SOME PROBLEMS OF RELATIVISTIC HYDRODYNAMICS OF RELAXING MEDIA

M. T. ZHUMARTBAEV

Institute of Nuclear Physics, Academy of Sciences, Kazakh S.S.R.

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A closed system of equations is obtained for relaxation relativistic hydrodynamics under the assumption that the resultant nonequilibrium states can be described thermodynamically. Both limiting cases—relativistic and ultrarelativistic—in which the relaxation process is due to the chemical reaction rate, on the one hand, and the rate of creation and annihilation of pairs, on the other, are considered. A study is made of the propagation of small perturbations in a relaxing medium. For small relaxation times, a limiting transition is made to equilibrium relativistic hydrodynamics. The structure of a weak relativistic shock wave is considered with account of relaxation. The equations of two-component relativistic hydrodynamics are generalized to cover multicomponent systems and are applied to multiple particle production.

1. FUNDAMENTAL EQUATIONS OF RELAXATION RELATIVISTIC HYDRODYNAMICS

To formulate the equations of relaxation hydrodynamics we introduce, as in nonrelativistic hydrodynamics, [1] an additional parameter ξ (in the general case, several such parameters), characterizing the deviation of the system from complete thermodynamic equilibrium. This parameter may be, for example, the concentration of one of the mixture components. We assume further that the resultant nonequilibrium states can be described thermodynamically, i.e., the following thermodynamic identities hold:

$$dw = \frac{1}{n}dp + Td\sigma + w_\xi d\xi, \quad d\varepsilon = nTd\sigma + wdn + \varepsilon_\xi d\xi, \quad (1.1)$$

where w and σ are the heat function and the entropy per particle; ϵ and n are the density of the internal energy and the number of particles per unit volume. In the state of equilibrium, when $\xi = \xi_0(n, \sigma)$, we have

$$w_\xi(p, \sigma, \xi_0) = 0, \quad \varepsilon_\xi(n, \sigma, \xi_0) = 0,$$

$$w_\xi \equiv (\partial w / \partial \xi)_{p,\sigma}, \quad \varepsilon_\xi \equiv (\partial \varepsilon / \partial \xi)_{n,\sigma}. \quad (1.2)$$

We shall assume that ξ characterizes the concentration of one of the components of the mixture. Let ξ be constant, and then the supplementary equation can be written in the form

$$d\xi / ds = 0.$$

Using the equation of continuity as a whole

$$\frac{\partial(nu_k)}{\partial x_k} = -\frac{\partial v_k}{\partial x_k}, \quad (1.3)$$

where ν_i is a certain four-vector, connected with the heat conduction and the relaxation process, we obtain

$$\frac{\partial(nu_k\xi)}{\partial x_k} = -\xi\frac{\partial v_k}{\partial x_k} \quad (1.4)$$

In the general case, however, one of the mixture components reacts with the other and the concentration ξ does not remain constant; therefore Eq. (1.4) should be written in the form

$$\frac{\partial(nu_k\xi + j_k)}{\partial x_k} = -\xi\frac{\partial v_k}{\partial x_k},$$

where f_i is the diffusion four-current.

The spatial component j has the meaning of the ordinary diffusion current, while the temporal component j_4 determines either the dissociation rate in the relativistic case or the rate of pair production and annihilation in the ultrarelativistic case. Again using Eq. (1.3), we ultimately obtain the relaxation equation

$$n\frac{d\xi}{ds} = -\frac{\partial j_k}{\partial x_k}. \quad (1.5)$$

The equation of motion and the entropy balance equation are

$$n\frac{d(wu_i)}{ds} + \frac{\partial p}{\partial x_i} = -\frac{\partial \tau_{ik}}{\partial x_k} + wu_i\frac{\partial v_k}{\partial x_k}, \quad (1.6)$$

$$nT\frac{d\sigma}{ds} = u_l\frac{\partial \tau_{lk}}{\partial x_k} + w\frac{\partial v_k}{\partial x_k} + w_\xi\frac{\partial j_k}{\partial x_k}. \quad (1.7)$$

We define the velocity by means of the same condition as in the book by Landau and Lifshitz. [2]

