INFRARED ASYMPTOTIC OF THE GREEN'S FUNCTIONS

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The infrared asymptotic form of the Green's function of a charged particle with spin 0 and ½ is found including all singular terms to all orders in e.

1. INTRODUCTION

In meson theory (without account of the electromagnetic interaction), the Green's function $G(p^2)$ of a particle with mass m has in the infrared region, $p^2 \approx m^2$, the form $(x = p^2m^{-2} - 1)$

$$G(p^2) \sim x^{-1} + \text{const.} \tag{1.1}$$

This formula follows from the Källén-Lehmann representation [1,2] and is valid in all orders of the coupling constant. However, if the particle is charged, the asymptotic form of its Green's function becomes considerably more complicated when the electromagnetic interaction is taken into account. This asymptotic form has been investigated in a great number of papers. It was shown by various methods—the renormalization group method,[2,3] the method of approximate solution of the functional equations of Schwinger,[4] by direct solution of the Dyson integral equation in the ladder approximation [5] and with the help of functional integration [6]—that the first term of the expansion of the Green's function in the infrared region has the form

$$G(p^2) \sim (-x)^{-1+\gamma}, \tag{1.2}$$

where γ is, in general, a series in the fine structure constant α, of which the first term has been determined. It was shown by the renormalization group method that γ contains no term of order α^2.[7]

The problem is now to find the exact expression for the exponent γ and to determine the higher terms in the expansion of the Green's function. This question has been investigated by Milekhin,[8] who used the method of functional integration.[9] The higher terms of the expansion were estimated by perturbation theory. Finally, it was shown by the author [10] that the infrared asymptotic form of the Green's function has, in all orders in the coupling constant, the form

$$G(p^2) \sim (-x)^{-1+\gamma} + O(x^\gamma) + \text{const}, \tag{1.3}$$

where, with a Feynman gauge,

$$\gamma = -\alpha/\pi \tag{1.4}$$

to all orders in α.

In the present paper we determine the explicit form of the function $O(x^\gamma)$ without use of perturbation theory. We obtain a formula which, in analogy to (1.1), contains explicitly all terms which are singular in the infrared region. It thus fully generalizes (1.1) to the case with electromagnetic interaction.

We consider the Green's function of a particle with spin 0 and ½. We shall use the Källén-Lehmann representation [1,2,10] and expansions of the matrix elements of the fields with respect to the momenta of the soft photons, obtained by the Low method [11] as generalized by the author,[12] who showed that the Low method is in general not applicable to the matrix elements for real processes in the higher orders in e. The graphs for such processes contain at least two external lines corresponding to charged particles. The exchange of soft photons between these leads to infrared divergences and makes the Low method invalid. In our case, however, we shall consider matrix elements of fields whose graphs contain only one line corresponding to a real charged particle. These matrix elements contain no infrared divergences, and the Low method can be applied to all orders in e.

2. PARTICLE WITH SPIN 0

Let us consider the Green's function for a charged spinless particle, which we shall call a meson for definiteness. We consider first the matrix element[1)]

1)As in[12], $\hbar = c = 1$, $ab = g^{mn}a_m b_n = a^0b^0 - \mathbf{ab}$,

$\langle k \mid k' \rangle = (2\pi)^3 2k^0\ \delta(\mathbf{k}-\mathbf{k}')$, $\tilde{d}\mathbf{k} = d\mathbf{k}/(2\pi)^3 2k^0$, $(F)_n = \prod_{i=1}^{n} F(k_i)$.

$$T_n = \langle 0|\Phi|r, k_1, \ldots, k_n\rangle, \quad (2.1)$$

where Φ is the Heisenberg operator of the meson field at the origin of the coordinate system, r and m are the momentum and the mass of the meson, and k_i is the momentum of the photon with polarization ϵ_i. This matrix element is represented by the graph of Fig. 1.

