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BOUNDARY CONDITIONS FOR THE SUPERCONDUCTIVITY EQUATIONS AT TEMPERATURES CLOSE TO CRITICAL

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Effective boundary conditions are obtained for the interface between superconducting and normal metals. All possible situations are considered, when the equations of superconductivity reduce to second-order differential equations.[7,8,11,16] It is shown that the boundary conditions obtained in the absence of a magnetic field can be easily generalized to include the case when there is a constant magnetic field directed along the interface. These results of the work, which admit of comparison with experimental data,[14] are in good agreement with the latter.

INTRODUCTION

RECENTLY there has been increased interest in phenomena occurring on the boundary between a normal and superconducting metal, and also between two superconducting metals. The theoretical papers published in this connection [1-5] are devoted to the introduction and not derivation of conditions on the interface. Inasmuch as it is not at all clear which of these conditions correspond to reality, we consider in this article cases when the equations of superconductivity admit of an exact solution, and the corresponding boundary conditions are derived directly from the microscopic equations. It turns out that not all the conditions "introduced" are correct, and that some of them are correct only in individual limiting cases.

As in our preceding paper,[6] we shall neglect throughout the differences of the effective masses, the Debye temperatures, and the level densities ζ ($\zeta = mp_0/2\pi^2$), and will assume that the metals differ only in the effective interaction between the electrons, which changes jumpwise.

1. CONDITIONS ON THE BOUNDARY BETWEEN TWO SUPERCONDUCTORS WITH DIFFERENT TRANSITION TEMPERATURES

We consider two semi-infinite superconductors with slightly differing T_c. If the temperature is close to their transition temperatures, then for each of them we can write the Ginzburg-Landau equation (G.L.).[7,8] The corresponding solution in the case when $T_{c1} > T_{c2}$ will be of the form

$$\Delta(z) = \begin{cases} \Delta_1 \,\mathrm{th}\left(\dfrac{\alpha_1 z}{\sqrt{2}\xi_1} + C_1\right) & \text{for} \quad z > 0, \\ -\Delta_2 \,\mathrm{cth}\left(\dfrac{\alpha_2 z}{\sqrt{2}\xi_2} + C_2\right) & \text{for} \quad z < 0, \end{cases} \qquad (1)^*$$

and in the case when $T_{c2} > T_{c1}$

$$\Delta(z) = \begin{cases} \Delta_1 \,\mathrm{cth}\left(\dfrac{\alpha_1 z}{\sqrt{2}\xi_1} + C_1\right) & \text{for} \quad z > 0, \\ -\Delta_2 \,\mathrm{th}\left(\dfrac{\alpha_2 z}{\sqrt{2}\xi_2} + C_2\right) & \text{for} \quad z < 0, \end{cases} \qquad (2)$$

where

$$\alpha_{1,2}^2 = \frac{12(T_{c\,1,2} - T)}{7\zeta(3)T_{c\,1,2}}, \qquad \xi_{1,2} = \frac{p_0}{2\pi m T_{c\,1,2}},$$

$$\Delta_{1,2}^2 = \frac{8(\pi T_{c\,1,2})^2 (T_{c\,1,2} - T)}{7\zeta(3) T_{c\,1,2}}$$

ζ is the Riemann ζ function.

The quantities $\xi_{1,2}$ and $\Delta_{1,2}^2$ can be replaced respectively by

$$\xi_0 = p_0 / 2\pi m T, \quad \Delta_{1,2}^2 = 8\pi T(T_{c\,1,2} - T) \,/\, 7\zeta(3) T_{c\,1,2}.$$

Such a substitution leads to increments of order α^3, whereas the G.L. equations are valid with accuracy to α^2.[9]

To find the constants C_1 and C_2 it is necessary to find the exact solution of the corresponding nonlinear integral equation (A.1) (see Appendix A). If $|T_{c1} - T_{c2}|/(T_{c1} + T_{c2}) \ll 1$, then it is clear that even near the boundary the wave function $\Delta(z)$ will differ little from $\Delta_{1,2}$. For this reason we shall seek the solution in the form

*th = tanh; cth = coth.

$$\Delta(z)=\begin{cases}\Delta_1+\varphi(z) & \text{for } z>0\\ \Delta_2+\varphi(z) & \text{for } z<0,\end{cases} \tag{3}$$

where $|\varphi(z)| \ll \Delta_{1,2}$. Thus, we again obtain a linear integral equation[1)], the solution of which is given in Appendix A.

In this case the small parameter is the quantity $|\alpha_1-\alpha_2|/(\alpha_1+\alpha_2)$. Accurate to terms of first order in this parameter, we obtain

$$\Delta(z)=\begin{cases}\Delta_1\left[1-\dfrac{\alpha_1-\alpha_2}{\alpha_1+\alpha_2}\exp\left(-\dfrac{\alpha_1+\alpha_2}{\xi_0\sqrt{2}}z\right)\right],\\ \Delta_2\left[1+\dfrac{\alpha_1-\alpha_2}{\alpha_1+\alpha_2}\exp\left(\dfrac{\alpha_1+\alpha_2}{\xi_0\sqrt{2}}z\right)\right].\end{cases} \tag{4}$$

