ON THE AXIOMATIC CONSTRUCTION OF THE SCATTERING MATRIX
3. THE HEISENBERG AND AXIOMATIC REPRESENTATIONS

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The concept of Heisenberg field operators is introduced and rules for the transition from the asymptotic representation to the Heisenberg representation are established. The relation between the axiomatic theory developed here and the theory of Lehmann, Symanzik, and Zimmermann [9] and the usual Lagrangian method for constructing the scattering matrix is investigated.

1. INTRODUCTION

In the last years much attention has been paid to the "axiomatic" approach to field theory, which is based on a certain system of basic physical assumptions or "axioms" instead of equations of motions. A system of this type has been proposed by Bogolyubov, Polivanov, and the author [1] 1) to establish the dispersion relations, and was further developed by the author [2,3] 2) in the direction of a systematic S matrix theory. In this approach there was no need of the concept of Heisenberg operators; the basic representation chosen for the field $\varphi(x)$ (for simplicity, we consider first a self-interacting scalar field) and in which the S matrix was found, was some special representation, in which the fields satisfy the free equations of motion and commutation relations [cf. PTDR, (2.4.2)], but at the same time describe real particles, so that they already take all interactions into account [PTDR, (2.2.—4.)]. We shall call this the asymptotic representation, or more precisely (as defined by the choice of the sign of the time in the causality condition), the out representation.

However, it should be noted that, together with the out representation, certain operators in PTDR, I, and II were also used in the Heisenberg representation, such as the current operator $j(x)$ and all current-like operators Λ_ν introduced in I.

In the present paper we use this approach to establish the general rules for the transition from one representation to the other. These rules allow us to clarify the relation between our axiomatic method and other axiomatic approaches as well as the usual Lagrangian method of constructing the scattering matrix.

1) In the following referred to as PTDR.

2) These papers are referred to as I and II in the following. We use, without explanation, the notation of these and the preceding papers.

2. HEISENBERG FIELD OPERATORS

We begin with the introduction of the Heisenberg field operators in analogy with the usual theory, basing ourselves on the known representation of the Heisenberg field $A(x)$ in terms of the asymptotic field $\varphi(x) = A^{\text{out}}(x)$ and the S matrix:

$$A(x) = T_W(\varphi(x)S)\cdot S^+. \tag{1}$$

We use this expression as our definition of the Heisenberg field. To this end we must, of course, also define the meaning of the symbol $T_W(\varphi(x)S)$. We shall interpret it in the sense of the Wick theorem; i.e., we shall consider the S matrix as given by its expansion I, (10) in terms of normal products of the asymptotic fields and define the chronological product of $\varphi(x)$ with each of the terms of this expansion as the usual Wick sum for the T product of normal products (of free fields!).3)

We emphasize that the possible appearance of derivatives of the fields in the expansion I (10) must be described by the coefficient functions

3) We note that this definition of the T product is not complete. There is still the well known arbitrariness when the arguments coincide. For a complete definition, one must specify the character of the singularities which appear in this case, or more precisely, give the rules for the integration of a T product with functions which are not sufficiently regular when the arguments coincide. It is known from the usual expansion (cf.[4], Ch. IV) that this arbitrariness may be traced back to the counter-terms added to the interaction Lagrangian.

Φ^{ν} (these must then contain derivatives of δ functions); as is easily seen, it follows from this that the chronological products of derivatives are equal to derivatives of chronological products:

$$\left\langle 0 \left| T_W\left(\frac{\partial^n \varphi(x)}{\partial x^n} \frac{\partial^m \varphi(y)}{\partial y^m} \right) \right| 0 \right\rangle = \frac{\partial^n}{\partial x^n} \frac{\partial^m}{\partial y^m} \langle 0 | T_W(\varphi(x)\varphi(y)) | 0 \rangle. \tag{2}$$

It is clear that the so-defined Wick T product, T_W, by no means has to coincide with the T product of Heisenberg operators (currents and Λ_ν operators) introduced in I and II, which we shall call a Dyson product and provide with the index D when necessary. Indeed, in the case of two Heisenberg operators, the second type of product implies a chronological ordering with respect to the explicitly appearing time variables, whereas the first type of product involves an ordering with respect to times which enter implicitly through the functional dependence on the asymptotic fields. The difference even remains in the case of free fields if derivatives are involved: the Dyson product does not satisfy (2) [a derivative can be regarded as a functional of the field with a kernel of a special kind; therefore in the Wick product the fields are ordered, according to (2), whereas in the Dyson product the derivatives themselves are ordered].

