DIFFRACTION SCATTERING AND REGGE POLES

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Submitted to JETP editor November 21, 1964; resubmitted January 15, 1965

J. Exptl. Theoret. Phys. (U.S.S.R.) 48, 1620-1624 (June, 1965)

The cross section for diffraction scattering of particles by a black nucleus is obtained by the method of complex angular momenta. The expression for the cross section takes into account the smearing of the nuclear boundary and some other features which are neglected in the usual diffraction theory. Comparison with the experiments on the scattering of α particles by Mg^{24} lends support to the theory.

1. The diffraction theory of the scattering of particles by complex nuclei leads to the well known expression for the cross section

$$\sigma(\vartheta)=\left[\frac{R}{\vartheta}J_1(kR\vartheta)\right]^2, \qquad (1)$$

which asymptotically (for $kR\vartheta \gg 1$) goes over into

$$\sigma(\vartheta)=\frac{2R}{\pi k\vartheta^3}\cos^2\left(kR\vartheta+\frac{\pi}{4}\right). \qquad (2)$$

Comparison with experiment (for example, α-nucleus scattering) shows that expression (2) describes the oscillations of the cross section correctly, although it has two important defects. First, the amplitude of the oscillations decreases, according to (2), like ϑ^{-3} as the scattering angle is increased, whereas the experimental data show an exponential decrease. Second, the cross section (2) vanishes at the minima, which is also in contradiction with experiment.

2. Let us consider the starting point of elementary diffraction theory, which leads to the expressions (1) and (2). This is the assumption that the S matrix is given by

$$S_l=\begin{cases}0 & \text{for } l<l_0=kR-{}^1/_2,\\ 1 & \text{for } l>l_0.\end{cases} \qquad (3)$$

It is, of course, also assumed that

$$kR\gg 1. \qquad (4)$$

Expression (3) evidently has the important shortcoming that it does not take account of the transitional boundary layer of the nucleus, which should lead to a gradual change of S_l from zero (complete absorption) to unity (absence of interaction) in some interval of angular momenta corresponding to impact parameters ΔR equal in order of magnitude to the magnitude of the diffuseness of the nucleus.

As a simple model of an S matrix which takes account of the nuclear boundary, we may consider instead of (3) the following expression:

$$S_l=1-\left[1+\exp\left(\frac{l-l_0}{\lambda}\right)\right]^{-1}, \qquad (5)$$

which goes over into (3) for $\lambda\to 0$. Defining the size of the transition layer Δl as the interval in which S_l changes from $(1+e^2)^{-1}$ to $1-(1+e^2)^{-1}$ (with l changing from $l_1=l_0-\Delta l$ to $l_2=l_0+\Delta l$), we obtain

$$\lambda={}^1/_2\Delta l. \qquad (6)$$

The quantity $\Delta l=k\Delta R$ will, under the condition (4), be equal to several units, if we assume that $\Delta R/R \approx 0.1$ to 0.2.

3. Let us now consider the quantity S_l defined by (5) as a function of the complex variable l. It is clear that the only singularities of this function are simple poles at the points

$$l_n=l_0\pm\lambda\pi n i \qquad (n=1, 3, 5, \ldots) \qquad (7)$$

with the residues

$$a_n=\lambda. \qquad (8)$$

The distance between neighboring poles is equal to $2\pi\lambda$, i.e., of order 10 in situations met in a real nucleus. The transition to expression (3) considered in the elementary diffraction theory corresponds to the case $2\pi\lambda \ll 1$, where the Regge poles defined by (7) approach each other so closely that they form a continuous line of poles, i.e., a cut in the complex l plane.

We see, therefore, that the consideration of the analytic properties of the matrix S_l reveals a sharp qualitative distinction between the case considered in the elementary diffraction theory and the real situation arising from the presence of a transition region at the nuclear boundary.