From (4) we see that, with the assumed accuracy, the wave function $\Delta(z)$ is effectively continuous together with its derivative ($\Delta_{1,2}\sim\alpha_{1,2}\ll 1$). We can show in general that the solution of the nonlinear integral equation (A.1) with accuracy to terms of order $\varphi^n(z)$ leads, with accuracy to $[|\alpha_1+\alpha_2|/(\alpha_1+\alpha_2)]^n$, to the same continuity conditions:

$$\Delta(0+)=\Delta(0-),\qquad \Delta'(0+)=\Delta'(0-). \tag{5}$$

For the case when $|T_{c1}-T_{c2}|/(T_{c1}+T_{c2})\sim 1$, the validity of condition (5) has so far not been successfully proved; nevertheless we can hope that these conditions are valid whenever the G.L. differential equations are satisfied on the left and on the right of the boundary (for $|z| \gg \xi_0$):

$$\frac{1}{4m}\frac{d^2\Delta}{dz^2}+\frac{1}{\eta}\left[\frac{T_c(z)-T}{T_c(z)}-\frac{7\zeta(3)}{8(\pi T)^2}\Delta^2(z)\right]\Delta=0,$$
$$\eta=\frac{7\zeta(3)\mu}{6(\pi T)^2};\qquad T_c(z)=\begin{cases}T_{c1} & \text{for } z>0,\\ T_{c2} & \text{for } z<0.\end{cases} \tag{6}$$

2. BOUNDARY CONDITIONS ON THE INTERFACE BETWEEN SUPERCONDUCTING AND NORMAL METALS

Let us consider the case when at a given temperature the first metal ($z>0$) is superconducting, while the second ($z<0$) is in the normal state. It is easy to understand that in this case the wave function $\Delta(z)$ may turn out to be small near the boundary even inside the superconductor. From Appendix C it follows that when the following conditions are satisfied

$$1\gg\left(\frac{T-T_{c2}}{T_{c2}}\right)^{1/2}\gg\left(\frac{T_{c1}-T}{T_{c1}}\right)^{1/2} \tag{7}$$

the wave function $\Delta(z)$ has the following expansions: for $\xi_0\ll z\ll\xi_0/\alpha_1$

$$\Delta(z)=C(\xi_0/\alpha_2+z), \tag{8}$$

and for $z<0$, $|z|\gg\xi_0$,

$$\Delta(z)=\frac{C\xi_0}{\alpha_2}\exp\{\alpha_2 z/\xi_0\}, \tag{9}$$

where C is an arbitrary constant of the order of α^2. Comparison of (8) and (9) shows that the function $\Delta(z)$ again satisfies the continuity conditions (5).

From formula (9) and condition (7) we see that inside a normal metal, at large distances from the boundary, the wave function $\Delta(z)$ varies slowly and is not too small (when $z\sim\xi_0$, $\Delta(z)\sim\Delta_1\alpha_1/\alpha_2\gg\alpha^3$). It follows therefore that even for a normal metal the G.L. equation (6) is valid, and in this case

$$T_c(z)=\begin{cases}T_{c1}>T & \text{for } z>0,\\ T_{c2}<T & \text{for } z<0.\end{cases}$$

The corresponding solution is of the form[2)]

$$\Delta(z)=\begin{cases}\Delta_1\,\mathrm{th}\left(\dfrac{\alpha_1 z}{\sqrt{2}\,\xi_0}+C_1\right) & \text{for } z>0,\\ -\Delta_2\dfrac{\sqrt{2}}{\mathrm{sh}(\alpha_2 z/\xi_0+C_2)} & \text{for } z<0.\end{cases} \tag{10*}$$

Thus, in this case, too, not only the differential equations but also the boundary conditions (5) are satisfied. We can hope that they are in general valid when the following requirements are satisfied:

$$\left|\frac{T-T_{c2}}{T_{c2}}\right|^{1/2}\ll 1,\qquad \left(\frac{T_{c1}-T}{T_{c1}}\right)^{1/2}\ll 1 \tag{11}$$

without additional limitations of the type (7).

Let us assume that the "normal" metal has a very low transition temperature, namely:

$$T_{c1}\gg T_{c2}>0. \tag{12}$$

It is shown in Appendix B that in this case for $\xi_0\ll z\ll\xi_0/\alpha$ we can obtain the following asymptotic expansion:

$$\Delta(z)=C(\beta+z), \tag{13}$$

where C is an arbitrary constant ($\alpha^3\ll C\ll\Delta_1$), and β is defined in (C.3) (see Appendix C). It follows from formula (13) that the wave function

1)The idea of such a linearization is due to A. I. Larkin.

*sh = sinh.

2)The question of the applicability of the G. L. equations to a normal metal is considered in the paper by Douglass,[10] but the purely phenomenological approach has led to different results.

satisfies the following effective boundary condition[3)]

$$\beta\Delta'(0+) = \Delta(0+), \quad (14)$$

which is sufficient to obtain $\Delta(z)$ inside the superconducting metal from the G.L. equations.

3. BOUNDARY CONDITIONS FOR STRONGLY CONTAMINATED METALS

Let us assume that the metals contain impurities of the same sort, and that the mean free path is much shorter than the dimensions of the Cooper pairs ($l \ll \xi_0$). To carry out the suitable calculations it is necessary to average in addition the integral equation (B.1) over the positions of the impurity atoms (see Appendix B). The technique of such averaging, as applied to superconductors, was developed by Abrikosov and Gor'kov,[9] while the analytic expression for the average value of the products of two Green's functions (B.3) was first obtained by de Gennes and Guyon [1]:

$$\langle T\sum_\omega \int_{-\infty}^{+\infty} G_\omega(\mathbf{r}', \mathbf{r}) G_{-\omega}(\mathbf{r}', \mathbf{r}) e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')} d\mathbf{r}'\rangle = \zeta\left[\ln\frac{2\gamma\tilde{\omega}}{\pi T} + \psi\left(\frac{1}{2}\right) - \psi\left(\frac{1}{2} + \frac{D^2k^2}{2}\right)\right];$$

$$\zeta = mp_0/2\pi^2, \quad \psi(x) = d\ln\Gamma(x)/dx; \quad D^2 = v_0^2\tau_{tr}/6\pi T, \quad (15)$$

where τ_{tr} is the "transport" time between collisions, while the term in the angle brackets denotes averaging over the impurity positions.