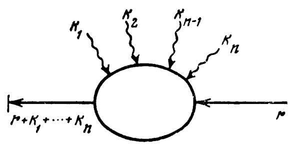

FIG. 1.

It has been shown earlier,[10] that the expansion of T_n in the momentum $k_n = k$ has the form

$$T_n \sim k^{-1} + O(1). \quad (2.2)$$

It can be shown in a similar fashion that the next term in this expansion has the form $O(k \ln k)$. Let us determine the explicit form of $O(1)$ in (2.2). To this end we use the following equality:

$$T_n(\varepsilon \to k) = eT_{n-1}, \quad (2.3)$$

which is a generalization of the Ward identity. This equality follows from the relations obtained by Kazes.[13] It is easy to prove it directly. For this purpose we note that insertion of a photon line with momentum k in the internal line corresponding to the charged particle or in the simple meson-photon vertex and the substitution $\epsilon \to k$ lead to the change

$$F(q) \to e[F(q) - F(q+k)], \quad (2.4)$$

where $F(q)$ is the factor corresponding to the above-mentioned line or vertex and q is the momentum of the charged particle on which it depends. The factor corresponding to an external line is simply multiplied by e.

Let us now consider an arbitrary graph for T_{n-1} corresponding to a renormalizable interaction with all counter terms, insert in it a photon with momentum k in all possible ways and make the replacement $\epsilon \to k$. Then we obtain Eq. (2.3) for the given graph of T_{n-1} and the corresponding class of graphs T_n. Summing over all graphs of T_{n-1}, we obtain this equation for all renormalizable matrix elements (2.1). We note that (2.3) is valid for an arbitrary charged particle with a renormalizable interaction.

Following Low,[11] we further consider the class of graphs $T_n^{(1)}$ in which a photon with momentum $k_n = k$ and the incoming meson can

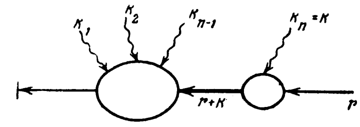

FIG. 2.

be separated from the remainder of the graph by cutting a single meson line (Fig. 2). They give the contribution

$$T_n^{(1)} = \Lambda_{n-1}(r+k)(2rk)^{-1}\varepsilon I(r+k, r). \quad (2.5)$$

We shall consider arbitrary directions ϵ, in particular such for which $\epsilon k \neq 0$. The function I has the following general structure:

$$I(r+k, r) = (2r+k)f + kg, \quad (2.6)$$

where f and g are invariant functions of $(r+k)^2$. It follows from (2.3) (for n = 1) that

$$(2rk)^{-1}kI(r+k, r) = e\langle 0|\Phi|r\rangle = eZ, \quad (2.7)$$

where Z corresponds to the external meson line with all inserts. (In the usual renormalization method, where the contribution of these inserts is equal to zero, Z = 0.) From (2.7) we obtain

$$f = eZ. \quad (2.8)$$

As far as the function g is concerned, it suffices to take it into account for k = 0, i.e., for the case of real meson lines. But in this case g = 0. Thus, to the accuracy of interest to us,

$$T_n^{(1)} = \Lambda_{n-1}(r+k)\frac{(2r+k)\varepsilon}{2rk}eZ. \quad (2.9)$$

In $\Lambda_{n-1}(r+k)$ it suffices to include two terms of the expansion in k:

$$Z\Lambda_{n-1}(r+k) = \left(1 + \sum_c ck\frac{\partial}{\partial rc}\right)T_{n-1} + 2rk\frac{\partial}{\partial r^2}\Lambda_{n-1}(r)Z, \quad (2.10)$$

where

$$c = k_1, \ldots, k_{n-1}, \quad \varepsilon_1, \ldots, \varepsilon_{n-1}.$$

The contribution of the remaining graphs of $T_n^{(2)}$ can be taken account of for k = 0. In order to determine its magnitude, we use (2.3). Substituting in it $T_n^{(1)}$ from (2.9) and (2.10) and $T_n^{(2)}(k=0)$, we find

