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SPINORS IN GRAVITATION THEORY

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Spinors in gravitation theory are treated as four-component objects which transform according to a nonlinear representation of the group of general covariant transformations. Interactions of a spinor field with gravitational, electromagnetic and other fields are constructed in accordance with the spinor transformation law thus derived. The interactions are expanded in a series in powers of the gravitational field, and this is convenient for the application of perturbation theory.

1. INTRODUCTION

1. Even now it is already of interest to discuss quantum gravitational effects not only in the weak-field approximation, but also in higher orders in terms of the gravitational coupling constant, and also the problem of renormalizations and the removal of divergences in gravitational interactions. According to Gupta[1] the problem of the quantization of the gravitational field within the framework of perturbation theory is essentially solved by expanding the nonlinear Einstein equations in an infinite series in the gravitational constant. Such an approach as applied to the gravitational field and to gravitational interactions of boson fields was adopted in reference[1].

2. But the gravitational interactions of fermions have not until now been discussed within the framework of this approach. The point is that the gravitational interactions of fermions are essentially more complicated[2-22]. Fock and Ivanenko[4,5] were the first to construct a theory of fermions in a gravitational field utilizing the tetrad formalism. In the majority of subsequent papers the same method is also used. However, within the framework of the tetrad formalism it is not clear how Gupta's program should be carried out, and how the gravitational interaction of fermions should be represented in explicit form in terms of the gravitational field as an expansion in terms of the gravitational coupling constant.

3. This has motivated us to seek another method of describing spinors in a gravitational field. We have used the group-theoretic approach and we have introduced spinors as objects transforming in accordance with a representation of that group according to which the fundamental tensor $g^{\mu\nu}$ is transformed. In this sense spinors turn out to be objects of the same type as tensors (scalar, vector etc.). At the same time there also exists an essential difference: although the law of transformation of spinors obtained below is linear and homogeneous in the spinor field, in contrast to the tensor case it depends on the gravitational field (the metric), and does so in a complicated nonlinear manner. In other words, spinors transform according to a nonlinear representation of the group mentioned previously.

On the other hand such an approach corresponds to Schrödinger's[10] idea of doing without orthogonal basis vectors, but he utilized γ-fields which, like tetrads, are related to the gravitational field only implicitly. On the other hand the "square root of the metric tensor" $r^{\mu\nu}$ appearing in our case can be regarded as a modification of a tetrad. In contrast to a tetrad, $r^{\mu\nu}$ is explicitly expressed in terms of the gravitational field, both its indices can be treated on an equal footing and refer to the same general basis (in the case of tetrads one index refers to a general basis, and the other one to a locally orthogonal one).

4. In the approach proposed here the gravitational interaction of fermions is expressed explicitly in terms of the gravitational field and, in accordance with Gupta's program, can be represented in the form of an infinite series in terms of the gravitational coupling constant (in the same way as the "self-interaction" of a gravitational field and the gravitational interactions between bosons in reference[1]). The interaction obtained in this manner in principle enables one to calculate gravitational effects involving fermions to any arbitrary order in the gravitational coupling constant. We emphasize that even for such simple

effects as the gravitational self-energy of the electron or the Compton-effect of a graviton on a fermion the weak field approximation is insufficient, and it is necessary to take into account interaction terms of the second order in the gravitational coupling constant.

2. THE GROUP PROPERTY OF GENERALLY COVARIANT TRANSFORMATIONS

1. In Riemannian geometry the law of transformation of the fundamental tensor $g^{\mu\nu}(x)$ can be represented in infinitesimal form by means of a local variation (cf., reference [23], p. 323)

$$\delta^* g^{\mu\nu} = g'^{\mu\nu}(x) - g^{\mu\nu}(x) = a(\partial_\rho \lambda^\mu g^{\rho\nu} + \partial_\rho \lambda^\nu g^{\mu\rho} - \lambda^\rho \partial_\rho g^{\mu\nu}), \tag{1}$$

where $\lambda^\mu(x)$ are four arbitrary infinitesimal functions. For convenience these functions are brought to dimensionless form by factoring out a constant a which has the dimension of length (in units of $\hbar = c = 1$). In future it will play the role of the gravitational coupling constant and will turn out to be related to the gravitational constant k in Newton's law by the expression $a^2 = 32\pi k$.

Local variations[24] can be regarded as transformations of the functions only, without a change in coordinates (of the type of gauge transformations in electrodynamics) and this, in particular, is useful in interpreting Einstein's theory in the language of a flat space. At the same time the use of local variations is equivalent to the use of substantive variations

$$\delta g^{\mu\nu} = g'^{\mu\nu}(x') - g^{\mu\nu}(x),$$

the definition of which involves the transformation of coordinates $x'^\mu = x^\mu + a\lambda^\mu(x)$.

Following Gupta[1] we shall describe the gravitational field by the quantity $h^{\mu\nu}(g^{\mu\nu} = \delta_{\mu\nu} + ah^{\mu\nu})$.[1)] From (1) it follows that

$$\delta^* h^{\mu\nu} = \partial_\mu \lambda^\nu + \partial_\nu \lambda^\mu + a(h^{\rho\nu}\partial_\rho \lambda^\mu + h^{\mu\rho}\partial_\rho \lambda^\nu - \lambda^\rho \partial_\rho h^{\mu\nu}). \tag{2}$$

We see that (2) differs from the tensor law by the additional term $\partial_\mu \lambda^\nu + \partial_\nu \lambda^\mu$ which is analogous to the gradient in gauge transformations of the electromagnetic field or of the Young-Mills field.