It is easy to check that all the general results obtained for pure superconductors are valid in this case too. Namely, when both metals are in the superconducting state and have nearly equal transition temperatures, the continuity conditions (5) are satisfied on the interface[4)]. The same conditions are valid when one of the metals is in the normal state, and conditions (7) are satisfied. On the other hand, if the transition temperature of the normal metal is close to zero, or if there is repulsion inside the metal, then condition (14) is valid. The value of β is calculated from the general formulas of Appendix B, using (15):

$$\beta = D(0.6 + 1.7/\ln(T_{c1}/T_{c2})). \quad (16)$$

If repulsion is present inside the normal metal, then β is obtained in analogy with the corresponding calculations in [6]:

$$\beta = D\left[0.6 - 1.7\left(\frac{1}{g_2\zeta} + \ln\frac{2\gamma\tilde{\omega}}{\pi T_c}\right)^{-1}\right]. \quad (17)$$

4. BOUNDARY CONDITIONS FOR METALS WITH PARAMAGNETIC IMPURITIES

Let us consider the most interesting case when the concentration of the impurities is close to the critical concentration of metal 1 ($z > 0$). For these conditions we can write a differential equation [11] which is valid for all temperatures:

$$\left[\pi^2(T^2 - T_c^2) + \frac{\Delta^2}{2}\right]\Delta(z) - \frac{v_0\tau_{tr}}{\tau_s}\frac{d^2\Delta}{dz^2} = 0,$$

$$T_c^2 = (6/\pi^2\tau_s^2)\ln(\pi T_{c0}\tau_s/2\gamma), \quad (18)$$

where τ_s is the "exchange" time between collisions, and T_{c0} is the critical temperature in the absence of impurities.

To find the boundary conditions, it is necessary to calculate the average value of the product of two Green's functions. Such an averaging is especially simple to perform if it is recognized that the interaction with the impurities has a δ-like character.[2] As a result of averaging we obtain

$$\langle T\sum_\omega \int_{-\infty}^{+\infty} G_\omega(\mathbf{r}', \mathbf{r}) G_{-\omega}(\mathbf{r}', \mathbf{r}) e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')} d\mathbf{r}'\rangle = \zeta\left[\ln\frac{2\gamma\tilde{\omega}}{\pi T} + \psi\left(\frac{1}{2}\right) - \psi\left(\frac{1}{2} + \frac{\rho}{2} + \frac{D^2k^2}{2}\right)\right], \quad (19)$$

where $\rho = 1/\pi T\tau_s \gg 1$, and D is defined in (15).

It can be shown that in this case the conditions (5) are satisfied, if the concentrations are close to critical for both the superconductor and the normal metal. If the concentration is much higher than the critical concentration of the normal metal, then conditions (14) should be satisfied. The value of β is calculated from the formulas of Appendix B and has a simple form, if terms of order $1/\rho \ll 1$ are discarded:

$$\beta = D_s\left[0.9 - \left(\frac{1}{g_2\zeta} + \ln\frac{2\gamma\tilde{\omega}}{\pi T_{c01}}\right)^{-1}\right], \quad D_s^2 = {}^1/_6 v_0^2\tau_{tr}\tau_s. \quad (20)$$

3) It is shown in a paper by the author[6] that for a metal which is normal at low temperatures ($g_2 > 0$), condition (14) is also satisfied. The value of β is also calculated there.

4) Werthamer[2] and the Gennes[4] introduced the conditions for the continuity of the quantities Δ/g and Δ'/g. These conditions are also valid in the region $|z| \sim l$, where the wave function changes strongly, so that it satisfies not the differential but the integral equation. On the other hand, if we are interested in a solution at distances much larger than $D \sim (l\xi_0)^{1/2}$, then we can use the differential equation, and its solution must be joined to the asymptotic expansion of the exact solution of the integral equation. Conditions (5) and (14) are the result of such joining and, strictly speaking, are valid in the region $D \ll z \ll D/\alpha$.

5. BOUNDARY CONDITIONS IN THE PRESENCE OF A CONSTANT MAGNETIC FIELD

To find the boundary conditions in the presence of a field directed along the boundary between the normal and superconducting metals, it is necessary to solve the following integral equation

$$\frac{\Delta^*(\mathbf{r})}{|g(z)|} = T\sum_{\omega}\int_{-\infty}^{+\infty} G_{\omega}(\mathbf{r}', \mathbf{r})\exp[ie(\mathbf{A}(\mathbf{r}) + \mathbf{A}(\mathbf{r}'), \mathbf{r}' - \mathbf{r})]$$

$$\times\, \Delta^*(\mathbf{r}')G_{-\omega}(\mathbf{r}', \mathbf{r})\,d\mathbf{r}'. \quad (21)$$