$$e\left[\left(1 + \sum_c ck\frac{\partial}{\partial rc}\right)T_{n-1} + 2rk\frac{\partial}{\partial r^2}\Lambda_{n-1}(r)Z\right]$$

$$+ T_n^{(2)}(\varepsilon \to k) = eT_{n-1}, \quad (2.11)$$

which leads to

$$T_n^{(2)} = -e\left[\sum_c c\varepsilon\frac{\partial}{\partial rc}T_{n-1} + 2r\varepsilon\frac{\partial}{\partial r^2}\Lambda_{n-1}(r)Z\right]. \quad (2.12)$$

Combining the expressions (2.9), (2.10), and (2.12) and including the order of the next term of the expansion, we obtain the desired expansion of T_n in terms of k:

$$T_n = \left(A + \sum_c B_c\right) T_{n-1} + O(k \ln k), \quad (2.13)$$

$$A = e\frac{(2r+k)\varepsilon}{2rk}, \qquad B_c = e\left(\frac{r\varepsilon}{rk}ck - c\varepsilon\right)\frac{\partial}{\partial rc}. \quad (2.14)$$

We note that this derivation is strictly valid only if a mass λ is introduced in the photon propagation function. Otherwise the derivative $\partial \Lambda_{n-1}(r)/\partial r^2$ does not exist. However, this derivative drops out from the final result, which remains true for $\lambda = 0$. As to the next term in the expansion, which is denoted by O in (2.13), it would be of order k for $\lambda \neq 0$. For $\lambda = 0$, it has the order k ln k.

In deriving (2.13) we assumed that $k = k_n$ is much smaller than all other momenta. Let now k_{n-1} be much smaller than all other momenta except k_n, with $k_n \ll k_{n-1}$ as before. Then we can expand T_{n-1} in (2.13) with respect to k_{n-1}. We keep in the pole terms in k_n two terms of this expansion and one term in the terms of the order of a constant. We thus obtain an expansion in k_n and k_{n-1}, in which all terms are symmetric under the interchange of the variables, except the term

$$B_{n,\,n-1} = (B_{k_{n-1}} + B_{\varepsilon_{n-1}})A. \quad (2.15)$$

However, this term can be easily symmetrized.[12] Using the condition $k_n \ll k_{n-1}$, we replace the factor rk_{n-1} in the denominator of (2.15) by $r(k_n + k_{n-1})$. Then we obtain the symmetric expression

$$B_{ij} = \frac{e^2}{r(k_i + k_j)}\left(k_i\frac{r\varepsilon_i}{rk_i} - \varepsilon_i\right)\left(\varepsilon_j - k_j\frac{r\varepsilon_j}{rk_j}\right), \quad (2.16)$$

and the expansion in k_n, k_{n-1} will be valid for arbitrary k_n, k_{n-1} much smaller than all remaining momenta. Extending this expansion to the remaining photon momenta, we obtain finally

$$T_n = (A)_n\left[\left(1 + \frac{1}{2}\sum_{i,\,j=1}^{n}\frac{B_{ij}}{A_iA_j}\right)Z + \sum_{i=1}^{n} O(k_i^2 \ln k_i)\right]. \quad (2.17)$$

We are now in the position to find the asymptotic form of the meson Green's function $G(p^2)$, using its spectral representation,[1,2] which we write in the form [10]

$$G(p^2) = \int_{m^2}^{a}\frac{g(r^2)\,dr^2}{p^2 - r^2 - i0} + v(p^2), \quad (2.18)$$

where a is arbitrarily close to m^2 and the function $v(p^2)$ is continuous in the neighborhood of m^2. The spectral function, which is a generalized function of the class S^*,

$$g(p^2) = (2\pi)^3\sum_N \delta(p - p_N)\langle 0|\Phi|N\rangle\langle N|\Phi^+|0\rangle \quad (2.19)$$