After these remarks on the meaning of the T product we can expand the right-hand side of (1), using the partial Wick theorem:

$$A(x) = (-i) \int dy D^c(x-y) \frac{\delta S}{\delta \varphi(y)} S^+ + :\varphi(x) S: S^+, \tag{3}$$

where the symbol: φS: denotes the sum of normal products of $\varphi(x)$ with normal products of each term of the expansion I (10) with the corresponding coefficient functions. On the other hand, we can write the identity $\varphi(x) = \varphi(x) S \cdot S^+$ and expand its right-hand side in the same way:

$$\varphi(x) = \varphi(x) S \cdot S^+ = (-i) \times \int dy D^{(-)}(x-y) \frac{\delta S}{\delta \varphi(y)} S^+ + :\varphi(x) S: S^+. \tag{4}$$

However, this is only the case for the scattering matrix itself. If we consider any other operator, for example A(x), we see that it contains new combinations of chronological products which do not enter in the S matrix, and the "compensation of the arbitrariness" in the T product and the Lagrangian is destroyed. As a result there remains some arbitrariness in the Heisenberg fields for given values of the S matrix elements. This circumstance, which was first noted by Borchers,[5] was investigated in detail by Slavnov and Sukhanov.[6]

Subtracting this from (3) and using

$$D^c(x-y) - D^{(-)}(x-y) = D^{(adv)}(x-y), \tag{5}$$

we get rid of the troublesome term: $\varphi S : S^+$, and introducing the current under the integral sign, write the result in the form

$$A(x) = \varphi(x) - \int D^{(adv)}(x-y) j(y) dy. \tag{6}$$

This formula could also have been written down immediately by noting that, according to (5), the function $-iD^{(adv)}$ plays the role of a contraction in the Wick theorem for the expansion of chronological products in terms of ordinary products (it is easy to see that the Wick theorems also hold for the expansion of any product occurring in quantum field theory in terms of any others).

Formula (6) is identical in form with the Yang-Feldman equations [this may serve as an additional argument in favor of our definition (1)]. It must be noted, however, that it has a different meaning (otherwise it would already contain the proof for the identity of the axiomatic and Hamiltonian approaches). Indeed, in the Yang-Feldman theory[7] there is, besides Eq. (6) which expresses the field A(x) through the current j(y), another, "trivial" equation, which expresses the current j(x) in the form of a function (usually, a polynomial) of the field A(x) at the same point x. Only these two equations together form a closed system, whereas in our method the second equation has so far not been introduced. We shall therefore call (6) the Yang-Feldman relation.

The Yang-Feldman relation provides an immediate proof that the transformation (1) does not change the hermitian properties of the field: the Heisenberg form A(x) of a hermitian field $\varphi(x)$ is a hermitian operator. This is seen from the hermiticity of the current and the reality of the function $D^{(adv)}$. We note that this circumstance is by no means trivial—the original formula (1) for the transition from the out field to the Heisenberg field provides no basis whatsoever for predicting this result earlier.

The above-mentioned incomplete definition of the T product shows up in the Yang-Feldman relation in the circumstance that the current whose matrix elements need, according to PTDR II (1), exist only as generalized functions, is multiplied by the (again generalized) function $D^{(adv)}$. The resulting indeterminacy becomes apparent if we take the matrix element of this equation between the states $\langle (p)_l |$ and $|(q)_s\rangle$ (cf.[8]):

$$\langle (\mathbf{p})_l | A(x) | (\mathbf{q})_s \rangle = \langle (\mathbf{p})_l | \varphi(x) | (\mathbf{q})_s \rangle - \int D^{(adv)}(x - x') \langle (\mathbf{p})_l | j(x') | (\mathbf{q})_s \rangle dx'.$$