We shall assume that the wave function $\Delta^*(\mathbf{r})$ depends only on z, and that the potential satisfies the condition $\mathbf{A}\cdot\mathbf{n} = 0$ ($\mathbf{n}$ is a vector normal to the interface). From this, and also from symmetry considerations, it follows that the potential can be taken in the form

$$\mathbf{A} = (A_x(z), A_y(z), 0). \quad (22)$$

At a temperature close to critical, the potential $\mathbf{A} \sim \alpha \ll 1$,[9] so that the phase factor in (21) can be expanded in powers of $\mathbf{A}$. As a result we obtain after simple transformations

$$\frac{\Delta^*(z)}{|g(z)|} = T\sum_{\omega}\int_{-\infty}^{+\infty} G_{\omega}(\mathbf{r}'\ \mathbf{r})\Delta^*(z')G_{-\omega}(\mathbf{r}', \mathbf{r})\,d\mathbf{r}'$$

$$+\, Te^2\sum_{\omega}\int_{-\infty}^{+\infty} A^2(z')G_{\omega}(\mathbf{r}', \mathbf{r})G_{-\omega}(\mathbf{r}', \mathbf{r})\,(r'-r)^2\Delta^*(z')\,d\mathbf{r}'$$

$$+\, Te^2\sum_{\omega}\int_{-\infty}^{+\infty} A^2(z)G_{\omega}(\mathbf{r}', \mathbf{r})G_{-\omega}(\mathbf{r}', \mathbf{r})\,(r-r')^2\Delta^*(z)'\,d\mathbf{r}', \quad (23)$$

where

$$A^2(z) = A_x^2 + A_y^2; \quad (r-r')^2 = (x-x')^2 + (y-y')^2.$$

To solve Eq. (23) in the region $\xi_0 \ll z \ll \lambda$ (λ is the depth of penetration of the field inside the superconductor), we can make the substitu- $A(z') \approx A(z) \approx A(0)$. Inclusion of the next higher terms of the expansion would lead to increments of the order of α^3. Equation (25) is again solved by the Wiener-Hopf method.[13] Calculations analogous to those carried out in Appendix B lead in all cases to the same results (5) and (14), with the same values of β.

We now carry out a canonical transformation

$$\mathbf{A} \to \mathbf{A} + \nabla\varphi, \qquad \varphi = \int_{z_0}^{z} A_z(x, y, s)\,ds,$$

where $A_z(x, y, z)$ is an arbitrary function. Then $\Delta^*(z)$ transforms in the following manner:

$$\Delta^*(z) \to \Delta^*(\mathbf{r}) = \Delta^*(z)e^{-2ie\varphi(\mathbf{r})},$$

so that the new function will satisfy the general conditions[5)]:

$$\beta\mathbf{n}\left(\nabla + \frac{2ie}{c}\mathbf{A}(0)\right)\Delta^*(0+) = \Delta^*(0+) \quad (24)$$

when $T_{c2} \ll T_{c1}$ or $g_2 > 0$;

$$\Delta^*(0+) = \Delta^*(0-), \quad \mathbf{n}\left(\nabla + \frac{2ie}{c}\mathbf{A}(0)\right)\Delta^*(0+)$$

$$= \mathbf{n}\left(\nabla + \frac{2ie}{c}\mathbf{A}(0)\right)\Delta^*(0-) \quad (25)$$

when

$$1 \gg \left(\frac{T - T_{c2}}{T_{c2}}\right)^{1/2} \gg \left(\frac{T_{c1} - T}{T_{c1}}\right)^{1/2},$$

which can be obtained directly from the requirement of gauge invariance.

6. SOLUTION OF THE GINZBURG-LANDAU-GOR'KOV EQUATIONS FOR A SUPERCONDUCTING FILM

Let a superconducting film occupy the volume $-d < z < d$, and let the normal metal occupy the remainder of space ($z > d$; $z < -d$). We assume that the thickness of the film is much larger than the distances over which the differential equations (6) or (18) are applicable (ξ_0—for pure metals, $D \sim (l\xi_0)^{1/2}$ for "contaminated" metals, and $D_S \sim (l l_S)^{1/2}$ for metals with paramagnetic impurities). Let us consider the case when conditions (14) are satisfied. To find $\Delta(z)$ inside the superconductor it is necessary to solve differential equation (6) with boundary conditions

$$\mp\beta\frac{d\Delta(\pm d)}{dz} = \Delta(\pm d). \quad (26)$$

The first integral of (6) is of the form

$$\frac{1}{4m}\left(\frac{d\Delta}{dz}\right)^2 + \frac{1}{\lambda_\tau}\left[\frac{T_c - T}{T_c} - \frac{7\zeta(3)\Delta^2}{16(\pi T_c)^2}\right]\Delta = C. \quad (27)$$

We introduce a new constant k^2

$$C = \frac{2\Delta_1^2(T_c - T)k^2}{\lambda_\tau T_c(1+k^2)^2} \quad (28)$$

[Δ_1 has been defined in (2)]. After this, as a result of the substitutions

$$z \to \left[\frac{\lambda_\tau T_c(1+k^2)}{2m(T_c - T)}\right]^{1/2} z, \qquad \Delta \to \Delta_1\left[\frac{2k^2}{1+k^2}\right]^{1/2}\Delta(z), \quad (29)$$

expression (27) reduces to the equation of the elliptic sine with parameter k, so that

$$\Delta(z) = \mathrm{sn}\,(z + C_1;\, k). \quad (30)$$

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5)Conditions similar to (24) and (25) were used by de Gennes,[5] but he confined himself to the introduction of the constant β, without indicating a method for its calculation.