2. In future it will be convenient for us to use the matrix notation

$$g = \| g^{\mu\nu} \|; \quad g^{-1} = \| g_{\mu\nu} \|; \quad h = \| h^{\mu\nu} \|;$$

$$\Lambda = \| \Lambda_{\alpha\beta} \| \equiv \| \partial_\alpha \lambda^\beta \|, \tag{3}$$

so that[2)]

$$g = 1 + ah, \quad g^{-1} = \sum_{n=0}^{\infty} (-ah)^n;$$

$$(h^n)^{\mu\nu} = h^{\mu\sigma_1} h^{\sigma_1\sigma_2} \ldots h^{\sigma_{n-1}\nu}. \tag{3'}$$

In terms of this notation we can write (2) in the following form:

$$\delta^* h = \Lambda + \widetilde{\Lambda} + a(\widetilde{\Lambda} h + h\Lambda - \lambda^\rho \partial_\rho h), \tag{4}$$

$$\delta^* g = a(\widetilde{\Lambda} g + g\Lambda - \lambda^\rho \partial_\rho g),$$

$$\delta^* g^{-1} = -a(\Lambda g^{-1} + g^{-1}\widetilde{\Lambda} + \lambda^\rho \partial_\rho g^{-1}). \tag{4'}$$

3. It is well known that transformations of the fundamental tensor form a group. In the language of infinitesimal transformations (1) this has the following meaning. We denote the result of transformation (1) with $\lambda^\mu = \lambda^\mu_i$ by $\delta^*_{\lambda_i} g^{\mu\nu}$ and we consider the bracket operation well-known in the theory of continuous Lie groups

$$\delta_{\lambda_2}{}^* \delta_{\lambda_1}{}^* g^{\mu\nu} - \delta_{\lambda_1}{}^* \delta_{\lambda_2}{}^* g^{\mu\nu}.$$

Corresponding to the fact that the transformations form a group the result of the bracket operation must be capable of being represented in the form (1) with a new "bracket" λ^μ_{br}:

$$(\delta^*_{\lambda_2}\delta^*_{\lambda_1} - \delta^*_{\lambda_1}\delta^*_{\lambda_2}) g^{\mu\nu} = \delta^*_{\lambda_{br}} g^{\mu\nu}. \tag{5}$$

It is not difficult to verify that this is indeed the case, and that

$$\lambda_{br}{}^\mu = -a(\lambda_1{}^\rho \partial_\rho \lambda_2{}^\mu - \lambda_2{}^\rho \partial_\rho \lambda_1{}^\mu). \tag{6}$$

The "not entirely" tensor quantity $h^{\mu\nu}$, and also all the tensors, transform in accordance with the representations of the same group. For all these objects relation (5) is satisfied, i.e.,

$$(\delta^*_{\lambda_2}\delta^*_{\lambda_1} - \delta^*_{\lambda_1}\delta^*_{\lambda_2}) T^{\mu_1\mu_2\ldots}_{\nu_1\nu_2\ldots} = \delta^*_{\lambda_{br}} T^{\mu_1\mu_2\ldots}_{\nu_1\nu_2\ldots}, \tag{7}$$

where λ_{br} is always expressed by formula (6). This can be easily verified in any special case. And conversely, by solving the relation of the structure (7) to λ^μ_{br} (6) one can infer the law of transformation of a tensor with a given number of indices if in addition we require that the law should not depend on other tensors, that it should

[1)]In this article the imaginary time coordinate $x^4 = it$ is used, and the Kronecker symbol $\delta_{\mu\nu}$ serves as the metric tensor of the special theory of relativity.

[2)]Here the upper indices turn out to be contracted with other upper indices, and this is related to our use of nontensor quantities. (Such contractions are unacceptable when one is constructing tensors from other tensors). In future we shall often use quantities with indices μ, ν,..., which are not tensors.

be linear and homogeneous in terms of the given tensor and, of course, that under Lorentz transformations it should reduce to the usual Lorentz law.

3. THE TRANSFORMATION LAW FOR A SPINOR

In this section we shall turn to just such an inference of the transformation law of an object which differs from a tensor, viz., a spinor. We shall take a spinor to mean a four-component object which transforms in accordance with a representation of the group of transformations of $g^{\mu\nu}$ under discussion in such a way that this law in the special case of Lorentz rotations and translations should, when

$$a\lambda^\mu = 2\omega_\nu{}^\mu x^\nu + c^\mu,$$
$$\omega_\nu{}^\mu = -\omega_\mu{}^\nu = \text{const}, \quad c^\mu = \text{const}, \tag{8}$$

go over into the Lorentz law for a Dirac spinor:

$$\delta^*\psi = -a\lambda^\rho\partial_\rho\psi + {}^1\!/_4 ia\langle\Lambda\cdot\sigma\rangle\psi \text{ with } a\lambda^\mu = 2\omega_\nu{}^\mu x^\nu + v^\mu, \tag{9}$$

where $\langle\Lambda\cdot\sigma\rangle = \Lambda_{\alpha\beta}\sigma_{\beta\alpha}$, i.e., we use angular brackets to denote (here and later) the operation of contraction with respect to the vector indices not involving the spinor indices. Further we have

$$\sigma_{\beta\alpha} = -i(\gamma_\beta\gamma_\alpha - \delta_{\beta\alpha}),$$

where γ_α are the usual Dirac matrices:

$$\gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha = 2\delta_{\alpha\beta}. \tag{10}$$

In seeking the transformation law we impose the following restrictions:

a) we assume in analogy with tensors that the desired law contains the spinor in a linear and homogeneous manner;

b) for simplicity we assume that the desired law differs from (9) only by a modification of the term containing the matrices $\sigma_{\beta\alpha}$, and that no new terms containing other matrices occur in it;

c) we assume that the desired law can contain in addition to the spinor field only the gravitational field $h^{\mu\nu}$, but must not contain any other fields, nor derivatives of $h^{\mu\nu}$. In other words, we assume that the spinor representation is nonlinear in $h^{\mu\nu}$.