Displacing now the argument of the current from x′ to x with the help of the translation invariance and introducing the Fourier transform of the function $D^{(adv)}$, we obtain for it

$$\langle (\mathbf{p})_l | A(x) | (\mathbf{q})_s \rangle = \langle (\mathbf{p})_l | \varphi(x) | (\mathbf{q})_s \rangle - \tilde{D}^{(adv)}(-(P - Q)) \langle (\mathbf{p})_l | j(x) | (\mathbf{q})_s \rangle,$$

an expression which loses its meaning for $p^2 - Q^2 = m^2$, i.e., in particular, for $l = 0$, $s = 1$ or for $l = 1$, $s = 0$ (cf. the effect, discussed earlier,[8] of this fact on the meaning of the so-called "asymptotic condition"[9]).

The above-given rule for the transition to the Heisenberg representation is taken over to the spinor case practically without any changes. If we are dealing with one spinor out field $\bar{\psi}(x)$, $\psi(x)$, we define the Heisenberg fields $\psi(x)$, $\bar{\psi}(x)$ by the equations

$$\chi(x) = T_W(\psi(x)S)\cdot S^+, \quad \bar{\chi}(x) = T_W(\bar{\psi}(x)S)\cdot S^+, \quad (7)$$

in which the T product is again, of course, understood as a sum of Wick contractions. By the same method as in the boson case we obtain from the first of these definitions, using the spinor current $\iota(y)$ introduced in PTDR (5.4),[4]

$$\chi(x) = \psi(x) + \int dx' S^{(adv)}(x - x')\,\iota(x') \quad (8a)$$

the Yang-Feldman relation for the spinor field. The same steps lead to the relation for the adjoint field

$$\bar{\chi}(x) = \bar{\psi}(x) + \int dx' \bar{\iota}(x') S^{(ret)}(x' - x). \quad (8b)$$

It is easy to see that the operator $\bar{\psi}(x)$ is indeed the Dirac adjoint of the operator $\psi(x)$, i.e., the transformation (7) conserves, as in the boson case, the hermitian properties of the transformed field.

3. PRODUCTS OF HEISENBERG FIELDS

The rules for the transition from the out fields to the Heisenberg fields can be immediately generalized for arbitrary operators in the out representation. Let O^{out} be some out operator. We define its Heisenberg form by the transformation

$$O^H = T_W(O^{out}S)\cdot S^+, \quad (9)$$

where, of course, O^{out} must, just like the S matrix, be expanded in normal products of out fields, and the entire product (9) reduces to products of the separate terms of these expansions.

Expanding now the T product in (9) according to the Wick theorem for the expansion of chronological products in terms of ordinary products[5] and recalling the definition of the radiation operators in I, we obtain

$$O^H = O^{out} - i\int (dy)_1 (dz)_1 \frac{\delta O^{out}}{\delta\varphi(y_1)} D^{(adv)}(y_1 - z_1) S^{(1)}(z_1) + \ldots + \frac{(-i)^s}{s!}\int (dy)_s (dz)_s \frac{\delta^s O^{out}}{(\delta\varphi(y))_s} [D^{(adv)}(y - z)]_s \times S^{(1)}(z_1, \ldots, z_s) + \ldots \quad (10)$$

the Yang-Feldman relation for an arbitrary Heisenberg operator having an out prototype. The factor s! in the denominators comes from the circumstance that each combination with $[D^{(adv)}]_s$ is taken s! times in the successive differentiation and differs only by the order of factors in this product. If the operator O^{out} is quasilocal (in the sense of Bogolyubov and Shirkov[4]) all integrations over y are removed; if it is a polynomial in the out fields, then the series in s terminates after a finite number of terms, just as it terminates after the first term in the usual Yang-Feldman relation for the field itself.

Important examples for the application of the generalized Yang-Feldman relation are provided by certain special kinds of products of Heisenberg fields, which we shall now introduce.

Let us begin with the "quasi-Wick T product of Heisenberg fields," which we define by the formula[6]

[4]For the convenience of the printer we replace here the Georgian letter «in» in PTDR by the Greek letter «iota.»