for p^2 sufficiently close to m^2, reduces to a sum over the states $|N\rangle = |r, k_1, \ldots, k_n\rangle$ containing a meson and an arbitrary number of soft photons. Here

$$(2\pi)^3\sum_N \delta(p - p_N) = (2\pi)^3\int \tilde{d}\mathbf{r}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum\eta\int\tilde{d}\mathbf{k}\right)_n \times \delta(p - r - k_1 - \ldots - k_n) = \sum\left(\sum\eta\right)_n, \quad (2.20)$$

where $\Sigma\eta$ denotes summation over all four polarizations of the photon[2)] (Feynman gauge) and

$$\sum = (2\pi)^3\int\tilde{d}\mathbf{r}\sum_n\frac{1}{n!}\left(\int\tilde{d}\mathbf{k}\right)_n\delta(p - r - k_1 - \ldots - k_n)$$

$$= \frac{1}{2\pi}\int d^4x\int\tilde{d}\mathbf{r}\,e^{-i(p-r)x}\sum_n\frac{1}{n!}\left(\int\tilde{d}\mathbf{k}\,e^{ikx}\right)_n. \quad (2.21)$$

In the last equation we have introduced the Fourier integral of the δ function from the conservation law in order to factorize the contributions from the different photons, which are then summed over n.

In order to avoid the singularities in (2.19) in integrating over the small momenta of the intermediate photons, we provide these with a small fictitious mass λ, i.e., we set $k^0 = (\mathbf{k}^2 + \lambda^2)^{1/2}$. We then take the limit $\lambda \to 0$ before the limit $p^2 \to m^2$. Substituting (2.17) in (2.19), we obtain

$$g(p^2) = Z^2\sum\left(\sum\eta\right)_n(A^2)_n \times\left[1 + \sum_{ij}\frac{B_{ij}}{A_iA_j} + \sum_i O(k_i^2\ln k_i)\right]. \quad (2.22)$$

Noting that

$$(A^2)_n = (a^2)_n\left[1 + \sum_i\frac{d_i}{a_i} + \sum_{ij} O(k_ik_j)\right], \quad (2.23)$$

where

$$a = \frac{er\varepsilon}{rk}, \qquad d = \frac{ek\varepsilon}{rk}, \quad (2.24)$$

we rewrite (2.22) in the form

$$g = Z^2(g_1 + g_2), \quad (2.25)$$

$$g_1 = \sum\left(\sum\eta\right)_n(a^2)_n\left[1 + \sum_{ij}\frac{B_{ij}}{a_ia_j} + \sum_i O(k_i^2\ln k_i)\right], \quad (2.26)$$

[2)] $\eta = -1$ for the time-like and $\eta = 1$ for the space-like polarizations, $\sum\eta\varepsilon_m\varepsilon_n = -g_{mn}$.

$$g_2 = \sum\Big(\sum\eta\Big)_n (a^2)_n \sum_i \frac{d_i}{a_i}. \tag{2.27}$$

Let us consider g_1. Summing over the polarizations and changing the notation of the photon momenta, we write (2.26) in the form

$$g_1 = \sum[(h)_n + (h)_{n-2}n(n+1)H_{n,\,n-1} + (h)_{n-1}nO(\ln k_n)], \tag{2.28}$$

where

$$h = -\left(\frac{em}{rk}\right)^2, \qquad H_{ij} = \frac{e^4m^2}{rk_ir(k_i+k_j)rk_j}\left(\frac{m^2k_ik_j}{rk_irk_j} - 1\right). \tag{2.29}$$