Inasmuch as the particle current $nu_i + \nu_i$ should give the particle density n in the proper system,

the four-vectors ν_i and j_i, as well as the four-tensor τ_{ik}, should satisfy the additional conditions

$$u_k v_k = 0, \quad u_k \tau_{ik} = 0, \quad u_k j_k = -j_0, \quad (1.8)$$

where j_0 is the temporal component of j_i, taken in the proper system. By virtue of these conditions, Eq. (1.7) can be represented in the form

$$\frac{\partial}{\partial x_k}\left(S_\partial u_k - \frac{\mu}{T}\nu_k - \frac{w_\xi}{T}j_k\right) = -\nu_k\frac{\partial}{\partial x_k}\left(\frac{\mu}{T}\right)$$
$$-\frac{1}{T}\tau_{lk}\frac{\partial u_l}{\partial x_k} - j_k\frac{\partial}{\partial x_k}\left(\frac{w_\xi}{T}\right) = \Sigma, \quad (1.9)$$

where μ is the chemical potential and S_e is the entropy per unit volume. The right side of (1.9) corresponds to the production of entropy in a unit volume as a result of the irreversible processes occurring in it. The quantity Σ should be positive.

According to the Onsager principle, the currents ν_i, j_i, and τ_{ik} should be linear functions of the forces

$$\frac{\partial}{\partial x_i}\left(\frac{\mu}{T}\right), \quad \frac{\partial}{\partial x_i}\left(\frac{w_\xi}{T}\right), \quad \frac{1}{T}\frac{\partial u_k}{\partial x_l}.$$

The coefficients in the expansions are determined under the following conditions:

a) The medium is assumed isotropic. b) The expressions for the currents, when substituted in (1.9) should ensure the positiveness of the right side. c) The currents should satisfy identically the conditions (1.8). d) In the nonrelativistic limit, the corresponding nonrelativistic expression should be obtained for the currents. e) The principle of symmetry of the kinetic coefficients applies. f) In the absence of relaxation, the equations should go over into the known relativistic equations with account of viscosity and thermal conductivity.

The last of these conditions is satisfied by virtue of (1.2) automatically if the preceding conditions are satisfied. We note that the foregoing conditions are completely analogous to the conditions imposed by Doĭch [3] for a unique determination of the coefficients in the expansions for the currents in relativistic magnetohydrodynamics.

As a result we obtain the following expressions for the currents:

$$\nu_i = -\varkappa\left(\frac{T}{w}\right)^2\left\{\frac{\partial}{\partial x_i}\left(\frac{\mu}{T}\right) + u_i u_k\frac{\partial}{\partial x_k}\left(\frac{\mu}{T}\right)\right\}$$
$$-D\left(\frac{T}{w}\right)^2\left\{\frac{\partial}{\partial x_i}\left(\frac{w_\xi}{T}\right) + u_i u_k\frac{\partial}{\partial x_k}\left(\frac{w_\xi}{T}\right)\right\}, \quad (1.10)$$

$$j_i = -D'\left(\frac{T}{w}\right)^2\left\{\frac{\partial}{\partial x_i}\left(\frac{\mu}{T}\right) + u_i u_k\frac{\partial}{\partial x_k}\left(\frac{\mu}{T}\right)\right\} - D_0$$
$$\times\left(\frac{T}{w}\right)^2\left\{\frac{\partial}{\partial x_i}\left(\frac{w_\xi}{T}\right) + u_i u_k\frac{\partial}{\partial x_k}\left(\frac{w_\xi}{T}\right)\right\} + u_i j_0, \quad (1.11)$$

where κ is the known thermal conductivity coeffi-

cient and D and D_0 are the diffusion coefficients, with c = 1. The viscosity tensor τ_{ik} has the same form as in [2].

Separating the pure diffusion part in (1.11), we can rewrite the relaxation equation (1.5) in the form

$$n\frac{d\xi}{ds} = -\frac{\partial j_k{}^{(D)}}{\partial x_k} - r, \quad (1.12)$$

where

$$r = \frac{\partial(u_k j_0)}{\partial x_k}. \quad (1.13)$$

The meaning of (1.13) becomes clear if we go over to the proper system, where $n\partial\xi/\partial t = -\partial j_0/\partial t$, i.e., the variation of the relaxation parameter ξ is determined by the variation of j_0 with time in the same volume element.