Inasmuch as the solution must be symmetrical with respect to z, it is clear that the constant C_1 must be chosen in the form $C_1 = K(k)$ [$K(k)$ is the complete elliptic integral of the first kind]. From the formulas for the transformation of the elliptic functions we obtain

$$\Delta(z) = \mathrm{cn}(z;k) / \mathrm{dn}(z;k). \quad (31)$$

In terms of the ordinary variables:

$$\Delta(z) = \Delta_1\left(\frac{2k^2}{1+k^2}\right)^{1/2} \times \mathrm{cn}\left\{\left[\frac{2m(T_c-T)}{\lambda_\tau(1+k^2)T_c}\right]^{1/2} z;\ k\right\} \times \left[\mathrm{dn}\left\{\left[\frac{2m(T_c-T)}{\lambda_\tau(1+k^2)T_c}\right]^{1/2} z;\ k\right\}\right]^{-1}. \quad (32)$$

We see from (32) that when $k = 0$ we get $\Delta \equiv 0$, i.e., a second-order phase transition takes place. It is easy to obtain the value of d corresponding to $k = 0$. When $k = 0$ the functions $\mathrm{cn}\, z \to \cos z$ and $\mathrm{dn}\, z \to 1$, so that

$$\Delta \sim \cos\left\{\left[\frac{2m(T_c-T)}{\lambda_\tau T_c}\right]^{1/2} z\right\},$$

and the boundary conditions (26) are of the form

$$\beta\left[\frac{2m(T_c-T)}{\lambda_\tau T_c}\right]^{1/2} = \mathrm{ctg}\left\{d\left[\frac{2m(T_c-T)}{\lambda_\tau T_c}\right]^{1/2}\right\}. \quad (33)^*$$

From this we obtain the critical thickness

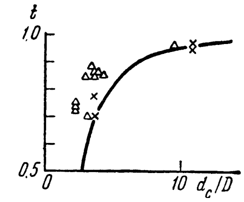
$$d = \left[\frac{\lambda_\tau T_c}{2m(T_c-T)}\right]^{1/2} \mathrm{arc\,ctg}\left\{\beta\left[\frac{2m(T_c-T)}{\lambda_\tau T_c}\right]^{1/2}\right\}. \quad (34)^\dagger$$

It can be shown that the remaining solutions of (33) correspond to the vanishing of the energetically unfavorable states which have nodes.

The solution (32) which we have obtained is symmetrical with respect to the origin, so that it is simultaneously a solution of the problem of a superconducting plate of thickness d, bordering at $z = d$ on a normal metal, and at $z = 0$ on vacuum.[12] The formula for the critical thickness remains in force here, but it must be remembered that in this case d denotes the thickness, whereas in the preceding case it denoted the half-thickness of the plate.

7. COMPARISON WITH EXPERIMENT

Experiments [14,15] have actually disclosed a dependence of the critical temperature on the thickness of a superconducting lead film which borders on a normal metal (copper, platinum).



Dependence of the quantity t on the dimensionless thickness d_c/D at temperatures close to the critical temperature of a bulk superconductor; Δ – experimental data[14]; × – experimental data[15]; curve – plot of (35).

Under the experimental conditions the metals were strongly contaminated ($l \ll \xi_0$), and the transition temperature of the normal metals was close to zero. Substituting the value of λ_T from the paper of Gor'kov [16] and the value of β from (16) in (35), we obtain[6]:

$$\frac{d_c}{D} = \frac{\pi}{2(1-t)^{1/2}}\,\mathrm{arc\,tg}\left[\frac{\pi}{1.2(1-t)^{1/2}}\right], \quad (35)^*$$

where $t = T/T_c$.

The value of D can be expressed in terms of the experimentally observed quantities:

$$D = (\pi k\hbar\sigma / 6e^2 T_c \gamma)^{1/2}, \quad (36)$$

where σ is the conductivity, γ the coefficient in the linear law of specific heat, and k is Boltzmann's constant. Under the conditions of the experiment in [14], $\rho = 1/\sigma = 2 \times 10^{-5}\ \Omega$-cm, $\gamma = 1.71 \times 10^3$ erg/deg^2—cm^3, so that D = 110 Å.

The figure shows the dependence of t on the dimensionless thickness d_c/D at temperatures close to the critical temperature of a bulky superconductor. The results of Hauser et al.[15] were recalculated with the same value D = 110 Å, since no data whatever are given there on the resistance of the lead film. It is seen from the figure that relation (35) is generally in fair agreement with experiment.

In conclusion I am sincerely grateful to Professor B. T. Geĭlikman for critical remarks and continuous interest in the work, and also to A. I. Larkin for numerous discussions. The author is also grateful to V. L. Ginzburg and D. A. Kirzhnits for a discussion of the results.

*ctg = cot.

†arc ctg = cot⁻¹.

[6]A somewhat different dependence was obtained by Werthamer,[2] who introduced the condition for the continuity of the logarithmic derivative of Δ(z) on the boundary; near T_c Werthamer's formula practically coincides with (35) if we put $\beta = D$.

*arc tg = $\tan^{-1}$.