Substituting in this (2.21) and summing over n, we find

$$g_1 = \frac{1}{2\pi}\int d^4x \int \tilde{d}\mathbf{r}\, e^{-i(p-r)x+F}\Big[1 + \int\int \tilde{d}\mathbf{k}_1\tilde{d}\mathbf{k}_2\, e^{i(k_1+k_2)x} H_{12} + \int \tilde{d}\mathbf{k}\, O(\ln k)\, e^{ikx}\Big], \tag{2.30}$$

where

$$F = \int \tilde{d}\mathbf{k}\, h e^{ikx} = \int^y \tilde{d}\mathbf{k}\, h + \int^y \tilde{d}\mathbf{k}\, h(e^{ikx} - 1) + \int_y \tilde{d}\mathbf{k}\, h e^{ikx}. \tag{2.31}$$

Here the integral to y (from y) is an integral over the region $pk \leq y\sqrt{p^2}$ ($pk > y\sqrt{p^2}$):

$$y = \sqrt{p^2} - m. \tag{2.32}$$

For $\lambda \to 0$ we have

$$\int^y \tilde{d}\mathbf{k}\, h = \gamma \ln\frac{2y}{\lambda e} + B, \quad B = \gamma\Big(1 - \int_0^1 \frac{dz}{1-bz}\Big), \quad b = 1 - \left(\frac{mp}{pr}\right)^2 \tag{2.33}$$

[γ is given by (1.4)]. In the remaining terms of (2.31) and (2.30) we can set $\lambda = 0$. Let us write

$$F = \gamma\ln\frac{2}{\lambda e} + B + D, \tag{2.34}$$

$$D = \gamma \ln y + \int \tilde{d}\mathbf{k}\, h(e^{ikx} - 1) + \int_y \tilde{d}\mathbf{k}\, h e^{ikx}. \tag{2.35}$$

The subsequent calculations are conveniently carried out in a coordinate system where $\mathbf{p} = 0$. It follows from the conservation of four-momentum in (2.21) that in this system actually $\mathbf{r}^2 < p^2 - m^2$. Therefore, all terms in (2.30) (except $e^{-i\mathbf{r}\cdot\mathbf{x}}$) can be expanded in $\mathbf{r}$, which after integration over $\mathbf{r}$ leads to $\delta(\mathbf{x})$ and its derivatives. After integration over $\mathbf{x}$ we can easily estimate each term of this expansion with the help of a change of variables,[3)] $x^0 \to x^0/y$, $k \to ky$. It is not difficult to see that H_{12} of (2.29) and B of (2.33) give no contribution of interest to us. We have

$$g_1 = \frac{1}{2m}\left(\frac{2}{\lambda e}\right)^\gamma \frac{1}{2\pi}\int d^4x\, e^{-iyx^0+D}\Big[1 - \frac{ix^0}{2m}\nabla^2\delta(\mathbf{x}) + i\nabla\delta(\mathbf{x})\frac{\partial D}{\partial r} + \delta(\mathbf{x})\int \tilde{d}\mathbf{k}\, O(\ln k)\, e^{ikx}\Big], \tag{2.36}$$

where all quantities are taken at $\mathbf{r} = 0$. Integrating over $\mathbf{x}$, we obtain

$$g_1 = \frac{1}{2m}\left(\frac{2}{\lambda e}\right)^\gamma \Big[R_1 + \frac{\gamma}{m}\Big(\frac{1}{2}R_2 - 2R_3\Big) + O(y^{1+\gamma}\ln y)\Big], \tag{2.37}$$

where

$$R_i = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \exp\{-ixy + \gamma S\}\, f_i(x), \tag{2.38}$$

$$S = \ln y + \int_0 \frac{dk}{k}(e^{ikx} - 1) + \int_y^\infty \frac{dk}{k} e^{ikx}, \tag{2.39}$$

$$f_1(x) = 1, \quad f_2(x) = ix\int_0^\infty dk\, ke^{ikx}, \quad f_3(x) = \int_0^\infty dk\, e^{ikx}. \tag{2.40}$$

