ON THE VORTEX STRUCTURE OF ROTATING HELIUM

L. V. KIKNADZE, Yu. G. MAMALADZE, and O. D. CHEĬSHVILI

Institute of Physics, Academy of Sciences, Georgian S.S.R.

Submitted to JETP editor December 26, 1964

J. Exptl. Theoret. Phys. (U.S.S.R.) 48, 1520-1525 (May, 1965)

It is shown that two-dimensional vortex lattices in rotating liquid helium are capable of rotating along with the vessel. It is also shown that regions of reverse rotation of the superfluid liquid exist in the intervortex volume (in the rotating reference system).

1. The basis of the hydrodynamics of rotating helium[1-3] is the concept that the motion of a superfluid is brought about by the presence in it of Onsager-Feynman vortex filaments. To be specific, the rotation of the vortices along with the vessel leads causes the superfluid to rotate with them, in the mean, like a rigid body with a nonzero curl of the average velocity, although the motion around each individual vortex is potential (see, for example, Sec. 1,4 and Fig. 1 in the review by Andronikashvili et al.[3]) However, a more detailed development of these concepts (determination of the geometric structure of the system of vortices, study of the velocity distribution in the intervortex regions, etc.) lies beyond the scope of such a theory, since all the physical quantities in it are averaged over regions containing sufficiently many vortices. In the present paper we consider such problems on the basis of the phenomenological theory of liquid helium.[4,5]

2. We consider equilibrium rotation of liquid helium in a vessel which is assumed to be sufficiently large that the effect of the walls can be neglected and the liquid can be considered to be infinite in extent. From the expression given by Pitaevskiĭ for the dissipation function[5] it follows that in helium rotation unaccompanied by energy dissipation we should have (in dimensionless form)

$$\mathbf{v}_n = [\boldsymbol{\omega}_0 \mathbf{r}], \qquad \nabla T = 0; \tag{1*}$$

$$(i\nabla + [\boldsymbol{\omega}_0 \mathbf{r}])^2\Psi + (\partial\varepsilon / \partial\rho_s)_{\rho, S}\Psi = 0. \tag{2}$$

Here ω_0 is the angular velocity, measured in units of $2\alpha/\hbar \approx 8.6 \times 10^{10}(T_\lambda - T)$ sec^{-1}; the coordinates are measured in units of $a_0 = \hbar/\sqrt{2m\alpha} \approx 4.3 \times 10^{-8}(T_\lambda - T)^{-1/2}$ cm; $\Psi = fe^{i\varphi}$ is the wave function measured in units of $(\alpha/\beta)^{1/2}$ ($\rho_s = f^2$, $\mathbf{v}_s = \nabla\varphi$); ϵ is the internal energy per unit volume, measured in units of α^2/β; S is the entropy, and T is the temperature. In the definition of the units we used the constants $\alpha \approx 4.5 \times 10^{-17}(T_\lambda - T)$ erg and $\beta \approx 4 \times 10^{-40}$ erg-cm^3, introduced in[4]; m is the mass of the helium atom and T_λ is the temperature of the λ transition in motionless helium, in the vicinity of which the phenomenological theory is valid.

If the conditions (1) and (2) are satisfied, the complete set of phenomenological equations of liquid helium[5] reduces to

$$2i\frac{\partial\Psi}{dt} = -\Delta\Psi + \left[\left(\frac{\partial\varepsilon}{\partial\rho_s}\right)_{\rho, S} + \left(\frac{\partial\varepsilon}{\partial\rho}\right)_{\rho_s, S}\right]\Psi, \tag{3}$$

$$\nabla(\partial\varepsilon/\partial\rho)_{\rho_s, S} = 2\omega_0^2\mathbf{r}, \tag{4}$$

$$\partial\rho_n / \partial t + [\boldsymbol{\omega}_0\mathbf{r}]\nabla\rho_n = 0 \tag{5}$$

and to the continuity equation for the entropy, which is unimportant in the following exposition. The continuity equation for ρ_s is the imaginary part of Eq. (3).