4. Let us use the Watson-Sommerfeld transformation[1] for the calculation of the scattering amplitude from S_l given by (5). Then the scattering amplitude is expressed through an integral over the contour B along the infinite straight line Re $l = -\frac{1}{2}$ and over the contour C along a semi-circle of infinite radius closing the contour B from the right, and the sum of the residues of the Regge poles defined by (7) and (8). It is easily seen that the integral along the contour C vanishes. The contribution from the integral along the contour B can also be neglected if

$$1-[1+\exp(-kR/\lambda)]^{-1}\ll 1. \tag{9}$$

This condition, which implies that the probability for absorption of a particle passing through the center of the nucleus is practically unity, is usually well satisfied. If this is the case, we can set $S_l = 0$ on the contour B. Then the resulting integral is equal to zero (cf. [1]) in virtue of the known relation between Legendre polynomials:

$$P_l = P_{-l-1},$$

which is valid for arbitrary (also complex) index l.

We have thus arrived at the result that the amplitude is determined only by the contributions from the poles of S_l in the half-plane Re $l > -\frac{1}{2}$. Since

$$|l_n| \gg 1 \tag{10}$$

for all n, we can use the asymptotic form of the Legendre polynomials and obtain directly the contribution from the n-th pole to the scattering amplitude in the form

$$f_n(\vartheta) = -\frac{i\pi\lambda(2l_n+1)}{k\sin(l_n\pi)(2\pi l_n\sin\vartheta)^{1/2}} \times \cos\left[\left(l_n+\frac{1}{2}\right)(\pi-\vartheta)-\frac{\pi}{4}\right]. \tag{11}$$

If we use the condition

$$e^{2\pi\lambda\vartheta} \gg 1, \tag{12}$$

which, in view of the preceding remarks, will be well satisfied for not too small angles ϑ, the main contribution to the amplitude will come only from the pair of complex conjugate poles closest to the real axis. As a result we obtain

$$\sigma(\vartheta) = \frac{8\beta^2 R}{\pi k\sin\vartheta}e^{-2\beta\vartheta}\cos^2\left(kR\vartheta+\frac{\pi}{4}+\frac{1}{2}\frac{\beta}{kR}\right), \tag{13}$$

where $\beta = \pi\lambda$, i.e., the imaginary part of l_1.

5. The expression (13) for the cross section is evidently in agreement with the experimentally observed exponential dependence of the amplitude of the oscillations on the scattering angle. However, an essential defect of formula (13) is the fact that it gives a cross section which vanishes in the minima, as in elementary diffraction theory. This is connected with the circumstance that the formula (5) chosen by us for S_l is purely real (here we are, of course, speaking of real values of l). Actually, the matrix S_l must be complex. Writing it in the form

$$S_l = \eta(l)e^{2i\delta(l)},$$

where $\eta(l)$ and $\delta(l)$ are real functions, we can, in generalization of the preceding discussion, formulate the following hypothesis, which is very natural in the present context.

This hypothesis is that the main contribution to the cross section comes from a pair of complex conjugate Regge poles. These poles are contained in the function $\eta(l)$, which can be regarded as being qualitatively similar to S_l as defined in (5). The real part of these poles is $\alpha = kR - \frac{1}{2}$, i.e., equal to l_0, the angular momentum of the particle on the grazing trajectory. The imaginary parts $\pm\beta$ are equal in order of magnitude to $\pm k\Delta R$, where ΔR is the size of the diffuseness region of the nucleus.

Denoting the residues at these poles by a and a*, we obtain in the same way as before the final expression for the elastic diffraction scattering cross section:

$$\sigma(\vartheta) = \frac{8\pi R}{k\sin\vartheta}|a|^2 e^{-2\beta\vartheta}\{\mathrm{sh}^2(\beta\theta_0)+\cos^2(kR\vartheta+\gamma)\}, \tag{14*}$$

$$\theta_0 = 2\beta^{-1}\,\mathrm{Im}\,\delta(l_0+i\beta),$$
$$\gamma = \pi/4+\arg a+\beta/2kR. \tag{15}$$

It is easy to see that the parameter θ_0 has a simple physical meaning: it is approximately equal to the classical scattering angle for a particle on the grazing trajectory. Indeed, assuming that the phase $\delta(l)$ can be sufficiently well represented by two terms of a Taylor series in a circle of radius $\rho \approx \beta$ and with center at the point l_0, we obtain

$$\delta(l_0+i\beta) = \delta(l_0)+i\left(\frac{d\delta}{dl}\right)_{l_0}\beta, \tag{16}$$

i.e.,

$$\mathrm{Im}\,\delta(l_0+i\beta) = \beta\left(\frac{d\delta}{dl}\right)_{l_0}, \tag{17}$$

*sh = sinh.

and recalling that in the quasi-classical case

$$\theta = \pm 2\frac{d\delta_l}{dl}, \quad (18)$$

where θ is the scattering angle of a particle with angular momentum l, we arrive at the desired result.

