EXACT INTEGRATION OF THE EQUATIONS OF MOTION OF RELATIVISTIC CHARGED PARTICLES FOR A CERTAIN CLASS OF VARIABLE ELECTROMAGNETIC FIELDS

L. G. GLIKMAN, V. M. KEL'MAN, and E. M. YAKUSHEV

Nuclear Physics Institute, Academy of Sciences, Kazakh SSR

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The exact solution is found for the equations of motion of relativistic charged particles in the median plane of an electromagnetic field having cylindrical symmetry. The magnetic potential of the field, expressed in cylindrical coordinates ρ, φ, z, has only one nonzero component in this plane—$A_\varphi = \rho^{-1}\omega(t/\rho)$, where ω is an arbitrary function and t is the time. The electrostatic potential of the field is equal to zero.

EXACT solutions of the equations of motion of charged particles in variable electromagnetic fields, taking account of the relativistic change of mass, have been obtained only in a very limited number of cases.[1,2] Aside from their intrinsic interest, such solutions are badly needed for treating various astrophysical problems and may find applications in the physics of high temperature plasma. In the present paper we look for a solution of the equations of motion of charged particles in a variable electromagnetic field having rotational symmetry, in which there is a median plane, perpendicular to the symmetry axis, which is a plane of antisymmetry for the magnetic field and a plane of symmetry for the electric field. We treat the motion in this plane. In addition it is assumed that the electric field of the charges is absent and that the electrostatic potential is zero.

In the cylindrical coordinate system ρ, φ, z, where the z axis is the symmetry axis of the field, we have (in the median plane)

$$A_\rho = A_z = 0, \qquad A_\varphi = A(\rho, t).$$

Here A_ρ, A_φ, and A_z are the components of the vector potential **A**, while t is the time. Under these conditions the relativistic Hamilton-Jacobi equation takes the following form:

$$-\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2+\left(\frac{\partial S}{\partial \rho}\right)^2+\left(\frac{1}{\rho}\frac{\partial S}{\partial \varphi}-\frac{e}{c}A\right)^2+m^2c^2=0,$$

where S is the action function, m and e are the rest mass and charge of the particle, and c is the velocity of light.

Introducing the notation

$$s=\frac{S}{mc^2}, \qquad r=\frac{\rho}{c}, \qquad f(r,t)=\frac{e}{mc^2}A,$$

we get

$$-p^2+q^2+\left(\frac{a_1}{r}-f\right)^2+1=0. \tag{1}$$

Here $p = \partial s/\partial t$ is the negative of the energy of the particle in units of the rest energy mc^2; $q = \partial s/\partial r$ is the generalized momentum corresponding to the coordinate r, in units of mc, while

$$a_1 = \partial s/\partial\varphi = -pr^2\dot{\varphi} + rf = \text{const} \tag{2}$$

is the generalized momentum conjugate to the coordinate φ, taken in units of mc^2.

Equation (1) is a nonlinear first order partial differential equation. It can be integrated by the Lagrange-Charpit method.[3] For the special case when $f(r, t) = \Phi/r$, where $\Phi = \Phi(t/r)$ is an arbitrary function of the argument t/r,

$$(pt+qr)^2+r^2-t^2=a_2 \tag{3}$$

is an integral of the motion (a_2 is an arbitrary constant). Because Φ is indeterminate at the point r = 0, t = 0, this point is excluded from consideration.

From Eqs. (1) and (3) we find

$$p=\frac{t}{t^2-r^2}\sigma_1(a_2+t^2-r^2)^{1/2} - \frac{r}{t^2-r^2}\sigma_2\left[a_2+\left(1-\frac{t^2}{r^2}\right)(a_1-\Phi)^2\right]^{1/2}, \tag{4}$$

$$q=-\frac{r}{t^2-r^2}\sigma_1(a_2+t^2-r^2)^{1/2} + \frac{t}{t^2-r^2}\sigma_2\left[a_2+\left(1-\frac{t^2}{r^2}\right)(a_1-\Phi)^2\right]^{1/2};$$

$$\sigma_1=\frac{pt+qr}{|pt+qr|}, \qquad \sigma_2=\frac{pr+qt}{|pr+qt|}. \tag{5}$$