Substitution of (4) in (3) (in which the integration constant can be discarded without limiting the generality of the presentation) reduces the solution of the entire system to the solution of the single equation (2), if only

$$\partial\Psi / \partial t = -[\boldsymbol{\omega}_0\mathbf{r}]\nabla\Psi, \tag{6}$$

which, together with (5), denotes the stationarity of the distribution of the values of ρ_n, ρ_s, and $\mathbf{v}_s$ in the rotating reference system. It will be shown below that Eq. (2) actually has a solution which describes motions of this type.

3. We substitute in (2) the expansion of $\partial\epsilon/\partial\rho_s$ in the form[4]

$$(\partial\varepsilon / \partial\rho_s)_{\rho, S} = -1 + f^2, \tag{7}$$

which gives

$$(i\nabla + [\boldsymbol{\omega}_0\mathbf{r}])^2\Psi - \Psi + |\Psi|^2\Psi = 0. \tag{2a}$$

It can be shown that Eq. (2a) admits a solution of the same type as obtained in the work of Ginzburg

*$[\boldsymbol{\omega}_0\mathbf{r}] \equiv \boldsymbol{\omega}_0 \times \mathbf{r}$.

and Pitaevskiĭ[4], which describes the vortex filament in a non-rotating liquid (see also [6]). However, in this case, the presence of the term $\omega_0 \times \mathbf{r}$ in (2a) leads to motion of the vortex filament itself about the axis of rotation of the normal component (the vessel). This is to be expected, for only in this case is it possible to avoid the energy dissipation that is inevitable when there is relative motion between the vortex and the normal liquid.[1-3]

Let us represent f in the vicinity of some point $\mathbf{r}_V$ by a series that is characteristic for the vortex point:

$$f = \sum_{k=0}^{\infty} A_{2k+1}\rho^{2k+1},$$

where $\rho = \mathbf{r} - \mathbf{r}_V$. We set

$$|\nabla\varphi - [\omega_0 \mathbf{r}]| = \sum_{k=-1}^{\infty} B_k \rho^k$$

and substitute this series in Eq. (2a). It is not difficult to establish the fact that the equation is actually satisfied, where

$$B_{-1} = 1, \quad B_0 = 0; \qquad \nabla\varphi = \rho_1/\rho^2 + [\omega_0 \mathbf{r}] + \ldots$$

($\rho_1 \perp \rho$, $\rho_1 = \rho$), and thereby confirm also the presence of rotation of the vortex point.

4. For the following exposition, it is extremely important that Eq. (2a) is analogous to one of the equations of the Ginzburg-Landau theory, used by Abrikosov[7] to explain the properties of type II superconductors. The well known analogy between the vortex filaments formed under the action of a magnetic field in such superconductors and the vortices in rotating liquid helium has been shown to be rather far reaching, permitting us to make wide use for our purposes of the methods and results of Abrikosov's paper[7], and of some of the many investigations stimulated by it on the theory of type II superconductors.

In a later paper we shall describe the limits of the region of the ω_0 – T diagram, occupied by the states of the liquid helium, which the Onsager-Feynman filaments penetrate. For the time being we limit ourselves to the remark that the vortices are formed in the rotating cylinder at very low angular velocities[8,9], but their accumulation with increase in ω_0 leads to a transition to the normal state when the distance between the vortices becomes comparable with the dimensions of their cores. Correspondingly, the value of ω_0 (the second critical velocity) is shown to be equal to $\omega_{02} = 0.5$ ($\omega_{02} = 4.3 \times 10^{10}(T_\lambda - T)\ \text{sec}^{-1}$, see below).