APPENDIX A

We consider the integral equation [9]

$$\frac{\Delta(z)}{|g(z)|} = T\sum_{\omega}\int_{-\infty}^{+\infty} G_\omega(\mathbf{l},\mathbf{r})\Delta(l_3)G_{-\omega}(\mathbf{l},\mathbf{r})\,d\mathbf{l} + T\sum_{\omega}\int_{-\infty}^{+\infty} G_\omega(\mathbf{l},\mathbf{m}) \times \Delta(m_3)G_\omega(\mathbf{s},\mathbf{r})\Delta(s_3)G_\omega(\mathbf{s},\mathbf{m})\Delta(l_3)G_{-\omega}(\mathbf{l},\mathbf{r})\,d\mathbf{l}\,d\mathbf{m}\,d\mathbf{s},$$

$$g(z) = \begin{cases} g_1 & \text{for } z > 0, \\ g_2 & \text{for } z < 0. \end{cases} \quad (A.1)$$

We shall seek the solution in the form

$$\Delta(z) = \Delta_1 + \varphi(z) \qquad (|\varphi(z)| \ll \Delta_1).$$

We use the circumstance that Δ_1 satisfies the integral equation (A.1), in which $g(z)$ is replaced by g_1. As a result we obtain in the first approximation in φ:

$$\Delta_1\left(\frac{1}{|g_1|} - \frac{1}{|g_2|}\right)\theta(-z) + \frac{\varphi(z)}{|g(z)|}$$

$$= T\sum_{\omega}\int_{-\infty}^{+\infty} G_\omega(\mathbf{l},\mathbf{r})\varphi(l_3)G_{-\omega}(\mathbf{l},\mathbf{r})\,d\mathbf{l} - 2T\Delta_1^2 \times \sum_{\omega}\int_{-\infty}^{+\infty} G_\omega(\mathbf{l},\mathbf{m})G_{-\omega}(\mathbf{l},\mathbf{r})G_{-\omega}(\mathbf{s},\mathbf{m})\varphi(s_3) \times G_\omega(\mathbf{s},\mathbf{r})\,d\mathbf{s}\,d\mathbf{m}\,d\mathbf{l} - T\Delta_1^2\sum_{\omega}\int_{-\infty}^{+\infty} G_\omega(\mathbf{r},\mathbf{s})G_{-\omega}(\mathbf{s},\mathbf{m})\varphi(m_3) \times G_\omega(\mathbf{m},\mathbf{l})G_{-\omega}(\mathbf{l},\mathbf{r})\,d\mathbf{s},\,d\mathbf{m}\,d\mathbf{l}. \quad (A.2)$$

For large $|z|$ the function φ is of the form

$$\varphi(z) \to 0 \text{ for } z \to \infty,$$

$$\varphi(z) \to \Delta_2 - \Delta_1 \text{ for } z \to -\infty \qquad (|\Delta_2 - \Delta_1| \ll \Delta_1). \quad (A.3)$$

We shall therefore seek it in the form

$$\varphi(z) = \frac{1}{2\pi}\int_{-\infty-i\delta}^{+\infty-i\delta} f(k)e^{ikz}dk \qquad (\delta > 0). \quad (A.4)$$

Going over to Fourier components in (A.2), we obtain

$$f^+(k)\left(\frac{1}{|g_1|} - K(k)\right) + f^-(k)\left(\frac{1}{|g_2|} - K(k)\right) = -\frac{i\Delta_1}{k}\left(\frac{1}{|g_2|} - \frac{1}{|g_1|}\right);$$

$$f^+(k) = \int_0^\infty e^{ikz}\varphi(z)\,dz; \qquad f^-(k) = \int_{-\infty}^0 e^{ikz}\varphi(z)\,dz;$$

$$K(k) = \frac{T}{(2\pi)^3}\sum_{\omega}\int_{-\infty}^{+\infty} G_\omega(\mathbf{p})G_{-\omega}(\mathbf{p}-\mathbf{k})\,d\mathbf{p} - \frac{2T\Delta_1^2}{(2\pi)^3}\sum_{\omega}\int_{-\infty}^{+\infty} G_{-\omega}(\mathbf{p}) \times G_\omega(\mathbf{p})G_\omega(\mathbf{p}-\mathbf{k})G_{-\omega}(\mathbf{p})\,d\mathbf{p} - \frac{T\Delta_1^2}{(2\pi)^3}\sum_{\omega}\int_{-\infty}^{+\infty} G_\omega(\mathbf{p}) \times G_{-\omega}(\mathbf{p})G_\omega(\mathbf{p}-\mathbf{k})G_{-\omega}(\mathbf{p}-\mathbf{k})\,d\mathbf{p}. \quad (A.5)$$

The functions $1/|g_{1,2}| - K(k)$ vanish when $k = \pm ib_{1,2}$,

$$b_1^2 = \frac{2\alpha_1^2}{\xi_0^2}; \qquad b_2^2 = \frac{3\alpha_1^2 - \alpha_2^2}{\xi_0^2}; \quad (A.6)$$

$\alpha_{1,2}$ have been defined in (2). It follows therefore that these functions can be approximately represented in the form

$$\frac{1}{|g_{1,2}|} - K(k) = \zeta\frac{k^2 + b_{1,2}^2}{k^2 + \xi_0^{-2}}. \quad (A.7)$$

It can be shown that more accurate calculations, similar to those made earlier [6] and in Appendix B, are not needed in this case, for they give rise to increments of the order of $\alpha_{1,2}^3$.