Assumption c) is weaker than the corresponding assumption for tensors: "the desired law must not involve any other fields in addition to the given field." This relaxation is necessary, for if it is not made then it is not possible to construct spinors,[3)] and this agrees with Cartan's[13] assertion. Indeed, if the spinor law does not contain any other fields, then, under assumptions a) and b) and with the necessary condition of reduction to the usual transformation law for a Dirac spinor with λ^μ given by (8), the desired law must necessarily have the form (9). But direct calculations show that for functions λ^μ having a general form different from (8) the transformations (9) do not form a group. Indeed, the use of the bracket operation utilizing the formula[4)]

$$[\langle A\cdot\sigma\rangle, \langle B\cdot\sigma\rangle] = 2i\langle(A-\tilde{A})\cdot(B-\tilde{B})\cdot\sigma\rangle \tag{11}$$

yields

$$(\delta^*_{\lambda_2}\delta^*_{\lambda_1} - \delta^*_{\lambda_1}\delta^*_{\lambda_2})\psi = -a\lambda^\rho_{\text{br}}\partial_\rho\psi + {}^1\!/_4 ia\langle\Lambda_{\text{br}}\cdot\sigma\rangle\psi$$
$$+ {}^1\!/_8 ia^2\langle(\Lambda_1+\tilde{\Lambda}_1)\cdot(\Lambda_2+\tilde{\Lambda}_2)\cdot\sigma\rangle\psi. \tag{12}$$

The whole expression in the second line of (12) is superfluous; only for the special choice of λ^μ in the form (8) when Λ_1 and Λ_2 are antisymmetric matrices does this expression vanish. Thus, the desired law cannot coincide with (9) and must contain additional terms which would compensate for the extra term arising in (12) and which would vanish if Λ were antisymmetric.

In accordance with c) we introduce the field $h^{\mu\nu}$ into the law. By a direct analysis it can be shown that the addition to (9) of the term

$$-{}^1\!/_4 ia^2\langle(\Lambda+\tilde{\Lambda})\cdot h\cdot\sigma\rangle\psi$$

compensates [because of the additive part of the variation of h(2)] for the extra term in (12), but in turn gives rise (as a result of the multiplicative part of the variation of h) to unnecessary additional terms in the bracket operation, which now contain a^3. In order to cancel the newly arisen additional terms it is necessary to add to (9) (at first with undetermined multipliers) terms which contain h quadratically, then cubically, etc. As a result of this frontal attack procedure involving repeated application of the bracket operation in such a way that the group property would be satisfied up to ever higher powers of the constant a, we found the first few terms of the desired transformation law:

$$\delta^*\psi = -a\lambda^\rho\partial_\rho\psi + {}^1\!/_4 ia\langle\Lambda\cdot\sigma\rangle\psi - {}^1\!/_{16} ia^2\langle(\Lambda+\tilde{\Lambda})\cdot h\cdot\sigma\rangle\psi$$
$$+ {}^1\!/_{32} ia^3\langle(\Lambda+\tilde{\Lambda})\cdot h^2\cdot\sigma\rangle\psi - {}^1\!/_{256} ia^4\langle h\cdot(\Lambda+\tilde{\Lambda})\cdot h^2\cdot\sigma\rangle\psi$$
$$- {}^5\!/_{256} ia^4\langle(\Lambda+\tilde{\Lambda})\cdot h^3\cdot\sigma\rangle\psi + \ldots \tag{13}$$

It is now natural to seek the exact infinitesimal transformation law for a spinor in the form

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3) This assertion will hold even if we drop assumption b) and modify the desired law by including in it all 16 Dirac matrices.

4) Formula (11) is obtained by multiplying $A_{\nu\mu}$ and $B_{\rho\lambda}$ by the commutation relation for the matrices $\sigma_{\mu\nu}$:

$$[\sigma_{\mu\nu}, \sigma_{\lambda\rho}] = 2i(\delta_{\mu\lambda}\sigma_{\nu\rho} + \delta_{\nu\rho}\sigma_{\mu\lambda} - \delta_{\mu\rho}\sigma_{\nu\lambda} - \delta_{\nu\lambda}\sigma_{\mu\rho}).$$

$$\delta^*\psi = -a\lambda^\rho\partial_\rho\psi + {}^1/_4 ia\langle[\Lambda + \Delta(\Lambda)]\cdot\sigma\rangle\psi, \tag{14}$$

where $\Delta(\Lambda)$ is a matrix of the form

$$\Delta(\Lambda) = \sum_{m,\, n=0}^{\infty} a^{m+n} c_{mn} h^m (\Lambda + \widetilde{\Lambda}) h^n. \tag{15}$$