[5]One could, of course, again take recourse to the stratagem used in the derivation of the Yang-Feldman relation, of a parallel expansion of the chronological and ordinary products in terms of normal products, but this would be too much computational work in this case.

[6]This definition is, of course, somewhat arbitrary, as indicated by the epithet «quasi Wick.» As a justification we recall that in the usual theory it is one of the possible forms of the definition of the T product of Heisenberg field operators which is particularly useful in the investigation of the Green's function.[10] It was this form which was used in PTDR in the study of the Källén-Lehmann spectral representations.

$$T_{QW}(A(x_1)\ldots A(x_n)) = T_W(\varphi(x_1)\ldots\varphi(x_n)S)\cdot S^+. \quad (11)$$

In order to transform this formula to a form which permits the application of (10), we note that the product of operators $\varphi(x)$ on the right-hand side of (11) can be included under the sign of an internal T product and the latter can be expanded in normal products by the usual Wick theorem. We thus arrive at the expansion

$$T_{QW}(A(x_1)\ldots A(x_n)) = N_Q(A(x_1)\ldots A(x_n)) + \sum_{i\neq j}\frac{-i}{2}D^c(x_i - x_j)N_Q(A(x_1)\ldots A(x_{i-1})A(x_{i+1}) \ldots A(x_{j-1})A(x_{j+1})\ldots A(x_n)) + \sum_{\substack{\text{contracted}\\ \text{pairs}}}\ldots + \ldots \quad (12)$$

of the entire quasi Wick product in terms of certain new products which we shall call quasinormal; they are defined by

$$N_Q(A(x_1)\ldots A(x_n)) = T_W(:\varphi(x_1)\ldots\varphi(x_n):S)\cdot S^+. \quad (13)$$

The quasinormal products already have the form required by (9). Applying (10) to them, we obtain the Yang-Feldman relation for quasinormal products of Heisenberg operators:

$$N_Q(A(x_1)\ldots A(x_n)) = :\varphi(x_1)\ldots\varphi(x_n): - iP\left(\frac{x_1}{x_2,\ldots,x_n}\right)\int dy_1 :\varphi(x_2)\ldots\varphi(x_n): D^{(adv)}(x_1 - y_1) \times S^{(1)}(y_1) + \ldots + (-i)^s P\left(\frac{x_1,\ldots,x_s}{x_{s+1},\ldots,x_n}\right)\int dy_1 \ldots dy_s :\varphi(x_{s+1})\ldots\varphi(x_n): D^{(adv)}(x_1 - y_1) \ldots D^{(adv)}(x_s - y_s) S^{(s)}((y)_s) + \ldots + (-i)^n\int dy_1\ldots dy_n D^{(adv)}(x_1 - y_1)\ldots D^{(adv)}(x_n - y_n)S^{(n)}(y_1,\ldots,y_n). \quad (14)$$

Turning now to the quasi Wick products (11), we easily see that if such a product is transformed into a sum of quasinormal products with the help of (12), and the Yang-Feldman relation (14) is used for each of the quasinormal products, then groups of terms appear with the same radiation operators and sets of advanced functions, which differ from the terms in (12) only by the replacement of part of the operators $\varphi(x)$ by chronological products of the free field. All such terms within each group can be summed, which leads precisely to the chronological products of free fields. As a result, the quasi Wick products of Heisenberg fields will obey generalized Yang-Feldman relations which are completely analogous to (14) except for the replacement of all normal products of out fields $\varphi(x)$ by chronological ones:

$$T_{QW}(A(x_1)\ldots A(x_n)) = T_W(\varphi(x_1)\ldots\varphi(x_n)) - iP\left(\frac{x_1}{x_2,\ldots,x_n}\right)\int dy_1 T_W(\varphi(x_2) \ldots\varphi(x_n))D^{(adv)}(x_1 - y_1)S^{(1)}(y_1) + \ldots + (-i)^s P\left(\frac{(x)_s}{(x)_{n-s}}\right)\int (dy)_s T_W(\varphi(x_{s+1}) \ldots\varphi(x_n))[D^{(adv)}(x-y)]_s S^{(s)}((y_s)) + \ldots + (-i)^n\int (dy)_n[D^{(adv)}(x-y)]_n S^{(n)}((y)_n). \quad (15)$$

