[ax^2 + (a\vartheta^2 - (x^2+\vartheta^2)(x^2+\vartheta^2+a))\sin^2\varphi], \quad \text{(I.5)}$$

$$(x^2+\vartheta^2+a)^2 - \frac{4x^2}{\alpha^2} = -b\,\frac{x^2+\vartheta^2+a}{x^2+\vartheta^2}(x^2+\vartheta^2\sin^2\varphi), \quad \text{(I.6)}$$

$$2(x^2+\vartheta^2+a) - \frac{4}{\alpha^2} = -b\left[1 + \frac{a\vartheta^2\cos^2\varphi}{(x^2+\vartheta^2)^2}\right]. \quad \text{(I.7)}$$

Substituting $4x^2/\alpha^2$ from (I.6) into (I.5) and (I.7) we get

$$(x^2+\vartheta^2+a)(\vartheta^2-x^2-a) < \frac{b}{x^2+\vartheta^2}\left\{x^2(x^2+\vartheta^2+a) + \frac{a\vartheta^2(x^2+\vartheta^2\sin^2\varphi)}{x^2+\vartheta^2}\right\}, \quad \text{(I.5}')$$

$$(x^2+\vartheta^2+a)(x^2-\vartheta^2-a) = \frac{b}{(x^2+\vartheta^2)^2}\{ax^4 + a\vartheta^2x^2\sin^2\varphi + (x^2+\vartheta^2+a)(x^2+\vartheta^2)\vartheta^2\sin^2\varphi\}. \quad \text{(I.7}')$$

It follows from (I.7′) that $x^2 > \vartheta^2 + a$ whence we get again that the left-hand side of (I.5′) is negative but since the right-hand side is positive, the inequality is proven.

In the case considered we can show that with increasing ϑ^2 both the y- and the z-curve move upwards. In particular this occurs when $\cos^2\varphi = 1$. However, the y-curve rises then faster than the z-curve and overtakes it. That this is, indeed, the case follows from the inequality

$$dy_1/d\vartheta^2 > dz_1/d\vartheta^2,$$

which is easily proven from the conditions (I.6) and (I.7).

APPENDIX II

Here we look for the solution of Eq. (6) in the case when $\sin^2\vartheta_q = 1$, i.e., of the equation

$$x(2+x) = \mp\alpha\{[x^2 + (1+x)\vartheta^2 + C + D] \times [x^2 + (1+x)\vartheta^2 + C - D]\}^{1/2}, \quad \text{(II.1)}$$

where $C = a + b/2$, $D = b/2$. When $D = 0$ this equation splits into two quadratic ones the roots of which are

$$x_{0j}^{(\pm)} = \delta_\pm + jR_\pm; \qquad j = \pm 1,$$

$$\delta_+ = -\frac{1+\alpha\vartheta^2/2}{\alpha+1}, \qquad \delta_- = \frac{1-\alpha\vartheta^2/2}{\alpha-1},$$

$$R_\pm = \frac{\alpha}{\alpha\pm 1}\left[\frac{1}{\alpha^2} - \vartheta^2 - C\left(1 \pm \frac{1}{\alpha}\right)\right]^{1/2}. \quad \text{(II.2)}$$

Taking the square of Eq. (II.1) we get a fourth degree equation which we can by means of (II.2) write in the form

$$y^4 - y^2(R_+^2 + R_-^2 + 2\Delta^2) + 2\Delta y(R_+^2 - R_-^2) + (\Delta^2 - R_+^2)(\Delta^2 - R_-^2) - D^2\alpha^2/(\alpha^2-1) \equiv y^4 + py^2 + qy + r = 0,$$

$$y = x - (\delta_+ + \delta_-)/2, \qquad \Delta = (\delta_- - \delta_+)/2. \quad \text{(II.3)}$$

To solve the fourth degree Eq. (II.3) we must find one of the solutions of the auxiliary cubic equation (see, e.g., Kurosh's book[9]):

$$z^3+pz^2+(p^2/4-r)z-q^2/8\equiv z^3-(R_+^2+R_-^2+2\Delta^2)z^2 + {}^1/_4[(R_+^2+R_-^2+2\Delta^2)-4(\Delta^2-R_+^2)(\Delta^2-R_-^2) + 4D^2\alpha^2/(\alpha^2-1)]z-\Delta^2(R_+^2-R_-^2)^2/2=0. \quad (II.4)$$

After that Eq. (II.3) reduces to two quadratic equations:

$$y^2\pm 2\sqrt{z/2}\,y+p/2+z\mp q/2\sqrt{2z}=0. \quad (II.5)$$

We assume that C and D are quantities of order $1/\alpha^2$ and, moreover, that $\vartheta^2 \lesssim 1/\alpha^2$. One then verifies easily that the coefficients of Eq. (II.4) have the following order of magnitude:

$$p\sim\alpha^{-2}+O(\alpha^{-4}),\quad p^2/4-r\sim\alpha^{-4},\quad q^2\sim\alpha^{-8},$$

whence we get immediately that at least one of the roots of (II.4) must be of order $1/\alpha^2$ and to evaluate it we can neglect the free terms. As a result we get

$$z_\pm={}^1/_2\{R_+^2+R_-^2+2\Delta^2\pm 2[(\Delta^2-R_+^2)(\Delta^2-R_-^2) - D^2\alpha^2/(\alpha^2-1)]^{1/2}\}\approx {}^1/_2\alpha^{-2}-{}^1/_4(\vartheta^2+C) \pm {}^1/_4[(\vartheta^2+C)^2-D^2]^{1/2}. \quad (II.6)$$

The approximate equality in the right-hand side of (II.6) occurs if we neglect terms of order α^{-4}. One can also easily verify that the corrections to $z_\pm$ caused by the free terms in (II.4) are also of order α^{-4}. Since, by definition, C > D, $z_\pm$ are real for all ϑ^2. Moreover, if $\vartheta^2 + C < 2/\alpha^2$, $z_+ > 0$, and furthermore, $z_+ > |z_-|$. In (II.5) we must substitute the solution of (II.4) which is of order of magnitude $1/\alpha^2$. It is clear that in that case z_+ is such a solution. Similarly, when $\vartheta^2 + C > 2/\alpha^2$, z_- is a proper solution, since in that case $z_- < 0$. Because of this fact the solutions of Eqs. (II.5) are now complex, i.e., when $\vartheta^2 + C > 2/\alpha^2$, scattering is not possible. In particular, it is impossible when $C > 2/\alpha^2$.

Substituting z_+ into (II.5) and solving the equations obtained, we get

$$x_j^{(\pm)}=\mp\left\{\frac{1}{\alpha^2}-\frac{\vartheta^2+C}{2}+\frac{[(\vartheta^2+C)^2-D^2]^{1/2}}{2}\right\}^{1/2}+\frac{1-\alpha^2\vartheta^2}{\alpha^2} + j\left\{\frac{1}{\alpha^2}-\frac{\vartheta^2+C}{2}-\frac{[(\vartheta^2+C)^2-D^2]^{1/2}}{2} \pm\left[2\left(\vartheta^2+C-\frac{1}{\alpha^2}\right)-C\right] \times\left[\frac{1}{\alpha^2}-\frac{\vartheta^2+C}{2}+\frac{[(\vartheta^2+C)^2-D^2]^{1/2}}{2}\right]^{-1/2}\right\}^{1/2}. \quad (II.7)$$

We verify easily that when D = 0 Eq. (II.7) turns into (II.2) if in both formulae we expand in powers of $1/\alpha$ and limit ourselves to the first two terms in such an expansion.

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