Since the matrix Δ is contracted with the antisymmetric matrix σ ($\langle\Delta\cdot\sigma\rangle = \Delta_{\mu\nu}\sigma_{\nu\mu}$ occurs in (14)), it is useful from the outset to take the matrix Δ as being antisymmetric: $\Delta = -\widetilde{\Delta}$, so that

$$c_{mn} = -c_{nm}. \tag{16}$$

Applying the bracket operation and at the same time utilizing (11) we obtain

$$(\delta_{\lambda_2}{}^*\delta_{\lambda_1}{}^* - \delta_{\lambda_1}{}^*\delta_{\lambda_2}{}^*)\psi = -a\lambda_{\mathrm{br}}{}^\rho\partial_\rho\psi - {}^1/_4 ia^2(\lambda_1{}^\rho\partial_\rho\langle\Lambda_2\cdot\sigma\rangle$$
$$-\lambda_2{}^\rho\partial_\rho\langle\Lambda_1\cdot\sigma\rangle)\psi - {}^1/_8 ia^2\langle[\Lambda_1 - \widetilde{\Lambda}_1 + 2\Delta(\Lambda_1)]\,[\Lambda_2$$
$$-\widetilde{\Lambda}_2 + 2\Delta(\Lambda_2)]\cdot\sigma\rangle\psi + {}^1/_4 ia\langle[\delta_{\lambda_2}{}^*\Delta(\Lambda_1)$$
$$-\delta_{\lambda_1}{}^*\Delta(\Lambda_2) - a\lambda_1{}^\rho\partial_\rho\Delta(\Lambda_2) + a\lambda_2{}^\rho\partial_\rho\Delta(\Lambda_1)]\cdot\sigma\rangle\psi, \tag{17}$$

where $\delta^*\Delta$ denotes the variation of Δ due to the variation of the h appearing in it which are varied in accordance with (2). Since we want the transformations (14) to form a representation of the group of transformations of $g^{\mu\nu}$, we must equate the result of the bracket operation to the variation (14):

$$(\delta_{\lambda_2}{}^*\delta_{\lambda_1}{}^* - \delta_{\lambda_1}{}^*\delta_{\lambda_2}{}^*)\psi = \delta^*_{\lambda_{\mathrm{br}}}\psi, \tag{18}$$

where λ_{br} is again given by formula (6). If in the left hand side we substitute expression (17) and write out the right hand side in detail, then in such form (18) will serve as a source for obtaining recurrence relations between the coefficients c_{mn}. (The principal steps in the calculation can be found in Appendix I to our paper[25].) From the recurrence relations it follows that c_{mn} can be defined by means of the generating function

$$G(x, y) = \sum_{m,\, n=0}^{\infty} c_{mn} x^m y^n,$$

$$G(x, y) = \frac{1}{2}\,\frac{(1+x)^{1/2} - (1+y)^{1/2}}{(1+x)^{1/2} + (1+y)^{1/2}}. \tag{19}$$

We reproduce the first few coefficients:

$$c_{01} = -\frac{1}{8};\quad c_{02} = \frac{1}{16};\quad \ldots;\quad c_{0n} = (-1)^n\frac{(2n-1)!!}{(2n+2)!!},$$

$$c_{12} = -\frac{1}{128};\quad c_{13} = \frac{1}{128};\quad \ldots;$$

$$c_{1n} = (-1)^{n+1}\frac{(2n-1)!!}{(2n+4)!!}(n-1), \tag{20}$$

. . . .

The appearance in (19) of square roots indicates that an important role must be played by the matrix $r = (1 + ah)^{1/2}$ which is uniquely defined by its expansion in series[5)]:

$$= 1 + \frac{1}{2}ah - \frac{1}{8}a^2h^2 + \ldots + \frac{1}{2}\left(\frac{1}{2} - 1\right)\ldots$$
$$\times\left(\frac{1}{2} - n + 1\right)\frac{1}{n!}a^n h^n + \ldots$$
$$= (1 + ah)^{1/2} = \sqrt{g}. \tag{21}$$

Representing the denominator of the generating function (19) as an integral of an exponential one can define the matrix Δ in the form

$$\Delta = {}^1/_2\int_0^\infty d\alpha\, e^{-\alpha r}[r, \Lambda + \widetilde{\Lambda}]\, e^{-\alpha r}, \tag{22}$$

where the square brackets denote a commutator.

Thus, it has been shown that the law of transformation of a spinor according to the representation of the group of transformations of the tensor $g^{\mu\nu}$ can be written in the form

$$\delta^*\psi = -a\lambda^\rho\partial_\rho\psi + {}^1/_4 ia\,\langle\Lambda\cdot\sigma\rangle\,\psi + {}^1/_8 ia\int_0^\infty d\alpha$$
$$\times\langle e^{-\alpha r}[r, \Lambda + \widetilde{\Lambda}]\, e^{-\alpha r}\cdot\sigma\rangle. \tag{23}$$

We emphasize that in (23) the field $h^{\mu\nu}$ appears in an essentially nonlinear manner, so that the spinor transforms according to a nonlinear representation of the group of generally covariant transformations.

We note that if we give up assumption b), then it is possible to introduce into (14) terms involving other matrices [over and above the terms which are always needed and which are present in (14)]. Thus, in seeking the law of transformation one could introduce additional terms which are multiples of the matrices 1 and γ_5. This lack of uniqueness can be interpreted as a transition to other objects. Thus, we introduce the quantity ψ_{vw} which transforms like $(\mathrm{Det}\, g)^{(v+w\gamma_5)/2}\psi$. We shall call this quantity ψ_{vw} a spinor of weight $v + w\gamma_5$. Its variation may be written in the form

$$\delta^*\psi_{vw} = a(v + w\gamma_5)\langle\Lambda\rangle\psi_{vw} - a\lambda^\rho\partial_\rho\psi_{vw}$$
$$+ {}^1/_4 ia\langle[\Lambda + \Delta]\cdot\sigma\rangle\psi_{vw}. \tag{24}$$

The first term on the right hand side is an additional term arising from the weight. It is not difficult to verify that the group property is again

5)I.e., in terms of indices:

$$r^{\mu\nu} = \delta_{\mu\nu} + {}^1/_2 ah^{\mu\nu} - {}^1/_8 a^2 h^{\mu\rho}h^{\rho\nu} + \ldots.$$

obeyed. Weighted spinors bear the same relationship to spinors as the well known weighted tensors do in relation to tensors[26].