6. Formula (14) is the final result of the present paper. It is valid only if the conditions

$$kR \gg 1, \qquad e^{2\beta\vartheta} \gg 1 \quad (19)$$

are fulfilled. A rigorous foundation of our basic hypothesis that only two complex conjugate Regge poles contribute to the amplitude requires, of course, further work. However, this hypothesis is completely natural and plausible, and it leads to a formula, (14), which is in complete agreement with the principal experimental facts. Indeed, formula (14) reproduces the characteristic oscillations of the cross section, it does not vanish at the minima, and the amplitude of the oscillations decreases exponentially with increasing scattering angle.

We add a few remarks on the parameters entering in (14). The angle θ_0 should, for charged particle scattering, essentially be given by the purely Coulomb scattering angle θ_C for the grazing trajectory, which is given by

$$\sin {}^1/_2\theta_c = (2E/B - 1)^{-1}, \quad (20)$$

where B is the Coulomb barrier and E is the energy of the particle in the center of mass system. One should expect, however, that θ_0 is somewhat smaller than θ_C owing to the nuclear attraction at the boundary of the nucleus.

The absolute value of the residue, $|a|$, can be estimated on the basis of the model for S_l represented by expression (5), which implies

$$|a| = \beta/\pi. \quad (21)$$

The quantity arg a, i.e., the phase of the residue, vanishes in this model. We note that the vanishing of the phase of the residue is a consequence of the symmetry of the derivative $d\eta/dl$ under the replacement $l \rightleftarrows l_0$, i.e., it will take place for a wide class of functions whose derivative belongs to the type of a smeared-out δ function which is symmetric in the argument $l - l_0$.

7. Postponing a detailed comparison of the present theory with experiment, we restrict our

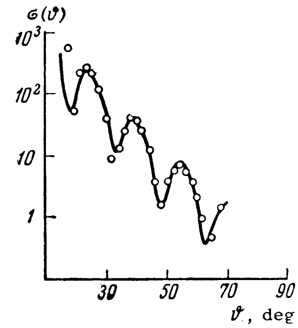

FIG. 1. Differential cross section (in millibarns per steradian) for the elastic scattering of α particles with energy 31.5 MeV by Mg^{24} nuclei.

analysis to the data on the elastic scattering of α particles with energy 31.5 MeV by Mg^{24} nuclei, obtained by Watters.[2] The figure shows the experimental points and the theoretical curve calculated by (14) with the following parameters: $kR = 12.15$, $\beta = 2.61$, $\theta_0 = 7°$, $|a| = 1.14$, $\gamma = 0.85$. We see that theory and experiment are in good agreement for scattering angles $\vartheta > 20°$, in accordance with condition (19). For the interaction radius we obtain $R = 5.75 \times 10^{-13}$ cm, and for the Coulomb scattering angle on the grazing trajectory, $\theta_C = 14°$. As we see, the value obtained, $\theta_0 = 7°$, satisfies the inequality $\theta_0 < \theta_C$, which follows from the physical considerations made above. The absolute value of the residue, $|a| = 1.14$, is rather close to the value $|a| = 0.83$ calculated by (21), which was obtained on the basis of the simple model (5). For the phase of the residue, arg a, we obtain -0.04, which is also in good agreement with the value zero which follows from the model (5). Thus, the preliminary comparison of our theory with experiment shows that formula (14) describes the experimental data well, and reasonable values are obtained for the parameters contained in it.

The author expresses his deep gratitude to D. V. Volkov for valuable remarks and to A. A. Kresnin for help with the comparison of the present theory with experiment.

[1] A. Sommerfeld, Partial Differential Equations of Physics, Academic Press, 1949.

[2] Harry J. Watters, Phys. Rev. 103, 1763 (1956).

Translated by R. Lipperheide
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