In Eqs. (1.3), (1.6), (1.7), the relaxation equation (1.12), together with Eqs. (1.10) and (1.11) and the equations of state form the closed system of equations for relaxation relativistic hydrodynamics.

We consider below the most important particular case, when we can neglect viscosity and thermal conductivity of the medium, and also diffusion ($D = D_0 = 0$). In this case the system of equations takes the form

$$n\frac{d(wu_i)}{ds} + \frac{\partial p}{\partial x_i} = 0, \quad \frac{\partial(nu_k)}{\partial x_k} = 0,$$
$$nT\frac{d\sigma}{ds} = w_\xi r, \quad n\frac{d\xi}{ds} = -r. \quad (1.14)$$

We consider small deviations from the state of local thermodynamic equilibrium.

For small deviations, the quantities w and r can be expanded in powers of $\xi - \xi_0$. It is obvious, that in the state of equilibrium $r|_{\xi - \xi_0} = 0$. Confining ourselves to first-order terms, we get

$$nT\frac{d\sigma}{ds} = w_{\xi\xi}r_\xi(\xi - \xi_0)^2, \quad \frac{d\xi}{ds} = -\frac{\xi - \xi_0}{\tau}, \quad (1.15)$$

where $\tau = n/r_\xi$ is the relaxation time. From (1.14) we see that the change in entropy, for small deviations from the equilibrium state, is of second order of smallness.

2. PROPAGATION OF SMALL DISTURBANCES IN A RELAXING MEDIUM AND TRANSITION TO EQUILIBRIUM HYDRODYNAMICS

We consider the propagation of small disturbances, such as sound waves, in a relaxing medium. For small deviations from the equilibrium state, confining ourselves to first-order terms, we obtain

$$r = r_\xi\delta\xi + r_n\delta n, \quad \delta p = p_\xi\delta\xi + p_n\delta n, \quad (2.1)$$

since the change in entropy is of second order of

smallness.

According to (2.1), we write (1.14) in the form

$$\frac{\partial \delta n}{\partial t} + n\frac{\partial \delta v_\beta}{\partial x_\beta} = 0,$$

$$n\frac{\partial \delta v_\alpha}{\partial t} + c_\infty^2 \frac{\partial \delta n}{\partial x_\alpha} + \frac{p_\xi}{w}\frac{\partial \delta\xi}{\partial x_\alpha} = 0,$$

$$\frac{\partial \delta\xi}{\partial t} = -\frac{1}{\tau}\Big(\delta\xi + \frac{r_n}{r_\xi}\delta n\Big), \qquad (2.2)$$

where

$$c_\infty^2 = (n/w)\partial w/\partial n$$

determines the rate of propagation of the sound vibrations with frequency $\omega \gg 1/\tau$ ($\alpha, \beta = 1, 2, 3$).

The solution of the last equation in (2.2) we seek in the form

$$\delta\xi = -\frac{r_n}{r_\xi}\Lambda^{-1}\delta n, \quad \Lambda = 1 + \tau\frac{\partial}{\partial t}.$$

Eliminating the derivative of δn, we obtain

$$\Lambda\Big\{\frac{\partial^2 \delta v_\alpha}{\partial t^2} - c_\infty^2 \frac{\partial^2 \delta v_\gamma}{\partial x_\gamma \partial x_\alpha}\Big\} + \frac{p_\xi r_n}{w r_\xi}\frac{\partial^2 \delta v_\gamma}{\partial x_\gamma \partial x_\alpha} = 0. \quad (2.3)$$

It can be shown that in this approximation the motion is potential, i.e., curl $\mathbf{v} = 0$. This denotes that

$$\partial^2 \delta v_\gamma / \partial x_\gamma \partial x_\alpha = \Delta \delta v_\alpha.$$