In conclusion of this section we note that the law of transformation of the quantity r defined by formula (21) can be written in the form

$$\delta^* r = -a\lambda^\rho \partial_\rho r + a\widetilde{\Lambda} r + {}^1/_2 ar(\Lambda - \widetilde{\Lambda} + 2\Delta). \qquad (25)$$

With the aid of relation

$$r\Delta + \Delta r = {}^1/_2 r(\Lambda + \widetilde{\Lambda}) - {}^1/_2(\Lambda + \widetilde{\Lambda})r \qquad (26)$$

we can write the right hand side of (25) in different forms and, in particular, we can make it explicitly symmetric. For the derivation of (25) and (26) cf. Appendix 2 in reference [25].

It contrast to the usual tetrads in which one index refers to the usual basis system, while the second one to the locally orthogonal one, the indices of $r^{\mu\nu}$ have equal standing and refer to one general basis system. At the same time, with the aid of $r^{\mu\nu}$ one can also introduce a specific locally orthogonal basis with the differentials $dy^\mu = r^{\mu\nu}dx_\nu$, in which $dy^\mu dy^\mu = g^{\mu\nu}dx_\mu dx_\nu$. This remark enables us to give another derivation of the spinor law in accordance with the following outline:

1) $r^{\mu\nu}$ is introduced by means of the equation $r^2 = g = 1 + ah$ and is represented in the form (21);

2) $\delta^* r$ in (25) is evaluated. As a result of this the matrix Δ (15) or (22) is produced;

3) taking into account the fact that $dx'_\mu = dx_\mu - a\partial_\mu\lambda^\rho dx_\rho$ we find that under general transformations dy^μ undergo an induced local orthogonal transformation

$$dy'^\mu = r'^{\mu\nu}(x')dx_\nu' = dy^\mu - {}^1/_2 a(\Lambda - \widetilde{\Lambda} + 2\Delta)^{\mu\nu}dy^\nu;$$

4) replacing in the usual law for the Dirac spinor (9) $a\Lambda$ by the "parameters" of the last transformation $\frac{1}{2}a(\Lambda - \widetilde{\Lambda} + 2\Delta)$ we again obtain the spinor law (14).[6)]

4. COVARIANT DERIVATIVE OF A SPINOR

Fock and Ivanenko[4] and Fock[5] have defined a spinor as a geometric object on the basis of its behavior under parallel translation and directly from this have obtained the covariant derivative of a spinor.

Our definition of the covariant derivative of a spinor will be the direct consequence of the law obtained above for the transformation of the spinor (23) under general transformations. The covariant derivative of a tensor with respect to $g^{\mu\nu}$ can be defined to be such a modification of the ordinary derivative the application of which to a tensor again leads to a tensor of rank higher by unity[26,27]. Such an approach is equivalent to the approach utilizing parallel translation, but in contrast to the latter it is more explicitly related to the group property of tensors. Similarly, we shall define the covariant derivative of a spinor $\nabla_\mu\psi$ as an object which transforms in accordance with the direct product of a vector (with respect to the index μ) and a spinor (with respect to the spinor index of ψ) representation, i.e.,

$$\delta^*\nabla_\mu\psi = -a\lambda^\rho\partial_\rho(\nabla_\mu\psi) - a\partial_\mu\lambda^\rho\nabla_\rho\psi + {}^1/_4 ia\langle(\Lambda + \Delta(\Lambda))\cdot\sigma\rangle\nabla_\mu\psi. \qquad (27)$$

We represent the symbol for the covariant derivative in the form

$$\nabla_\mu = \partial_\mu - \Gamma_\mu. \qquad (28)$$

We now substitute (28) into (27) and utilize the law of transformation of a spinor (14). We then obtain the following transformation law:

$$\delta^*\Gamma_\mu = -a\lambda^\rho\partial_\rho\Gamma_\mu - a\partial_\mu\lambda^\rho\Gamma_\rho + {}^1/_4 ia[\langle(\Lambda + \Delta)\cdot\sigma\rangle, \Gamma_\mu] + {}^1/_4 ia\langle\partial_\mu(\Lambda + \Delta)\cdot\sigma\rangle. \qquad (29)$$

On the assumption that Γ_μ is completely expressed in terms of $h^{\mu\nu}$ (similarly to the affine relation for tensors in Riemannian geometry) (29) represents an inhomogeneous equation for the determination of Γ_μ.

We expand Γ_μ in terms of the complete system of Dirac matrices:

$$\Gamma_\mu = a_\mu I + a_\mu{}^\alpha\gamma_\alpha + a_\mu{}^{\alpha\beta}\sigma_{\beta\alpha} + a_\mu{}^{\alpha5}i\gamma_\alpha\gamma_5 + a_\mu{}^5\gamma_5. \qquad (30)$$

If we substitute the expansion (30) into (29) we can obtain a particular solution of the inhomogeneous equation (29) in the form (cf., Appendix 3 to reference [25])

$$\Gamma_\mu = -{}^1/_4 ir^{\alpha\beta}[\partial_\beta g_{\mu\gamma} + (r^{-1}\partial_\mu r^{-1})_{\beta\gamma}]r^{\gamma\delta}\sigma_{\delta\alpha}$$
$$= -{}^1/_4 ir^{\alpha\beta}[-[\beta, \gamma\mu] + (r^{-1}\partial_\mu r^{-1})_{\beta\gamma}]r^{\gamma\delta}\sigma_{\delta\alpha}. \qquad (31)$$

The last expression is written in terms of the three index Christoffel symbol of the first kind[26,27] in order to demonstrate the relationship of the expression obtained above for Γ_μ with the expression which was obtained in tetrad formalism [cf., reference [19], formula (8)]. We emphasize that Γ_μ (31) does not contain any new quantities and represents a series in powers of the gravi-