5. In order to avoid mathematical difficulties associated with the solution of the nonlinear equation (2a), we carry out, following Abrikosov[7], a detailed analysis of the geometric structure of the system of vortices in the region of the second critical velocity $\omega_{02} = 0.5$, although this region is but slightly accessible to experimental investigation. The purpose of this analysis is not to obtain quantitative data to be compared with experiments, but definite conclusions, the qualitative character of which is not connected in principle with the enormous velocity of rotation or with the unrealistic closeness of T to T_λ.

For $\omega_0 \approx \omega_{02}$ (in connection with the high density of vortices on which $|\Psi| = 0$) one can neglect the next term of Eq. (2a) and consider the linearized equation

$$(i\nabla + [\omega_0 \mathbf{r}])^2 \Psi - \Psi = 0. \tag{2b}$$

Its solutions are connected by means of the simple formula

$$\Psi(x, y) = \Psi_1(x + x_0, y + y_0) \exp[-i\omega_0(x + 2x_0)y] \tag{8}$$

with the solutions $\Psi_1(x, y)$ of the equation

$$(i\nabla + 2\omega_0 x\mathbf{k})^2 \Psi_1 - \Psi_1 = 0, \tag{9}$$

which, according to [7], describe a two-dimensional vortex lattice. In Eqs. (8) and (9), x_0 and y_0 are arbitrary numbers, the presence of which causes the lattices just mentioned to be arbitrarily shifted relative to the origin of the coordinates; this origin is fixed by Eq. (2b) on the axis of rotation of the normal component; $\mathbf{k}$ is the basis vector of the y axis; to convert from the notation of Abrikosov[7] to ours, we must set $\kappa = 1$, $H_0 = 2\omega_0$.

Nonzero solutions of Eq. (2b) exist for

$$\omega_0 = \tfrac{1}{2}(2n + 1) \qquad (n = 0, 1, \ldots),$$

whence it follows that $\omega_{0\,\max} = \omega_{02} = 0.5$. Solutions for intermediate values of ω_0 must be sought by means of the nonlinear equation. In particular, for $\omega_0 \lesssim \omega_{02}$, we use a method of successive approximations in which the zeroth approximation serves as the solution of the linear equation for $\omega_0 = \omega_{02}$ (see the case $H_0 \lesssim \kappa$ in [7]).

According to (8), $f \equiv |\Psi| = |\Psi_1|$. Therefore, the lines f = const, shown in the drawing in [7] and in Fig. 2 in [10], describe the density distribution of the superfluid component in cases of vertex lattices with square and triangle symmetries respectively. Then the requirement of the existence of solutions found by the method of successive approximations (see Eq. (14) in [7]), and minimization of the free energy, lead in our case to the condition

$$\overline{f^4} / (\overline{f^2})^2 = \min,$$

which contains only f. Therefore, one can directly apply to the rotating helium the conclusion of the work of Kleiner et al.[10] on the preference for the case of triangle symmetry (see also [11]).

The imaginary part of Eq. (2a) (and, consequently, of (2b) also),

$$(\nabla f, \nabla\varphi - [\omega_0 \mathbf{r}]) = 0 \qquad (10)$$

shows that the lines f = const are simultaneously lines of flow for the superfluid component in the rotating reference system. Considering from this viewpoint the drawings we have referred to from [7] and [10], it is impossible not to note the fact that, in the vicinity of the points where f is maximized, the superfluid component rotates in a direction opposite to the direction of its rotating about the vortex points (where f = 0). This fact deserves a detailed study.