After substituting (A.7) in (A.5) we can easily obtain $\varphi(z)$. We write out $\Delta(z)$ for $z > 0$:

$$\Delta(z) = \Delta_1\left\{1 - \frac{(\alpha_1^2 - \alpha_2^2)\exp(-\sqrt{2}\,\alpha_1 z/\xi_0)}{(3\alpha_1 - \alpha_2)^{1/2}[\alpha_1\sqrt{2} + (3\alpha_1 - \alpha_2^{1/2}]}\right\}. \quad (A.8)$$

The expression obtained must be further expanded in powers of $[|\alpha_1 - \alpha_2|/(\alpha_1 + \alpha_2)]$ accurate to terms of first order, for in this case it is precisely this quantity which is a small parameter. As a result we obtain the simple formula

$$\Delta(z) = \Delta_1\left\{1 - \frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2}\exp\left[-\frac{(\alpha_1 + \alpha_2)z}{\sqrt{2}\,\xi_0}\right]\right\}. \quad (A.9)$$

For $z < 0$ a solution can be found by making a simple substitution $1 \rightleftarrows 2$, $z \to -z$. As a result we obtain formula (4).

APPENDIX B

As shown in [6], to study the behavior of the wave function of the pair near the boundary we can confine ourselves to the linear integral equation

$$\Delta(z) = -Tg(z)\sum_{\omega}\int_{-\infty}^{+\infty} G_\omega(\mathbf{l},\mathbf{r})\Delta(l_3)G_{-\omega}(\mathbf{l},\mathbf{r})\,d\mathbf{l}, \quad (B.1)$$

where

$$g(z) = \begin{cases} g_1 & \text{when } z > 0, \\ g_2 & \text{when } z < 0. \end{cases}$$

We shall seek the solution in the form

$$\Delta(z) = \frac{1}{2\pi}\int_{-\infty+i\delta}^{+\infty+i\delta} f(k)e^{ikz}dk,$$

$$f(k) = f^+(k) + f^-(k) = \int_0^\infty e^{ikz}\Delta(z)\,dz + \int_{-\infty}^0 e^{ikz}\Delta(z)\,dz,$$

$$\delta > 0. \quad (B.2)$$

Substituting (B.2) in (B.1), we obtain the following equation

$$\frac{K_1(k)}{g_1} f^+(k) = -\frac{K_2(k)}{g_2} f^-(k);$$

$$K_{1,2}(k) = 1 + Tg_{1,2} \sum_\omega \int_{-\infty}^{+\infty} \exp\{ik(z - l_3)\} G_\omega(\mathbf{l}, \mathbf{r}) G_{-\omega}(\mathbf{l}, \mathbf{r})\, dl. \tag{B.3}$$

Let the functions $K_{1,2}(k)$ be analytic in the strip $|\operatorname{Im} k| < a$, and let them have at $k = \pm ia$ singularities (poles or branch points). We assume also that $K_{1,2}(k)$ are even and have zeros at $k = \pm c$ and $k = \pm ib$ respectively, with $c \ll a$ and $b < a$. As a result they can be represented in the form

$$K_1(k) = \frac{(k^2 - c^2) N_1^+(k)}{(k^2 + a^2) N_1^-(k)}, \qquad K_2(k) = \frac{(k^2 + b^2) N_2^-(k)}{(k^2 + a^2) N_2^+(k)}, \tag{B.4}$$

where $N_{1,2}^+(k)$ are analytic and have no zeros for all $\operatorname{Im} k > -a$, and $N_{1,2}^-(k)$ are analytic and have no zeros for all $\operatorname{Im} k < a$.

Substituting (B.4) in (B.3) and carrying out the usual regrouping, we obtain the values of $f^\pm(k)$:

$$f^+(k) = \frac{(k + ib) |N_1^+(c) N_2^+(c)| iC}{(k^2 - c^2) N_1^+(k) N_2^+(k) b},$$

$$f^-(k) = -i \frac{|N_1^+(c) N_2^+(c)| C g_2}{(k - ib) N_1^-(k) N_2^-(k) b g_1} \tag{B.5}$$

(C is an arbitrary constant). Inasmuch as $c \ll a$, we can obtain in the region $1/a \ll x \ll 1/c$ the following expansion:

$$\Delta(z) = C(\beta + z), \tag{B.6}$$

$$\beta = \frac{1}{b} + \frac{\varphi_1 + \varphi_2}{c}, \tag{B.7}$$

where $\varphi_{1,2}$ are the phases of the functions $N_{1,2}^+(k)$ at $k = c$.

For negative z, if $b \ll a$, the asymptotic expansion will be determined only by the pole at $k = ib$:

$$\Delta(z) = \frac{g_2 C |N_1^+(c) N_2^+(c)|}{g_1 b N_1^-(ib) N_2^-(ib)} e^{bz} \tag{B.8}$$

($|z| \gg 1/a$). On the other hand, if $b \lesssim a$, then it is necessary to add to expression (B.8) the term connected with the singularity at $k = ia$.