It is instructive to operate on the Yang-Feldman relation for the quasinormal product (14) with the Klein-Gordon operator $\Box - m^2$ for each of the variables. All terms containing normal products will then be annihilated, and the functions $D^{(adv)}$ in the only remaining last term are transformed into δ functions, so that the integration becomes trivial. In this way we obtain an expression for the radiation operator in terms of Heisenberg fields:

$$i^n S^{(n)}(x_1,\ldots,x_n) = (\Box_1 - m^2)\ldots(\Box_n - m^2)N_Q(A(x_1) \ldots A(x_n)). \quad (16)$$

In particular, if we take account of I (12), the coefficient functions of the scattering matrix will also be expressed in terms of Heisenberg fields:

$$\Phi^{(n)}(x_1,\ldots,x_n) = (\Box_1 - m^2)\ldots(\Box_n - m^2)\langle 0|N_Q(A(x_1) \ldots A(x_n))|0\rangle. \quad (17)$$

The relations (16) and (17) permit a comparison of the present theory and the well-known formulation of axiomatic theory due to Lehmann, Symanzik, and Zimmermann.

4. CONNECTION WITH THE THEORY OF LEHMANN, SYMANZIK, AND ZIMMERMANN

In constructing a quantum field theory based on the system of axioms of Lehmann, Symanzik, and Zimmermann,[9] where the Heisenberg fields are the basic objects of the theory, the relation (17) plays a fundamental role for the determination of the coefficient functions of the scattering matrix. However, in this approach it is not proved for quasinormal products but for the so-called φ products, which are connected with the Dyson T products of Heisenberg fields by the same relations (12) which relate our quasinormal products to the quasi Wick products. The proof in [9] is essentially based on the asymptotic condition and is, from our point of view, not exhaustive.[8]

We see now that the validity of relation (17) in the form required of Lehmann et al.[9] would imply

$$T_D(A(x_1)\dots A(x_n)) = T_{QW}(A(x_1)\dots A(x_n))$$
$$\equiv T_W(\varphi(x_1)\dots\varphi(x_n)S)\cdot S^+ \qquad (18)$$

if only after application of n Klein-Gordon operators and averaging over the vacuum. Let us see now whether this relation is fulfilled.

The Dyson T product of Heisenberg operators on the left-hand side of (18) consists, according to the Yang-Feldman relation (6), of two parts of different type: the free field $\varphi(x)$ and a term which involves an integral over the current, i.e., $A(x) = \varphi(x) + A'(x)$. Their T product is therefore written in the form of a binomial expansion or, if $A'(x)$ is replaced by the integral over the current, in the form

$$T_D(A(x_1)\dots A(x_n)) = T_W(\varphi(x_1)\dots\varphi(x_n)) + \dots$$
$$+(-i)^s P\left(\frac{(x)_s}{(x)_{n-s}}\right)\int (dz)_s P_T(x_1,\dots,x_n)$$
$$\times (D^{(adv)}(x-z)S^{(1)}(z))_s\varphi(x_{s+1})\dots$$
$$\dots\varphi(x_n) + \dots + (-i)^n\int (dz)_n P_T(x_1,\dots,x_n)$$
$$\times [D^{(adv)}(x-z)S^{(1)}(z)]_n, \qquad (19)$$

where we have introduced the chronological symmetrizing operator $P_T(x_1,\dots,x_n)$, which acts on a product of operators depending on $x_1,\dots,x_n$ according to the rule

$$P_T(x_1,\dots,x_n)O_1(x_1)\dots O_n(x_n)$$
$$= P(1,\dots,n)\vartheta(x_1-x_2)\dots\vartheta(x_{n-1}-x_n)O_1(x_1)$$
$$\dots O_n(x_n). \qquad (20)$$

Here the operator $P(1,\dots,n)$ in (20) denotes the sum over all permutations of the arguments $x_1,\dots,x_n$ together with the operators depending on these.