It is easy to show that

$$S = -C + i\frac{\pi}{2} - \ln(x + i0), \tag{2.41}$$

where C is the Euler constant,[14] and

$$-f_2(x) = f_3(x) = i(x + i0)^{-1}. \tag{2.42}$$

We are therefore, dealing with integrals R_i which, as shown by Gel'fand and Shilov,[15] uniquely define, in the class S^*, the generalized step functions:

$$R_1 = e^{-C\gamma + i\gamma\pi/2}\frac{1}{2\pi}\int_{-\infty}^{\infty} dx e^{-ixy}(x + i0)^{-\gamma} = e^{-C\gamma}\frac{y_+^{\gamma-1}}{\Gamma(\gamma)}, \tag{2.43}$$

$$-R_2 = R_3 = e^{-C\gamma}\frac{y_+^\gamma}{\Gamma(1+\gamma)}. \tag{2.44}$$

In the last expression the index + may be omitted, since it is integrable at zero in the usual sense for the value (1.4) of γ. Thus

$$g_1 = \frac{1}{2m}\left(\frac{2}{\lambda e}\right)^\gamma e^{-C\gamma}\frac{y_+^{\gamma-1}}{\Gamma(\gamma)}\Big[1 - \frac{5}{2}\frac{y}{m} + O(y^2\ln y)\Big]. \tag{2.45}$$

Introducing

$$x = p^2m^{-2} - 1 \tag{2.46}$$

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3) We note that the integral over x^0 in (2.30) defines, in general, a generalized function of y in the neighborhood of $y = 0$. Therefore, this change of variables is possible only for $y \neq 0$. The integrals remaining after that are defined as the values of this generalized function at the regular point $y = 1$. This remark refers to the terms written in (2.36). The remaining terms are continuous at $y = 0$.

and noting that

$$y = \frac{1}{2} mx\left(1 - \frac{1}{4}x\right) + O(x^3),$$

we obtain finally

$$g_1 = \frac{1}{m^2}\left(\frac{m}{\lambda e}\right)^\gamma e^{-C\gamma}\frac{x_+^{\gamma-1}}{\Gamma(\gamma)}\left[1 - x - \frac{\gamma}{4}x + O(x^2\ln x)\right]. \quad (2.47)$$

Let us now consider g_2 of (2.27). Summing over polarizations, we have

$$g_2 = \sum (h)_{n-1} n\left(-\frac{e^2}{rk_n}\right). \quad (2.48)$$

The subsequent calculations are analogous to these just discussed. We obtain finally

$$g_2 = \frac{1}{m^2}\left(\frac{mx}{\lambda e}\right)^\gamma e^{-C\gamma}\frac{1}{\Gamma(\gamma)}\left[\frac{1}{2} + O(x)\right]. \quad (2.49)$$

Thus the spectral density of the meson Green's function (2.25) is equal to

$$g(p^2) = \frac{Z^2}{m^2}\left(\frac{m}{\lambda e}\right)^\gamma e^{-C\gamma}\frac{x_+^{\gamma-1}}{\Gamma(\gamma)}\left[1 - \left(\frac{1}{2} + \frac{\gamma}{4}\right)x + O(x^2\ln x)\right]. \quad (2.50)$$

Substituting this expression in (2.18) and integrating it as a generalized step function for $p^2 \neq m^2$,[15,12] we obtain for the meson Green's function the following asymptotic expression:

$$G(p^2) = Z_1\frac{1}{m^2}(-x)^{\gamma-1}\left[1 - \left(\frac{1}{2} + \frac{\gamma}{4}\right)x\right] + \text{const}, \quad (2.51)$$

$$Z_1 = Z^2\left(\frac{m}{\lambda e}\right)^\gamma e^{-C\gamma}\Gamma(1-\gamma). \quad (2.52)$$

We note that at the point $x = 0$ this function must be regarded as a generalized step function $(-x + i0)^{\gamma-1}$.[15]

3. PARTICLE WITH SPIN ½

Let us now find, in exactly the same fashion, the infrared asymptotic form of the Green's function of a charged particle with spin ½ (proton). We consider the matrix element (2.1), where now Φ is the proton field, and r and m are the momentum and the mass of the proton. We write it in the form

$$T_n = \mathcal{G}_n Z u, \quad (3.1)$$

where u is the proton spinor, and the constant Z corresponds, as before, to inserts in the external proton line.