Thus, Eq. (2.3) can be rewritten as

$$\tau\frac{\partial}{\partial t}\Big\{\frac{\partial^2 \delta v_\alpha}{\partial t^2} - c_\infty^2 \Delta \delta_\alpha\Big\} + \frac{\partial^2 \delta v_\alpha}{\partial t^2} - c_0^2 \Delta \delta v_\alpha = 0, \quad (2.4)$$

where

$$c_0^2 = c_\infty^2 - p_\xi r_n/w r_\xi. \qquad (2.5)$$

Equation (2.4) is formally identical with the corresponding equation (see [1]) in nonrelativistic hydrodynamics, while (2.5) determines the equilibrium speed of sound c_0.

We see from (2.4) that absorption of sound waves takes place. Indeed, let us represent a small velocity disturbance in the form of a plane wave; then we can readily arrive at the well-known relation between the frequency and the wave vector

$$k = \omega\Big(\frac{1 - i\omega\tau}{c_0^2 - c_\infty^2 i\omega\tau}\Big)^{1/2},$$

i.e., the propagation of sound is accompanied by dispersion.

Let us now consider the question of the transition to equilibrium hydrodynamics. The equations of the characteristics of the system (1.14) for one-dimensional flow are of the form

$$\frac{dx}{dt} = \frac{v \pm c_\infty}{1 \pm v c_\infty}, \quad \frac{dx}{dt} = v. \qquad (2.6)$$

We see therefore that as τ tends to zero the char-

acteristics of the relaxation hydrodynamics do not go over into the characteristics of equilibrium hydrodynamics, which are determined by the speed c_0.

A direct series expansion of all the hydrodynamic quantities in powers of the small parameter τ does not lead to the desired results, since the next higher approximations give expressions that diverge in time. Therefore, to go over to equilibrium relativistic hydrodynamics we must consider the flow starting from the instant when the relaxation process in the zeroth approximation in the parameter $\delta_0 = c_0\tau/L$ ($\delta_0 \ll 1$), where L is the characteristic macroscopic length, has already been completed. Under these conditions, the following expansions will be valid at subsequent instants of time

$$\xi = \xi_0 + \delta_0\xi^{(1)} + \delta_0^2\xi^{(2)} + \ldots,$$

$$\frac{\partial}{\partial t}\frac{n}{(1-v^2)^{1/2}} = \delta_0 N^{(1)} + \delta_0^2 N^{(2)} + \ldots,$$

$$(1-v^2)^{-1/2}\frac{\partial}{\partial t}\frac{wv_\alpha}{(1-v^2)^{1/2}} = \delta_0 V_\alpha^{(1)} + \delta_0^2 V_\alpha^{(2)} + \ldots.$$

$$\frac{\partial\sigma}{\partial t} = \delta_0\Sigma^{(1)} + \delta_0^2\Sigma^{(2)} + \ldots, \qquad (2.7)$$

where $\xi^{(1,2)}$, N, V_α, and Σ are functions of the variables n, v_α, and σ, and ξ depends also on x.

Explicit expressions for the coefficients N, V_α, and Σ can be readily obtained by comparing (2.7) with the system (1.14) and by recognizing that the spatial derivatives increase the order of smallness of the quantities by unity. Just as in the paper of Stakhanov and Stupochenko, [1] we obtain

$$\frac{\delta_0}{\tau}\xi^{(1)} = \frac{\partial\xi_0}{\partial n}n\frac{\partial}{\partial x_\beta}\frac{v_\beta}{(1-v^2)^{1/2}}$$

In addition, taking into consideration the expansions

$$w = w_0(n, \sigma) + \delta_0^2 w_{\xi\xi}\xi^{(2)} + \ldots,$$

$$p = p_0(n, \sigma) + \delta_0 (p_\xi)_{\xi=\xi_0}\xi^{(1)} + \ldots,$$

where w_0 and p_0 are the equilibrium values of the heat function and of the pressure, we ultimately obtain the equations of motion in first approximation, in the form

$$n\Big\{\frac{d}{ds}(wu_\alpha) + \frac{\partial w}{\partial x_\alpha}\Big\} = -\frac{\partial}{\partial x_\alpha}\Big(\zeta\frac{\partial}{\partial x_\alpha}u_\beta\Big), \quad (2.8)$$

where $\zeta = \tau n w c_0^2$ plays the role of the viscosity coefficient.