6)If by means of such an approach one generalizes the tensors of the special theory of relativity, one obtains not the usual tensors, but quantities which also transform according to nonlinear representations, laws of the type (42) – (44), cf., below. For example, instead of the vector law one would obtain (42).

tational field (cf., expansion (21) for r):

$$\Gamma_\mu = -{}^1/_4 i\{-a\partial_\alpha h^{\mu\beta} + a^2[h^{\mu\sigma}\partial_\alpha h^{\sigma\beta} + {}^1/_2\partial_\alpha h^{\mu\sigma}h^{\sigma\beta} - {}^1/_2 h^{\alpha\sigma}\partial_\sigma h^{\mu\beta} - {}^1/_4\partial_\mu h^{\alpha\sigma}h^{\sigma\beta}] + O(a^3)\}\sigma_{\beta\alpha}. \tag{32}$$

The general solution of (29) consists of (31) and of the general solution of the homogeneous equation corresponding to (29):

$$\Gamma^\mu_{\text{gen}} = \Gamma_\mu + a_\mu I + b_{\mu\alpha}r^{\alpha\beta}\gamma_\beta + c_{\mu\alpha\beta}r^{\alpha\gamma}r^{\beta\delta}\sigma_{\delta\gamma} + d_{\mu\alpha}r^{\alpha\beta}i\gamma_\beta\gamma_5 + a_\mu^5\gamma_5, \tag{33}$$

where a_μ and a_μ^5, $b_{\mu\alpha}$, $d_{\mu\alpha}$ and $c_{\mu\alpha\beta}$ are covariant vectors and tensors of the second and the third rank respectively (cf., Appendix 3 to reference[25]). In contrast to Γ_μ (the simplest possible solution of the inhomogeneous equation), all the remaining terms in (33) can either be considered equal to zero, or different from zero and constructed from some suitable fields. The introduction of the covariant derivative is useful for the construction of invariant interactions. The inclusion into the covariant derivative of the solution of the inhomogeneous equation, for example (31), is necessary for this. As regards the remaining terms appearing in (33), they are not necessary, and the corresponding interactions can always be written in invariant form separately.

Let us make an analogy. The electromagnetic interactions are always also included through the "covariant derivative" $\partial_\mu - ieA_\mu$. However, the choice of this covariant derivative is not unique to the same extent. Nothing prevents us from taking in place of $\partial_\mu - ieA_\mu$ the "covariant derivative" $\partial_\mu - ieA_\mu + f\gamma_\nu F_{\nu\mu}$, etc. The lack of uniqueness in (33) is of exactly the same type. For example, the tensor $c_{\mu\alpha\beta}$ can be realized in the form

$$c_{\mu\alpha\beta} = g_{\mu\alpha}\partial_\beta R - g_{\mu\beta}\partial_\alpha R,$$

where R is the scalar curvature. This would lead to an additional "non-minimal" interaction with the gravitational field through its higher derivatives.

We shall assume that the covariant derivative of a spinor is $\nabla_\mu = \partial_\mu - \Gamma_\mu$, where the affine relation of Γ_μ is given by expression (31). The interactions to which this leads we shall call "minimal". Such a choice of a covariant derivative is also convenient because in this case

$$\nabla_\mu(\bar\psi\psi) = \partial_\mu(\bar\psi\psi), \qquad \nabla_\mu(\bar\psi\gamma_5\psi) = \partial_\mu(\bar\psi\gamma_5\psi), \tag{34}$$

$$\nabla_\mu(r^{\nu\lambda}\bar\psi\gamma_\lambda\psi) = (\delta_\rho{}^\nu\partial_\mu + \Gamma_{\mu\rho}{}^\nu)(r^{\rho\lambda}\bar\psi\gamma_\lambda\psi), \tag{35}$$

$$\nabla_\mu(r^{\nu\lambda}\bar\psi\gamma_\lambda\gamma_5\psi) = (\delta_\rho{}^\nu\partial_\mu + \Gamma_{\mu\rho}{}^\nu)(r^{\rho\lambda}\bar\psi\gamma_\lambda\gamma_5\psi), \tag{36}$$

$$\nabla_\mu(r^{\nu\lambda}r^{\sigma\tau}\bar\psi\sigma_{\lambda\tau}\psi) = (\delta_\alpha{}^\nu\delta_\beta{}^\sigma\partial_\mu + \delta_\alpha{}^\nu\Gamma_{\mu\beta}{}^\sigma + \delta_\beta{}^\sigma\Gamma_{\mu\alpha}{}^\nu)(r^{\alpha\lambda}r^{\beta\tau}\bar\psi\sigma_{\lambda\tau}\psi). \tag{37}$$

The covariant derivatives in (34)—(37) are calculated taking into account the distributive property, for example,

$$\nabla_\mu(r^{\nu\lambda}\bar\psi\gamma_\lambda\psi) = (\nabla_\mu r^{\nu\lambda})\bar\psi\gamma_\lambda\psi + r^{\nu\lambda}(\nabla_\mu\bar\psi)\gamma_\lambda\psi + r^{\nu\lambda}\bar\psi\gamma_\lambda\nabla_\mu\psi.$$

In this calculation the covariant derivative of $r^{\nu\lambda}$ as a function of $g^{\alpha\beta}$ is equal to zero:

$$\nabla_\mu r^{\nu\lambda} = 0, \tag{38}$$

and this can be regarded as a consequence of the easily verified important identity

$$\partial_\mu r^{\sigma\rho} + \Gamma_{\mu\lambda}{}^\sigma r^{\lambda\rho} - ir^{\sigma\alpha}\,\mathrm{Sp}(\sigma_{\alpha\rho}\Gamma_\mu) = 0. \tag{39}$$

The form of the covariant derivatives (34)—(37) agrees with the fact that the combinations written out in this form transform like scalars, vectors and a tensor (cf., the next section). At the corresponding point of the formalism of orthogonal basis vectors Fock[5] has fixed the form of the covariant derivative with the aid of conditions (34) and (35). A requirement of this nature leaves in (33) an arbitrariness only in the choice of a_μ, while $b = c = d = a^5 = 0$. However, we note that the terms eliminated from the covariant derivative can be introduced into the invariant Lagrangian as new independent interactions.