In order to prove or disprove the validity of (18), one would have to proceed in the following way: transform, by permutation of operators, each term in (19) in such a way that the T products of the free fields stand to the left, which then can be replaced by Wick products. Then (19) would go over into an expansion constructed analogously to (15), and one would only have to compare the coefficients of each T product. This procedure would involve very complicated combinatorics. We shall not do this, using the fact that to disprove (18) it suffices to show that, for example, the current-like operator Λ_n enters in different ways in the coefficients of the T products of the free fields in (19) and (15).

In transforming (19) to a form analogous to (15) we must commute the operators $S^{(1)}$ with each other and with the free operators $\varphi(x)$. It is easy to see that only the last commutations will lead to an increase of the order. Therefore the current-like operator of n th order Λ_n can only come from the second term

$$P\left(\frac{x_1}{(x)_{n-1}}\right)(-i)P(1,\dots,n)\int dz_1\vartheta(x_1-x_2)$$
$$\dots\vartheta(x_{n-1}-x_n)(D^{(adv)}(x_1-z_1)S^{(1)}(z_1))\varphi(x_2)\dots\varphi(x_n)$$

in (19), which contains n-1 operators of the free field. Even in this case the required operator Λ_n is obtained only from those terms in the sum over commutations in which the time x_1 is larger than all other times, since only in these terms the operator $S^{(1)}$ will originally stand all the way to the left, and it takes exactly n-1 commutations to bring them into the form (15).

Thus we shall only be interested in the terms

$$(-i)P\left(\frac{x_1}{(x)_{n-1}}\right)\int dz_1\,\Theta(x_1;x_2,\dots,x_n)$$
$$\times D^{(adv)}(x_1-z_1)S^{(1)}(z_1)\ P(2,\dots,n)\vartheta(x_2-x_3)$$
$$\dots\vartheta(x_{n-1}-x_n)\varphi(x_2)\dots\varphi(x_n). \qquad (21)$$

Further, in the product $S^{(1)}(z_1)\varphi(x_2)\dots\varphi(x_n)$ we are again only interested in the term with the maximal order of variational differentiations, i.e., in changing the order of the operators $S^{(1)}$ and φ it suffices to keep only the term with the commutator. As a result we find

$$S^{(1)}(z_1)\varphi(x_2)\dots\varphi(x_n) = \dots + (-i)^{n-1}\int dz_2$$
$$\dots dz_n[D(x-z)]_{n-1}\frac{\delta^{n-1}S^{(1)}(z_1)}{(\delta\varphi(z))_{n-1}}.$$

If we also use the equations of motion for the current-like operators II (18), we see that the total contribution proportional to Λ_n to the Dyson T product of the fields $A(x)$ is

$$(-i)^2 i^{n-1}P\left(\frac{x_1}{(x)_{n-1}}\right)\int (dz)_n\,\Theta(x_1;x_2,\dots,x_n)$$
$$\times D^{(adv)}(x_1-z_1)[D(x-z)]_{n-1}\Lambda_n((z)_n), \qquad (22)$$

where we took account of the fact that the sum over the permutations $P(2,\dots,n)$ goes only over the ϑ functions owing to the symmetry of the operator Λ_n, and gives unity by completeness. In the quasi Wick product (15), on the other hand, the current-like operator Λ_n can come only from the last term containing $S^{(n)}$ and gives the contribution

$$-i(-i)^n\int (dz)_n[D^{(adv)}(x-z)]_n\Lambda_n(z_1,\dots,z_n).$$

We thus obtain finally

$$T_W(\varphi(x_1)\dots\varphi(x_n)S)\cdot S^+ - T_D(A(x_1)\dots A(x_n))$$

$$= \text{terms with lower-order current-like operators}$$

$$+(-i)^{n+1}\int (dz)_n\Big\{[D^{(adv)}(x-z)]_n$$

$$+(-1)^n P\Big(\frac{x_1}{x_2,\dots,x_n}\Big)D^{(adv)}(x_1-z_1)\,\vartheta(x_1-x_2)$$

$$\dots\vartheta(x_1-x_n)D(x_2-z_2)\dots D(x_n-z_n)\Big\}\Lambda_n((z)_n). \quad (23)$$