To go over to the ultrarelativistic case, when the chemical potential in the zeroth approximation is $\mu = 0$, it is necessary to make in (2.8) a substitution $n \to S$, $w \to T$. [4] The latter case is of interest in the hydrodynamic theory of multiple particle production.

3. WIDTH OF SHOCK WAVES WITH ACCOUNT OF RELAXATION

To determine the distribution of the thermodynamic quantities over the width of a shock wave, we make use of the ordinary hydrodynamic conservation laws for the mass, energy, and momentum flux with account of the fluxes brought about by the irreversible terms. The corresponding conservation laws for a stationary discontinuity are of the form (see [5])[1]:

$$nu_x - \left(\frac{T}{w}\right)^2 (1 + u_x^2) \left\{\varkappa \frac{d}{dx}\left(\frac{\mu}{T}\right) + D \frac{d}{dx}\left(\frac{w_\xi}{T}\right)\right\}$$

$$= n_0 u_x^0 = i, \tag{3.1}$$

$$nw u_x^2 + p - \left(\frac{4}{3}\eta + \zeta\right)(1 + u_x^2)\frac{du_x}{dx}$$

$$= n_0 w_0 u_x^{02} + p_0, \tag{3.2}$$

$$nw u_x u_4 - \left(\frac{4}{3}\eta + \zeta\right) u_x u_4 \frac{du_x}{dx} = n_0 w_0 u_x^0 u_4^0, \tag{3.3}$$

$$nu_x \frac{d\xi}{dx} = -\frac{dj_x}{dx} \tag{3.4}$$

(the shock wave moves from right to left, the state ahead of the shock wave is designated by a zero index).

For shock waves of low intensity, all the quantities in the shock wave can be expanded in powers of the entropy discontinuities $\Delta\sigma = \sigma - \sigma_0$, the pressure discontinuities $\Delta p = p - p_0$ and the concentration discontinuities $\Delta\xi = \xi - \xi_0$. Taking into account the thermodynamic equations (1.1) we get, as in [5]

$$w - w_0 = \frac{1}{n_0}\Delta p + T\Delta\sigma + w_\xi\Delta\xi + \frac{1}{2}\frac{\partial}{\partial p_{\sigma,\xi}}\left(\frac{1}{n}\right)\Delta p^2,$$

$$n - n_0 = -n_0^2 \frac{\partial}{\partial p_{\sigma,\xi}}\left(\frac{1}{n}\right)\Delta p - n_0^2 \frac{\partial}{\partial\sigma}\left(\frac{1}{n}\right)_{p,\xi}\Delta\sigma$$

$$- n_0^2 \frac{\partial}{\partial\xi_{p,\sigma}}\left(\frac{1}{n}\right)\Delta\xi$$

$$- n_0^2 \left[\frac{1}{2}\frac{\partial^2}{\partial p_{\sigma,\xi}^2}\left(\frac{1}{n}\right) - n_0\left(\frac{\partial}{\partial p_{\sigma,\xi}}\left(\frac{1}{n}\right)\right)^2\right]\Delta p^2. \tag{3.5}$$

Let us express the derivatives in (3.1) in terms of derivatives of p, σ, and ξ. It must be borne in mind here that differentiation with respect to x increases the order of smallness of the quantities by unity (since the width of the shock wave is inversely proportional to the amplitude of the wave). Therefore the derivative dp/dx is a quantity of second order of smallness, whereas the derivatives $d\sigma/dx$ and $d\xi/dx$ are of third order. Thus, on the

whole, the mass flux due to the thermal conductivity and diffusion is of second order of smallness. As a result we obtain (3.1) in the form

$$nu_x = \left(\frac{T}{w}\right)^2 (1 + u_x^2)$$

$$\times \left\{\frac{\varkappa}{T}\left(\frac{\partial w}{\partial p_{\sigma,\xi}} - \frac{w}{T}\frac{\partial T}{\partial p_{\sigma,\xi}}\right) + D\frac{\partial}{\partial p}\left(\frac{w_\xi}{T}\right)\right\}\frac{dp}{dx} + i.$$

From this we get the expansions for u_x and u_4.