In conclusion we briefly discuss weighted spinors ψ_{vw} (24). For these quantities additional terms will appear in (29)

$$a(v + w\gamma_5)\partial_\mu\langle\Lambda\rangle - aw\langle\Lambda\rangle[\Gamma_\mu, \gamma_5].$$

As a result for a weighted spinor one should take for the simplest affine relation

$$\Gamma_\mu{}^{(v,w)} = \Gamma_\mu - (v + w\gamma_5)\Gamma_{\mu\alpha}{}^\alpha \tag{40}$$

incomplete analogy with what occurs for tensors (cf., reference[26], p. 55). The application of the covariant derivative to the bilinear combinations of ψ_{vw} will yield results different from (34)—(37).

5. PROPERTIES OF BILINEAR COMBINATIONS

From (14) it follows that

$$\delta^*(\bar\psi\psi) = -\alpha\lambda^\rho\partial_\rho(\bar\psi\psi), \tag{41}$$

$$\delta^*(\bar\psi\gamma_\mu\psi) = -a\lambda^\rho\partial_\rho(\bar\psi\gamma_\mu\psi) - {}^1/_2 a\,(\Lambda - \tilde\Lambda + 2\Delta)_{\mu\beta}\bar\psi\gamma_\beta\psi, \tag{42}$$

$$\delta^*(\bar\psi\sigma_{\mu\nu}\psi) = -a\lambda^\rho\partial_\rho(\bar\psi\sigma_{\mu\nu}\psi) - {}^1/_2 a\,(\Lambda - \tilde\Lambda + 2\Delta)_{\mu\beta}\bar\psi\sigma_{\beta\nu}\psi - {}^1/_2 a(\Lambda - \tilde\Lambda + 2\Delta)_{\nu\beta}\bar\psi\sigma_{\mu\beta}\psi, \tag{43}$$

$$\delta^*(\bar\psi\gamma_\mu\gamma_5\psi) = -\alpha\lambda^\rho\partial_\rho(\bar\psi\gamma_\mu\gamma_5\psi) - {}^1/_2 a\,(\Lambda - \tilde\Lambda + 2\Delta)_{\mu\beta}\bar\psi\gamma_\beta\gamma_5\psi, \tag{44}$$

$$\delta^*(\bar\psi\gamma_5\psi) = -a\lambda^\rho\partial_\rho(\bar\psi\gamma_5\psi), \tag{45}$$

$$\delta^*(\bar\psi\gamma_\mu\nabla_\nu\psi) = -a\lambda^\rho\partial_\rho(\bar\psi\gamma_\mu\nabla_\nu\psi) - a\partial_\nu\lambda^\rho\bar\psi\gamma_\mu\nabla_\rho\psi - {}^1/_2 a\,(\Lambda - \tilde\Lambda + 2\Delta)_{\mu\beta}\bar\psi\gamma_\beta\nabla_\nu\psi. \tag{46}$$

Consequently, the quantities $(\bar{\psi}\psi)$ and $\bar{\psi}\gamma_5\psi$ transform like scalars, while the remaining quantities do not transform according to tensor laws. Combinations obtained by multiplying by a nontensor quantity $r^{\mu\nu}$ will transform like tensors; the quantities $r^{\mu\nu}\bar{\psi}\gamma_\nu\psi$ and $r^{\mu\nu}\bar{\psi}\gamma_\nu\gamma_5\psi$ are, in terms of the usual terminology, contravariant vectors, $r^{\mu\alpha}r^{\nu\beta}\psi\sigma_{\alpha\beta}\psi$ is a contravariant antisymmetric tensor of the second rank, while $r^{\mu\alpha}\bar{\psi}\gamma_\alpha\nabla_\nu\psi$ is a mixed tensor of the second rank; the contraction of the latter tensor is a scalar which appears in the Lagrangian for the spinor field.

6. INTERACTIONS OF A SPINOR FIELD

The Lagrangian density must be not a scalar which transforms in accordance with $\delta^*\varphi = -a\lambda^\rho\partial_\rho\varphi$, but a relative scalar of weight 1 which under general transformations changes infinitesimally by a divergence:

$$\delta^*\mathcal{L} = -a\partial_\rho(\lambda^\rho\mathcal{L}).$$

Then the total Lagrangian will be an invariant. Knowing the transformation properties of bilinear combinations of spinors one can easily construct a Lagrangian density for the interaction of a spinor field simultaneously with the gravitational field a and an electromagnetic field A

$$\mathcal{L} = -\{{}^1\!/_2 r^{\mu\nu}[\bar{\psi}\gamma_\mu(\partial_\nu - \Gamma_\nu - ieA_\nu)\psi - \bar{\psi}(\overleftarrow{\partial_\nu} + \Gamma_\nu + ieA_\nu)\gamma_\mu\psi] + m\bar{\psi}\psi\}(\operatorname{Det} g)^{-1/2}. \tag{47}$$