For real k, the functions $N_{1,2}^\pm(k)$ are obtained from the usual formulas:

$$N_1^\pm(k) = \exp\left\{\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \ln\left[\frac{K_1(\zeta)(\zeta^2 + a^2)}{\zeta^2 - c^2}\right] \frac{d\zeta}{\zeta - k \mp i\delta}\right\};$$

$$N_2^\pm(k) = \exp\left\{-\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \ln\left[\frac{K_2(\zeta)(\zeta^2 + a^2)}{\zeta^2 + b^2}\right] \frac{d\zeta}{\zeta - k \mp i\delta}\right\}.$$

Carrying out simple transformations, we get

$$N_1^\pm(k) = \left[\frac{K_1(k)(k^2 + a^2)}{k^2 - c^2}\right]^{\pm 1/2} \times \exp\left\{-\frac{ik}{\pi} \int_0^\infty \ln\left[\frac{K_1(\zeta)(\zeta^2 + a^2)(k^2 - c^2)}{K_1(k)(k^2 + a^2)(\zeta^2 - c^2)}\right] \frac{d\zeta}{\zeta^2 - k^2}\right\},$$

$$N_2^\pm(k) = \left[\frac{K_2(k)(k^2 + a^2)}{k^2 + b^2}\right]^{\mp 1/2} \times \exp\left\{\frac{ik}{\pi} \int_0^\infty \ln\left[\frac{K_2(\zeta)(\zeta^2 + a^2)(k^2 + b^2)}{K_2(k)(k^2 + a^2)(\zeta^2 + b^2)}\right] \frac{d\zeta}{\zeta^2 - k^2}\right\}. \tag{B.9}$$

Confining ourselves to terms of order c/a, we obtain the functions $\varphi_{1,2}$ for two limiting cases:

1) $a \gtrsim b \gg c$

$$\varphi_1 = -\frac{c}{\pi} \int_0^\infty \left[\frac{K_1'(\zeta)}{\zeta K_1(\zeta)} - \frac{2}{\zeta^2}\right] d\zeta + c, \tag{B.10}$$

$$\varphi_2 = \frac{c}{a} - \frac{c}{b} + \frac{c}{\pi} \int_0^\infty \frac{K_2'(\zeta)}{\zeta K_2(\zeta)} d\zeta; \tag{B.11}$$

2) $a \gg b \sim c$

$$\varphi_1 = -\frac{c}{\pi} \int_0^\infty \left[\frac{K_1'(\zeta)}{\zeta K_1(\zeta)} - \frac{2}{\zeta^2}\right] + c, \tag{B.12}$$

$$\varphi_2 = -\varphi_1, \tag{B.13}$$

where in the functions K_1 and K_1' it is necessary to put $c = 0$ ($T = T_{c1}$), so that for small k their expansion is of the form

$$K_1(k) \sim k^2; \qquad K_1'(k) \sim k.$$

Formulas (B.6)—(B.8) and (B.10)—(B.13) solve our problem. From (B.6) it follows that the function $\Delta(z)$ satisfies the general boundary condition (14), and the constant β is determined from formula (B.7). If $a \gg b \sim c$, the phases φ_1 and φ_2 differ only in sign. Therefore in this case $\beta = 1/b$, and when $z < 0$ and $|z| \gg 1/a$ we have

$$\Delta(z) = Ce^{bz}/b. \tag{B.14}$$

Comparison of (B.14) with (B.6) shows that the wave function satisfies the continuity conditions (5). Condition (14) is in this case a consequence of formula (5) and is of no interest in itself.

APPENDIX C

Let us consider the conditions on the interface between two metals which do not contain impurities. The function $K(k)$ is of the form [6]

$$K_{1,2}(k) = 1 + g_{1,2}\zeta\left\{\ln\frac{\tilde{\omega}}{2\pi T} + \frac{2\pi imT}{p_0} \times \ln\left[\frac{\Gamma(1/2 + ip_0k/4\pi mT)}{\Gamma(1/2 - ip_0k/4\pi mT)}\right]\right\} \tag{C.1}$$

We shall assume that the following conditions are satisfied:

$$(T_{c1} - T) / T_{c1} \ll 1; \quad T_{c2} / T_{c1} \ll 1.$$

In this case

$$a = 1 / \xi_0, \quad b = a(1 - T_{c2} / 2\gamma T_{c1}); \quad c = a\alpha_1. \quad \text{(C.2)}$$

Therefore it is easy to obtain p from formulas (B.7), (B.10), and (B.11):

$$\beta = \xi_0(0.7 + 0.5 / \ln(T_{c1} / T_{c2})). \quad \text{(C.3)}$$

Let the temperature be close both to the temperature of the transition of the metal into the superconducting state, and to the transition temperature of the normal metal:

$$\left(\frac{T_{c1} - T}{T_{c1}}\right)^{1/2} \ll 1; \quad \left(\frac{T - T_{c2}}{T_{c2}}\right)^{1/2} \ll 1. \quad \text{(C.4)}$$

In this case

$$b = a|\alpha_2| \ll a, \quad \text{(C.5)}$$

so that the continuity conditions (5) are satisfied.

We have $\beta = 1/b$. If we substitute this quantity in (B.6) it turns out that when the temperature is decreased $\Delta(z)$ may become comparable with Δ_1, so that the linear equation can no longer be used for its determination. As shown in [6], the constant $C \sim \Delta_1 [(T_{c1} - T)/T_{c1}]^{1/2}$, so that the function $\Delta(z)$ obtained in (B.6) is of the order of

$$\Delta(z) \sim \Delta_1 \left[\frac{(T_{c1} - T) T_{c2}}{T_{c1}(T - T_{c2})}\right]^{1/2}.$$

It follows therefore that the condition for the applicability of the linear approximation is as follows:

$$1 \gg \left(\frac{T - T_{c2}}{T_{c2}}\right)^{1/2} \gg \left(\frac{T_{c1} - T}{T_{c1}}\right)^{1/2}. \quad \text{(C.6)}$$

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