Substituting the obtained expansions in (3.2) and (3.3), we get two equations relating the jumps in pressure, entropy, and concentration. Subtracting one equation from the other, we obtain the following relation between the discontinuities of the entropy and concentration:

$$n_0 T\Delta\sigma + n_0 w_\xi\Delta\xi = -\frac{(1 + u_x^{02})}{u_x^0}w_0\left(\frac{T}{w_0}\right)^2$$

$$\times \left\{\frac{\varkappa}{T}\left(\frac{\partial w}{\partial p_{\sigma,\xi}} - \frac{w}{T}\frac{\partial T}{\partial p_{\sigma,\xi}}\right) + D\frac{\partial}{\partial p}\left(\frac{w_\xi}{T}\right)\right\}\frac{dp}{dx}.$$

We took account here of the fact that

$$n_0^2 w_0 \frac{u_x^{02}}{1 + u_x^{02}}\frac{\partial}{\partial p}\left(\frac{1}{n}\right) + 1$$

is a quantity of first order of smallness and tends to zero as $\Delta p \to 0$. In the acoustic approximation $v \to c_\infty$ and

$$\frac{\partial}{\partial p}\left(\frac{1}{n}\right)_{\sigma,\xi} = -\frac{1}{n_0^2 w_0 c_\infty^2}.$$

To determine the discontinuity of the concentration, we make use of (3.4). Integrating directly, we obtain

$$i\Delta\xi = -j_x + \xi_0 v_x.$$

Further, inasmuch as $j_0 = (\partial j_0/\partial\xi)\Delta\xi$ (the first term of the expansion in the equilibrium state is $j_0^{(1)} = 0$), we ultimately find the following equations for the concentration and entropy discontinuities:

$$\Delta\xi = \frac{P(1 + i^2/n_0^2)}{i(1 + j_{0\xi}/n_0)}\frac{dp}{dx},$$

$$T\Delta\sigma = \left[Q - \frac{w_\xi P}{1 + j_{0\xi}/n}\right]\frac{1 + i^2/n_0^2}{i}\frac{dp}{dx}, \tag{3.6}$$

where

$$P = \frac{1}{w^2}\frac{\partial T}{\partial p}\left\{w\left(1 - \frac{T}{nw\,\partial T/\partial p}\right)(\varkappa\xi_0 - D)\right.$$

$$\left. + \left(\frac{T\,\partial w_\xi/\partial p}{\partial T/\partial p} - w_\xi\right)(D_0 - D\xi_0)\right\},$$

$$Q = \frac{\partial T}{\partial p}\left\{\varkappa\left(1 - \frac{T}{nw\,\partial T/\partial p}\right)\right.$$

$$\left. - \frac{D}{w}\left(\frac{T\,\partial w_\xi/\partial p}{\partial T/\partial p} - w_\xi\right)\right\}. \tag{3.7}$$

Now, in exactly the same manner as in [5], we

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[1] The earlier paper by the author[5] contains an error: the time derivatives should be omitted.

can easily find the following differential equation

$$\frac{i^2}{2n_0}\left\{3\frac{\partial}{\partial p}\frac{1}{n}+n_0 w_0\frac{\partial^2}{\partial p^2}\frac{1}{n}\right\}(p-p_0)(p-p_1)$$
$$= a_1 i\left(1+\frac{i^2}{n_0{}^2}\right)\frac{dp}{dx}\frac{\partial}{\partial p}\frac{1}{n}, \tag{3.8}$$

where the coefficient a_1 depends not only on the viscosity and thermal conductivity coefficients but also on the diffusion coefficients; p_0 and p_1 are the pressures at large distances on both sides of the discontinuity surface. Integrating (3.8), we can easily find an expression for the width of the shock wave δ_x, and also expressions for the variation of the entropy and concentration inside the discontinuity. Inasmuch as the width of the shock wave is inversely proportional to the amplitude of the wave, it turns out, as expected, that the changes in entropy and concentration on the shock wave are quantities of second order of smallness in Δp.