The equation (2.3) is also valid in this case. The contribution from the graph of Fig. 2 is now equal to[4)]

$$\mathcal{G}_n^{(1)} = \Lambda_{n-1}(r+k)(2rk)^{-1}(\hat{r} + \hat{k} + m)I(k). \quad (3.2)$$

As shown in [11,16], we have, up to terms in (3.1) which do not depend on k,

$$I(k) = e\hat{\varepsilon} + \frac{\mu'}{2}[\hat{k}, \hat{\varepsilon}], \quad (3.3)$$

where μ' is the anomalous magnetic moment of the proton.

Let us consider, instead of (3.2),

$$\mathcal{G}_n^{(1)} = \Lambda_{n-1}(P)(2rk)^{-1}(\hat{P} + M)I(k), \quad (3.4)$$

$$P = r + k, \quad M^2 = P^2, \quad M = m + rk/m + O(k^2). \quad (3.5)$$

The difference of (3.2) and (3.4) does not contain k^{-1}, and we include it in $\mathcal{G}_n^{(2)}$. Since $\hat{P}(\hat{P} + M) = \hat{M}(\hat{P} + M)$, we see that $\Lambda_{n-1}(P)$ in (3.4) has the same matrix structure as $\mathcal{G}_{n-1}$ and does not contain $\hat{P}$. Therefore, the expansion of $\Lambda_{n-1}(P)$ in k has the form (2.10) (with T replaced by $\mathcal{G}$ and $Z = 1$). Substituting $T_n^{(1)}$ and $T_n^{(2)}(k = 0)$ in (2.3), we have

$$eZ\left[\left(1 + \sum_c ck\frac{\partial}{\partial rc}\right)\mathcal{G}_{n-1} + 2rk\frac{\partial}{\partial r^2}\Lambda_{n-1}(r)\right] \times (2rk)^{-1}(\hat{P} + M)\hat{k}u + T_n^{(2)}(\varepsilon \to k) = eT_{n-1}. \quad (3.6)$$

Noting that

$$(2rk)^{-1}(\hat{P} + M)\hat{k}u = \left(1 + \frac{\hat{k}}{2m}\right)u, \quad (3.7)$$

we obtain

$$T_n^{(2)} = -eZ\left(\mathcal{G}_{n-1}\frac{\hat{\varepsilon}}{2m} + 2r\varepsilon\frac{\partial}{\partial r^2}\Lambda_{n-1}(r) + \sum_c c\varepsilon\frac{\partial}{\partial rc}\mathcal{G}_{n-1}\right)u. \quad (3.8)$$

Combining this expression with $T_n^{(1)}$, we find the following expansion for T_n:

$$T_n = \left(a + \sum_c B_c\right)T_{n-1} + Z\mathcal{G}_{n-1}\delta u + O(k\ln k), \quad (3.9)$$

where a and B_c are given by (2.24) and (2.14) and

$$\delta = \frac{\hat{r} + m}{2rk}\frac{\mu'}{2}[\hat{k}, \hat{\varepsilon}] + e\frac{\hat{k}\hat{\varepsilon}}{2rk}. \quad (3.10)$$

Extending the expansion to the remaining photon momenta, we obtain

$$T_n = (a)_n\left[\left(1 + \sum_{i=1}^n \frac{\delta_i}{a_i} + \frac{1}{2}\sum_{i,j=1}^n{}' \frac{B_{ij}}{a_i a_j}\right)Zu + \sum_{i=1} O(k_i^2\ln k_i)\right]. \quad (3.11)$$