If this relation is averaged over the vacuum, the locality and symmetry properties (cf. II) of current-like operators imply that the vacuum expectation value $\langle 0|\Lambda_n|0\rangle$ must have the form of a symmetric polynomial of the differential operators with respect to all coordinates $P(\dots\partial/\partial_z\dots)$ acting on the product of δ functions

$$\delta(z_1-z_2)\dots\delta(z_{n-1}-z_n). \quad (24)$$

It is now easily seen that if the product (24) without the polynomial $P(\partial/\partial_z)$ is substituted in the average value (23), the latter will, since all z are equal, effectively contain the chain of ϑ functions

$$\vartheta(z_1-x_1)\vartheta(x_1-x_2)\dots\vartheta(x_1-x_n)$$

which can be supplemented with the product

$$\vartheta(z_2-x_2)\dots\vartheta(z_n-x_n),$$

then all D functions reduce to $D^{(adv)}$ functions, and the resulting symmetric combination of these functions can be taken out from under the symmetrization operator, which, acting on the remaining ϑ functions, gives unity by completeness. Thus the second term in the last member of (23) cancels against the first. However, if the polynomial $P(\dots\partial/\partial_z\dots)$ is not identically equal to unity but contains differentiation operators of sufficiently high order, then this compensation breaks down, as is easy to verify; the two T products—the quasi Wick and the Dyson products—will no longer coincide.

We thus convince ourselves that the axiomatics of PTDR which we developed in I and II describes a wider class of theories than the axiomatics of Lehmann, Symanzik, and Zimmermann: in our case there is no restriction on the possible order of derivatives in current-like operators, which would follow from the requirement (18).[7)]

[7)]It is interesting to note that an attempt is described in the literature[11] to treat condition (18) (after application of n Klein-Gordon operators and averaging over the vacuum) as a basic equation of the theory.

5. CONNECTION WITH THE LAGRANGIAN FORMALISM

The rules (1) and (9) introduced in the preceding sections for the transition from the out operators to Heisenberg operators via Wick T products have a rather inconvenient form. First of all, in this form it is not possible to invert the transformation explicitly; although the Heisenberg form of each out operator can easily be found, we do not know of an effective method of obtaining the out form of a Heisenberg operator. Similarly, we can in general not count on the preservation of the group property: the Heisenberg form of a product of operators is not necessarily equal to the product of the Heisenberg forms of the factors. (We received a lesson concerning this last circumstance in the preceding section, when we investigated the relation between the Dyson and quasi Wick products of Heisenberg operators.) As already noted, we can not even be sure that the transformation (9) preserves the hermiticity of the operators. It would, of course, be very desirable to avoid these troubles, at the price of certain restrictions if need be, and to establish that the transition can be described in the usual form of a unitary transformation. Is that possible?

Thus, let us assume that the transformation (1) (for definiteness, we consider the field operators themselves) is unitary, i.e., there exists some unitary operator, which it is natural to denote by $S(\infty, x)$, such that

$$A(x) = S(\infty, x)\varphi(x)S^+(\infty, x), \quad (25)$$

and consider the consequences of this assumption. Commuting φ with $S(\infty, x)$ in (25), we obtain an alternative form to (6) for the Yang-Feldman relation:

$$A(x) = \varphi(x) - \int D(y-x)\left[i\frac{\delta S(\infty, x)}{\delta\varphi(y)}S^+(\infty, x)\right]dy \quad (26)$$

From the requirement that both expressions be equal for the Heisenberg field we get the condition

$$\int dy\left[\vartheta(y-x)j(y) - i\frac{\delta S(\infty, x)}{\delta\varphi(y)}S^+(\infty, x)\right]D(y-x) = 0, \quad (27)$$

from which the operator $S(\infty, x)$ is to be obtained.