4. EQUATIONS OF MULTICOMPONENT RELATIVISTIC HYDRODYNAMICS AND THEIR APPLICATION TO MULTIPLE PARTICLE PRODUCTION

To describe multicomponent systems it is necessary to introduce several additional parameters ξ. The thermodynamic equations are then of the form

$$dw = \frac{1}{n}dp + T\,d\sigma + \sum_m w_{\xi_m}d\xi_m,$$
$$d\varepsilon = nT\,d\sigma + w\,dn + \sum_m \varepsilon_{\xi_m}d\xi_m, \tag{4.1}$$

where m runs through values whose number is one less than the total number of components. As before, in the state of equilibrium, when $\xi_m = \xi_{0m}(n,\sigma)$, we have

$$w_{\xi_m}(p,\sigma,\xi_{0m}) = 0, \quad \varepsilon_{\xi_m}(n,\sigma,\xi_{0m}) = 0. \tag{4.2}$$

It is obvious that for multicomponent systems we shall have several relaxation equations

$$n\frac{d\xi_m}{ds} = -\frac{\partial j_{hm}}{\partial x_k}. \tag{4.3}$$

The additional conditions (1.8) will take the form

$$u_h v_k = 0, \quad u_h \tau_{ik} = 0, \quad u_k j_{km} = -j_{0n}. \tag{4.4}$$

Further, to determine the coefficients in the expansions for the fluxes, we impose the same conditions as in Sec. 1. As a result we have

$$v_i = -\varkappa\left(\frac{T}{w}\right)^2 j\left\{\frac{\partial}{\partial x_i}\left(\frac{\mu}{T}\right)+u_i u_k\frac{\partial}{\partial x_k}\left(\frac{\mu}{T}\right)\right\}$$
$$-\sum_m D_m\left(\frac{T}{w}\right)^2\left\{\frac{\partial}{\partial x_i}\left(\frac{w_{\xi_m}}{T}\right)+u_i u_k\frac{\partial}{\partial x_k}\left(\frac{w_{\xi_m}}{T}\right)\right\}, \tag{4.5}$$

$$j_{im} = -D\left(\frac{T}{w}\right)^2\left\{\frac{\partial}{\partial x_i}\left(\frac{\mu}{T}\right)+u_i u_k\frac{\partial}{\partial x_k}\left(\frac{\mu}{T}\right)\right\}$$
$$-\sum_{m'} D'_{m'}\left(\frac{T}{w}\right)^2\left\{\frac{\partial}{\partial x_i}\left(\frac{w_{\xi_{m'}}}{T}\right)+u_i u_k\frac{\partial}{\partial x_k}\left(\frac{w_{\xi_{m'}}}{T}\right)\right\}$$
$$+u_i j_{0m}. \tag{4.6}$$

Some of the coefficients D_m, D, and $D'_{m'}$ may be equal to one another by virtue of the Onsager principle.

Let us write out the complete system of equations of the multicomponent system, neglecting viscosity and thermal conductivity of the medium, and also omitting the diffusion terms. For small relaxation times this system becomes

$$n\left\{\frac{d}{ds}(wu_\alpha)+\frac{\partial w}{\partial x_\alpha}\right\}= -\frac{\partial}{\partial x_\alpha}\left(\zeta'\frac{\partial}{\partial x_\beta}u_\beta\right),$$
$$\frac{\partial(nu_k)}{\partial x_k}=0,$$
$$nT\frac{d\sigma}{ds}\cong 0, \quad \frac{d\xi_m}{ds}=-\frac{\xi_m-\xi_{0m}}{\tau_m}, \tag{4.7}$$

where ζ' is the analog of the viscosity coefficient and τ_m are the relaxation times.