Let us now consider the proton Green's function

$$G(p) = \hat{p}G_1(p^2) + mG_2(p^2). \quad (3.12)$$

The spectral representation for $G_i(p^2)$ has the form (2.18):[1,2]

4) $\hat{a} = \gamma a$, $\{\gamma^m, \gamma^n\} = 2g^{mn}$.

$$G_i(p^2) = \int\limits_{m^2}^{a} \frac{s_i(r^2)\,dr^2}{p^2 - r^2 - i0} + v_i(p^2), \qquad (3.13)$$

$$\hat{p}s_1(p^2) + ms_2(p^2) = s(p)$$

$$= (2\pi)^3 \sum_N \delta(p - p_N) \langle 0|\Phi|N\rangle\langle N|\overline{\Phi}|0\rangle. \qquad (3.14)$$

Substituting in this the expansion (3.11) and summing over the spin directions of the intermediate proton according to the formula

$$\sum u\bar{u} = \hat{r} + m,$$

we obtain

$$s(p) = Z^2 \sum (\sum \eta)_n (a^2)_n \Big\{\Big[1 + \sum_{i,j} \frac{B_{ij}}{a_i a_j} + \sum_i O(k_i^2 \ln k_i)\Big] + (\hat{r} + m) + \sum_i [\delta_i(\hat{r} + m) + (\hat{r} + m)\gamma^0\delta_i^+\gamma^0]/a_i\Big\}. \qquad (3.15)$$

The terms with μ' in this expression cancel, and we have

$$s(p) = Z^2 \Big\{g_1(\hat{p} + m) + \sum \Big(\sum \eta\Big)_n (a^2)_n \times \sum_i [-\hat{k}_i + (\hat{k}_i\hat{\varepsilon}_i(\hat{r} + m) + (\hat{r} + m)\hat{\varepsilon}_i\hat{k}_i)/2r\varepsilon_i]\Big\}, \qquad (3.16)$$

where g_1 is given by (2.26). Summing over photon polarizations, we obtain

$$s(p) = Z^2[g_1(\hat{p} + m) + g_2 m], \qquad (3.17)$$

where g_2 is given by (2.48). Thus the present case is reduced to the previous one:

$$s_1 = Z^2 g_1, \quad s_2 = Z^2(g_1 + g_2). \qquad (3.18)$$

From (2.47) and (2.49) we obtain the spectral densities of the proton Green's function:

$$s_i(p^2) = \frac{Z^2}{m^2}\left(\frac{m}{\lambda e}\right)^{\gamma} e^{-C\gamma} \frac{x_+^{\gamma-1}}{\Gamma(\gamma)}[1 + L_i x + O(x^2 \ln x)], \qquad (3.19)$$

$$L_1 = -1 - \frac{1}{4}\gamma, \qquad L_2 = -\frac{1}{2} - \frac{1}{4}\gamma. \qquad (3.20)$$

The infrared asymptotic of this function has the form

$$G_i(p^2) = Z_1 \frac{1}{m^2}(-x)^{\gamma-1}(1 + L_i x) + \text{const}, \qquad (3.21)$$

where Z_1 is given by (2.52). Instead of (3.12) and (3.21) one can also write

$$G(p) = Z_1\left(\frac{1}{m - \hat{p}} + \frac{2\hat{p} + m}{2m^2} + \gamma\frac{\hat{p} + m}{4m^2}\right)\left(\frac{m^2 - p^2}{m^2}\right)^{\gamma} + \text{const}. \qquad (3.22)$$

The first two terms in this formula agree with the result obtained by Milekhin by functional integration.[8]

In conclusion we note that the method just presented can also be employed to find the infrared asymptotic forms of the vertex functions and scattering matrix elements to all orders in the coupling constant.[10,12] In particular, it turns out that the scattering of charged particles into small angles is described (except for a phase factor) by the simplest one-photon graph in all orders in e. This means that the scattering into small angles follows the Coulomb law for arbitrarily large energies.

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246