We begin with the case of square symmetry. Then, according to (8),

$$\Psi(x, y) = C_0 \exp[-{}^1/_2(x + 2x_0)y] \sum_{n=-\infty}^{\infty} \exp[in\sqrt{2\pi}(y + y_0) - {}^1/_2(x + x_0 - n\sqrt{2\pi})^2], \qquad (11)$$

where c_0 is a normalizing constant, $\omega_0 = \omega_{02}$, and the lattice parameter is equal to $\sqrt{2\pi}$. We consider the phase of this function in the vicinity of the vortex points

$$x_v = (2p + 1)\sqrt{\pi/2} - x_0,$$
$$y_v = (2q + 1)\sqrt{\pi/2} - y_0$$

and of the maxima

$$x_m = p\sqrt{2\pi} - x_0,$$
$$y_m = q\sqrt{2\pi} - y_0 \qquad (p, q = \ldots -1, 0, 1, \ldots).$$

Expanding Ψ in powers of $x - x_v$, $y - y_v$ and consequently in powers of $x - x_m$ and $y - y_m$, and making use of the equations

$$\sum_{n=-\infty}^{\infty} (n^2 - n)\exp[in\pi - \pi(0.5 - n)^2] = 0$$

and

$$\sum_{n=-\infty}^{\infty} \exp[in\pi - \pi(0.5 - n)^2] = 0,$$

we get

$$\varphi_v = \tan^{-1}\frac{y - y_v}{x - x_v} + {}^1/_2 x_v(y - y_v) - {}^1/_2 y_v(x - x_v) + \ldots, \qquad (12)$$

$$\varphi_m = {}^1/_2 x_m(y - y_m) - {}^1/_2 y_m(x - x_m) + \ldots \qquad (13)$$

The first of these equations describes the already noticed rotation (with angular velocity $\omega_{02} = 0.5$) of the vortex points around the axis of rotation of the normal component (the vessel). The second describes the analogous motion of the maxima. Here, the relative motion of the superfluid adjacent to the maxima is a reverse rotation (with velocity ω_{02}). In the fixed reference system, the motion is a rotation about the axis of the container with a constant linear velocity:[1)]

$$\nabla\varphi_m \approx [\omega_0 \mathbf{r}_m] = [\omega_0 \mathbf{r}] - [\omega_0, \mathbf{r} - \mathbf{r}_m].$$

In this connection, it is of interest to recall the work of Pellam[12] in which he contrasted the vortex model of rotating helium with a model of domains whose centers rotate together with the vessel, while a reverse rotation around these centers takes place inside the domains. This leads to a constant linear velocity inside the domain, to the equation curl $\mathbf{v}_s = 0$, and to a mean velocity $\mathbf{v}_s = \omega_0 \times \mathbf{r}$. It is curious that the presence of such domains (near the maxima of the wave functions) turned out to be not in contradiction, but in close connection with the presence of vortices.

Similar results are also obtained for lattices with triangle symmetry, when

$$\Psi = C_0 \exp[-{}^1/_2 i(x + 2x_0)y]\left(\sum\nolimits_1 \pm i\sum\nolimits_2\right)\exp[in(\sqrt{3}\pi)^{1/2} \times (y + y_0) - {}^1/_2(x + x_0 - n(\sqrt{3}\pi)^{1/2})^2], \qquad (14)$$

where Σ_1 and Σ_2 denote sums over all even and odd n, respectively.

Thus both the considered lattices are "rigid" in the sense that they possess the property of rotating as a unit about an arbitrarily placed axis in the lattice, the axis about which the normal component is moving.

6. This property of the lattices of quantum vortices, which seemingly is a trivial result of their interaction with the normal component rotating as a whole (see Fig. 3), turns out to be very significant, demonstrating a difference in principle between the phase of the wave function φ and the velocity potential φ_0 of the vortex lattice in the classical ideal incompressible liquid.

The coincidence of φ and φ_0, which occurs in the presence of a single vortex in a fixed unbounded liquid (when $\varphi = \varphi_0 = \tan^{-1}[(y - y_v)/(x - x_0)]$, could have created the false impression that the quantum and classical vortices differ only in the

---

1)Here it is essential that, owing to the fact that

$$\sum_{n=-\infty}^{\infty} (4\pi n^2 - 1)e^{-\pi n^2} = 0,$$

there are no terms of second order in $x - x_m$, and $y - y_m$ in the expansion (13).

density distribution, but not in the velocity. Actually, in the general case, φ and φ_0 can differ strongly from each other. The point is that φ is determined by a set of equations which connect this quantity with the velocity distribution in the normal component. But φ_0 is expressed by the equation

$$\varphi_0 = \sum \tan^{-1}[(y - y_v)/(x - x_v)]$$

(summed over all vortices), thanks to which the velocity distribution is determined only by the distribution of the vortices themselves. The resultant difference between φ and φ_0 is clearly evident in the motion of the just considered unbounded, two-dimensional lattice vortex formed in liquid helium and the lattice of classical point vortices which is geometrically identical with it.