In the ultrarelativistic case the number of degrees of freedom of this system decreases by unity, and Eqs. (4.7) take the form

$$S\left\{\frac{d}{ds}(Tu_\alpha)+\frac{\partial T}{\partial x_\alpha}\right\}=-\frac{\partial}{\partial x_\alpha}\left(\zeta'\frac{\partial}{\partial x_\beta}u_\beta\right),$$
$$\frac{\partial(Su_k)}{\partial x_k}=0, \quad \frac{d\xi_m}{ds}=-\frac{\xi_m-\xi_{0m}}{\tau_m},$$
$$\zeta'=\sum\tau_m ST c_{0m}{}^2. \tag{4.8}$$

In the zeroth approximation, which we shall consider for simplicity in what follows, the right side of the equation of motion can be neglected; then, using the usual procedure, [6] we can readily obtain the characteristic relations on a simple wave. On the characteristic of the sound wave, as before, we have

$$c_\infty z = \pm\eta, \quad z=\ln(S/S_0), \quad \eta=\text{Arth }v. \tag{4.9}*$$

In addition, on the characteristic of the particle current line, there is one more relation:

$$d\ln(\xi_m-\xi_{0m})=-\frac{dt}{\tau_m}(1-v^2)^{1/2}, \tag{4.10}$$

which obviously is connected with the presence of the relaxation equation. Conditions (4.8) and conditions (4.9) constitute the initial system of equations when solving the problem of multiple particle production.

From Landau's hydrodynamic theory [7] it is

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*Arth = tanh^{-1}.

known that the total multiplicity is given by the expression

$$N = K(E / 2M)^{1/4}, \qquad (4.11)$$

where E is the energy of the colliding nucleons in the laboratory frame. However, in multicomponent hydrodynamics we can also specify the multiplicity of individual species of secondary particles, which is obviously expressed by the formula

$$N_i = \xi_{0i} N, \qquad (4.12)$$

where ξ_{0i} are the equilibrium values of the component concentrations. The calculation is carried out for the stage of free scattering of the system, when the temperature of the system decreases to the critical value $kT_k \approx m_\pi$. During this stage we can employ, as is customary, the formulas of statistics. As shown by Belen'kiĭ,[8] the percentage of heavy particles is small compared with the number of produced pions.

Formula (4.12) contains the equilibrium values of the concentration. It can be shown that at large distances from the collision plane ξ_i approaches exponentially the equilibrium value. Indeed, at large distances, when inertial scattering takes place, the relaxation equation can be written in the one-dimensional case in the form

$$x \frac{\partial \xi_i'}{\partial r} + t \frac{\partial \xi_i'}{\partial t} + \frac{\xi_i'}{\tau_i} (t^2 - x^2)^{1/2} = 0, \qquad (4.13)$$

where

$$\xi_i' = \xi_i - \xi_{0i}, \quad v = x / t.$$

It is easy to see that (4.13) has a solution

$$\xi_i - \xi_{0i} = C_i \exp \left[-(t^2 - x^2)^{1/2} / \tau_i \right], \qquad (4.14)$$

which decreases exponentially with time. It can be shown that at the start of the three dimensional scattering stage, at relaxation times $\tau_i \approx 10^{-23}$ sec, the right side of (4.14) decreases by a factor e, i.e., $\xi \approx \xi_{0i}$.

In a multicomponent system we consider also the angular distributions of individual species of particles. Inasmuch as at high temperatures the densities of the particles are proportional to the entropy, we have

$$dS_i \sim \xi_i \exp \left[-\frac{1}{2}(\sqrt{\tau} - \sqrt{\eta})^2 \right] d\eta, \qquad (4.15)$$

where τ and η are the Landau variables. Thus, the angular distributions of both the pions and the heavy particles will be represented with corresponding weights ξ_i. As to the energy distribution of the secondary particles, it is determined by the same already known expression. The same pertains to the number of particles carried away by the traveling wave (see [9]).

It must be noted that all formulas now contain not the equilibrium speed of sound c_0, but the speed of sound c_∞, which in the ultrarelativistic case is equal to $1/\sqrt{3}$.

Thus, summarizing the foregoing, we can state that, compared with ordinary hydrodynamic theory of multiple particle production, it is possible in the analysis of multicomponent systems to effect a more detailed judgement of the character of the secondary particles and their angular distribution. On the whole, the quantitative relations remain unchanged.

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