By means of integration by parts this Lagrangian can be reduced to a simpler form, which, however, is not selfconjugate:

$$\mathcal{L} = -\{r^{\mu\nu}\bar{\psi}\gamma_\mu(\partial_\nu - \Gamma_\nu - ieA_\nu)\psi + m\bar{\psi}\psi\}(\operatorname{Det} g)^{-1/2}, \tag{48}$$

and from this the Dirac equation in a gravitational field immediately follows:

$$r^{\mu\nu}\gamma_\mu(\partial_\nu - \Gamma_\nu - ieA_\nu)\psi + m\psi = 0. \tag{49}$$

The Lagrangian density (47) (in contrast to the Lagrangian density in the formalism of an orthogonal basis system) is explicitly expressed in terms of the gravitational and other fields and represents an infinite series in powers of the gravitational field $h^{\mu\nu}$. We obtain the initial terms of the series. In order to do this we utilize the expansions of $r^{\mu\nu}$ (21), Γ_μ (32) and[7)]

[7)]Formula (50) follows from the relation

$$\operatorname{Det}\|g^{\mu\nu}\| = {}^1\!/_{24}\{(g^{\alpha\alpha})^4 - 6(g^{\alpha\alpha})^2 g^{\beta\gamma}g^{\gamma\beta} + 3(g^{\alpha\beta}g^{\beta\alpha})^2 + 8g^{\alpha\alpha}g^{\beta\gamma}g^{\gamma\delta}g^{\delta\beta} - 6g^{\alpha\beta}g^{\beta\gamma}g^{\gamma\delta}g^{\delta\alpha}\}.$$

$$\operatorname{Det} g = 1 + ah_1 + {}^1\!/_2a^2(h_1^2 - h_2) + {}^1\!/_6a^3(h_1^3 - 3h_1h_2 + 2h_3) + {}^1\!/_{24}a^4(h_1^4 - 6h_1^2h_2 + 8h_1h_3 + 3h_2^2 - 6h_4), \tag{50}$$

$$(\operatorname{Det} g)^{-1/2} = 1 - {}^1\!/_2ah_1 + {}^1\!/_8a^2(h_1^2 + 2h_2) + \ldots;$$
$$h_1 = h^{\alpha\alpha},\quad h_2 = h^{\alpha\beta}h^{\beta\alpha},\quad h_3 = h^{\alpha\beta}h^{\beta\gamma}h^{\gamma\alpha},\quad h_4 = h^{\alpha\beta}h^{\beta\gamma}h^{\gamma\delta}h^{\delta\alpha}. \tag{51}$$

Then

$$\mathcal{L} = -{}^1\!/_2[\bar{\psi}\gamma_\mu(\partial_\nu - ieA_\nu)\psi - (\partial_\nu + ieA_\nu)\bar{\psi}\gamma_\mu\psi]\{\delta_{\mu\nu} + {}^1\!/_2a(h^{\mu\nu} - \delta_{\mu\nu}h_1) + {}^1\!/_8a^2[\delta_{\mu\nu}(h_1^2 + 2h_2) - h^{\mu\nu}h_1 - h^{\mu\rho}h^{\rho\nu}]\} - m\bar{\psi}\psi[1 - {}^1\!/_2ah_1 + {}^1\!/_8a^2(h_1^2 + 2h_2)] + {}^1\!/_{16}a^2\varepsilon_{\mu\nu\lambda\rho}\bar{\psi}\gamma_5\gamma_\mu\psi h^{\nu\sigma}\partial_\lambda h^{\sigma\rho} + O(a^3). \tag{52}$$

Continuing with the expansion of (47) one can also easily obtain further terms of the series. In the linear approximation in $ah^{\mu\nu}$ the interaction of fermions with gravitons has been considered already[1,28]. The terms of the second order in a written out in (52) enable us to calculate the gravitational self-energy, the Compton-effect of gravitons, etc. One can also easily write down the interactions with other fields. For example, pseudoscalar coupling with a pseudoscalar field φ can be written down in the following manner:

$$\mathcal{L} = if\bar{\psi}\gamma_5\psi\varphi(\operatorname{Det} g)^{-1/2}. \tag{53}$$

The four-fermion interactions also have a very simple appearance; the presence of a gravitational field leads only to the appearance of $(\operatorname{Det} g)^{-1/2}$:

$$\mathcal{L} = \{f_S(\bar{\psi}_1\psi_2)(\bar{\psi}_3\psi_4) + f_V(\bar{\psi}_1\gamma_\mu\psi_2)(\bar{\psi}_3\gamma_\mu\psi_4) + f_T(\bar{\psi}_1\sigma_{\mu\nu}\psi_2)(\bar{\psi}_3\sigma_{\mu\nu}\psi_4) + \ldots\}(\operatorname{Det} g)^{-1/2}. \tag{54}$$

We emphasize that the γ-matrices in (54) are general numerical Dirac matrices:

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}.$$

In conclusion we shall state the result of squaring the Dirac equation in the electromagnetic and the gravitational fields (49):

$$[(\operatorname{Det} g)^{1/2}(\partial_\mu - \Gamma_\mu - ieA_\mu)(\operatorname{Det} g)^{-1/2}g^{\mu\nu}(\partial_\nu - \Gamma_\nu - ieA_\nu) + {}^1\!/_4R + {}^1\!/_2er^{\mu\lambda}r^{\nu\rho}F_{\mu\nu}\sigma_{\lambda\rho} - m^2]\psi = 0,$$

where R is the scalar curvature. This equation was first derived by Fock (in the orthogonal basis formalism), while a convenient system for the intermediate calculations needed for its derivation may be found in Schrödinger's paper[10]. In our formalism the calculations are carried out in an analogous manner.

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235