The former, as shown in Sec. 5, rotate with the normal component. The situation with the latter is as follows. First, it must be noted that they generally do not have a selected point about which they could rotate. In obvious connection with this fact, the expression

$$\mathbf{v}_v = \{\nabla\varphi_0 - \nabla \tan^{-1}[(y - y_v)/(x - x_v)]\}_{\mathbf{r}=\mathbf{r}_v},$$

which should define the velocity of the proper motion of the vortices, is not single-valued. One could attempt to get around this difficulty by selecting in the expression for φ_0 a sequence of summation corresponding to a system of vortices in a large, but not infinite, cylindrical vessel. To be precise, one of the vortices is considered central; then one sums the potentials of four or six of its nearest neighbors (depending on the character of the lattice), located on one circle, etc. However, it is not difficult to verify that such an artificial method does not prevent the "spreading apart" of the vortex lattice constructed in a classical, ideal incompressible liquid, a lattice geometrically identical with the two dimensional lattices considered by us in Sec. 5.[2)] The rotation of such classical lattices can be obtained only by replacement of the ambiguous expression for $\mathbf{v}_V$ by the analogous (but unique) expression connected with the Weierstrass ζ-function (private communication from V. K. Tkachenko).

7. It is evident that statements concerning the possibility of an essential difference between the phase wave function and the classical velocity potential of the geometrically identical systems of classical and quantum vortices, of the capacity of the quantum vortices to create "rigid" (relative to "spreading apart" during the motion) two-dimensional lattices, and of the presence in the intervortex regions of a reverse relative rotation of the superfluid, have a rather general character, and there is no basis for doubt in their validity at much lower velocities and temperatures than those for which Eq. (2b) is applicable.

[2)]The authors are grateful to N. P. Kogoniya, Ts. T. Tarkashvili, and O. B. Maladze, who confirmed this fact by direct machine computation.

[1] H. E. Hall, Phil. Mag. Suppl. **9**, 89 (1960).

[2] I. L. Bekarevich and I. M. Khalatnikov, JETP **40**, 920 (1961), Soviet Phys. JETP **13**, 643 (1961).

[3] Andronikashvili, Mamaladze, Matinyan, and Takadze, UFN **73**, 3 (1961), Soviet Phys. Uspekhi **4**, 1 (1961).

[4] V. L. Ginzburg and L. P. Pitaevskii, JETP **34**, 1240 (1958), Soviet Phys. JETP **7**, 858 (1958).

[5] L. P. Pitaevskii, JETP **35**, 408 (1958), Soviet Phys. JETP **8**, 282 (1959).

[6] L. P. Pitaevskii, JETP **40**, 646 (1961), Soviet Phys. JETP **13**, 451 (1961).

[7] A. A. Abrikosov, JETP **32**, 1442 (1957), Soviet Phys. JETP **5**, 1174 (1957).

[8] R. G. Arkhipov, JETP **33**, 116 (1957), Soviet Phys. JETP **6**, 90 (1958).

[9] W. F. Vinen, Nature **181**, 1524 (1958).

[10] Kleiner, Roth, and Autler, Phys. Rev. **133**, A1226 (1964).

[11] J. Matricen, Phys. Lett. **9**, 289 (1964).

[12] J. R. Pellam, Phys. Rev. Lett. **9**, 281 (1962).

Translated by R. T. Beyer
218