We would now like to get rid of the integral and rewrite (27) in the form of an "equation of motion" for $S(\infty, x)$:

$$i\frac{\delta S(\infty, x)}{\delta\varphi(y)} = \vartheta(y-x)j(y)S(\infty, x). \quad (28)$$

As is seen, we arrive then at a typical Tomonaga-Schwinger equation in terms of variational derivatives, where the current operator plays the role of the Hamiltonian (cf. Sec. II). Of course, Eq. (28) is only a sufficient condition for the possibility of writing (1) in the form of a unitary transformation, but by no means a necessary one. Let us see whether the integrability conditions for this equation are fulfilled. Differentiating the right-hand side of (28) with respect to $\varphi(z)$, then doing the differentiation in the reverse order, forming the difference of the second derivatives and using the integrability condition II (21) for the current operator, we reduce the integrability condition (28) to the form

$$\vartheta(y-x)\vartheta(x-z)\frac{\delta j(y)}{\delta\varphi(z)}-\vartheta(z-x)\vartheta(x-y)\frac{\delta j(z)}{\delta\varphi(y)}=0. \quad (29)$$

Rewriting the variational derivatives of the current with the help of the equations of motion II (18), we see that the terms with commutators cancel and only the term with the current-like operator Λ_2 remains. We obtain then the condition

$$(\vartheta(y-x)-\vartheta(z-x))\Lambda_2(z, y)=0. \quad (30)$$

Since the operator Λ_2 differs from zero only for coinciding arguments, it can depend on the difference z-y only through a δ function and its derivatives; by invariance, it must be a polynomial of $(\Box_{z-y}-m^2)$ acting on this function. As in the very similar discussion of the preceding section, we see at once from (30) that if no current-like operator contains derivatives of δ functions [here we must consider all operators, since (30) must hold for all matrix elements, and the higher matrix elements contain higher operators Λ_ν), the integrability condition for $S(\infty, x)$ will be fulfilled. But physically all operators Λ_ν can be free of derivatives only in an unrenormalizable theory without derivative couplings.

Thus for unrenormalizable theories without derivative couplings the transformation (1) can be formally written in unitary form. The term "formally" is added deliberately - from the fulfilment of the integrability conditions it does not follow at all that the result of the integration is also finite. Nevertheless, one may hope so in an unrenormalizable theory.

The further course of the considerations would be very simple. Since a unitary transformation can be inverted, we would define with its help the out form of the operators $\Lambda_\nu(x_1, \ldots, x_\nu)$:

$$\Lambda_\nu^{out}(x_1, \ldots, x_\nu)=S^+(\infty, x)\Lambda_\nu(x_1, \ldots, x_\nu)S(\infty, x)$$
$$(x=x_1=\ldots=x_\nu). \quad (31)$$

In the variation of (31) with respect to the out field, one would have to vary Λ_ν as well as $S(\infty, x)$. Owing to the "equation of motion" (28), the terms coming from the variation of $S(\infty, x)$ would just cancel the terms with a commutator in the equations of motion II (18) for the current-like operators. As a result we would obtain very simple equations of motion for the operators Λ_ν^{out}:

$$\frac{\delta\Lambda_\nu^{out}(x_1, \ldots, x_\nu)}{\delta\varphi(y)}=\Lambda_{\nu+1}^{out}(x_1, \ldots, x_\nu, y), \quad (32)$$

which would imply the quasilocality of these operators:

$$\frac{\delta\Lambda_\nu^{out}(x_1, \ldots, x_\nu)}{\delta\varphi(y)}=0, \text{ unless } x_1=\ldots=x_\nu=y, \quad (33)$$

and the property that they are all successive derivatives of a single local Lagrangian.

Thus, in an unrenormalizable theory without derivative couplings the axiomatic formulation can, if only formally, be reduced to the Lagrangian form. The situation is considerably less clear for renormalizable and, particularly, for unrenormalizable theories. It is easy to see that if, say, the operator Λ_2 contains just one Klein-Gordan operator, then condition (30) ceases to hold. However, since it was only a sufficient and not a necessary condition for the existence of the unitary transformation (25), this result by itself does not yet lead to any conclusions to the opposite effect. One must investigate the integral equation of motion (27), which is less perspicacious. We were not able to reach any definite conclusions. Clear is only that, since (28) is not fulfilled, the basic equations cannot in this case be reduced to the simple form (32).

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