Using (4) and (5), we bring the expression for s

$$s = a_1\varphi + \int_{r_0,\, t_0}^{r,\, t} (p\,dt + q\,dr)$$

to the following form:

$$s = a_1\varphi + \sigma_1(a_2 - r^2(1-x^2))^{1/2} + \sigma_2\frac{a_2^{1/2}}{2}\ln\frac{r^2(1+x)^2}{[a_2^{1/2} + \sigma_1\sigma_2(a_2 - r^2(1-x^2))^{1/2}]^2} + \sigma_2\int_{x_0}^{x}\frac{(a_1-\Phi)^2 dx}{a_2^{1/2} + [a_2 + (1-x^2)(a_1-\Phi)^2]^{1/2}}, \quad (6)$$

where $x = t/r$, and the subscript 0 denotes the initial value of the corresponding quantity. In deriving (6) it was assumed that along the path of integration the value of σ_2 remains constant and that there are no singularities of Φ leading to divergences of the integral.

The equations of motion, which are obtained by differentiating s with respect to the arbitrary constants a_1 and a_2, have the form

$$r = \frac{2a_2^{1/2}}{\psi(1+x)/\mu + \mu(1-x)/\psi}, \quad (7)$$

$$\varphi = \varphi_0 - \sigma_2\int_{x_0}\frac{(a_1-\Phi)\,dx}{[a_2 + (1-x^2)(a_1-\Phi)^2]^{1/2}} \quad t = xr(x). \quad (8)$$

Here

$$\psi(x_0, x) = \exp\left[-\int_{x_0}^{x}\frac{(a_1-\Phi)^2\,dx}{[a_2 + (1-x^2)(a_1-\Phi)^2]^{1/2}[a_2^{1/2} + (a_2 + (1-x^2)(a_1-\Phi)^2)^{1/2}]}\right], \quad (9)$$

$$\mu = \frac{r(1+x)\psi}{a_2^{1/2} + \sigma_1\sigma_2[a_2 - r^2(1-x^2)]^{1/2}} = \text{const.} \quad (10)$$

Equations (7), (9), and (10) enable us to rewrite the expressions (4) and (5) for the energy and momentum of the particle in the following form:

$$p = \frac{\sigma_2}{2a_2^{1/2}}\left\{\left(\frac{\mu}{\psi}\right)^{x/|x|}\frac{[a_2 + (1-x^2)(a_1-\Phi)^2]^{1/2} + |x|a_2^{1/2}}{1+|x|} + \left[\left(\frac{\mu}{\psi}\right)^{x/|x|}\frac{[a_2 + (1-x^2)(a_1-\Phi)^2]^{1/2} + |x|a_2^{1/2}}{1+|x|}\right]^{-1} \times (a_2 + (a_1-\Phi)^2)\right\}, \quad (11)$$

$$q = -\frac{\sigma_2}{2a_2^{1/2}}\frac{x}{|x|}\left\{\left(\frac{\mu}{\psi}\right)^{x/|x|} \times \frac{a_2^{1/2} + |x|[a_2 + (1-x^2)(a_1-\Phi)^2]^{1/2}}{1+|x|} - \left[\left(\frac{\mu}{\psi}\right)^{x/|x|}\frac{a_2^{1/2} + |x|[a_2 + (1-x^2)(a_1-\Phi)^2]^{1/2}}{1+|x|}\right]^{-1} \times (a_2 - x^2(a_1-\Phi)^2)\right\}. \quad (12)$$

In making calculations using (7)—(12) one must take account of the variation of σ_2 along the path of integration. The value of σ_2 always changes when the particle passes through the symmetry axis since, because of the peculiarities of the cylindrical coordinates, the sign of $\dot{r}$ then changes. In addition the value of σ_2 changes when we pass through a root of the expression $pr + qt$, if the order of the first nonzero derivative of $pr + qt$ with respect to the time is odd.

In case of a change of sign of $pr + qt$, we can proceed as follows: choose the point at which the sign changes as the initial point for the further motion and, having determined the initial values of t, r, φ, p, and q there and having computed the constant μ for the changed σ_2, continue the integration. Since

$$\frac{dx}{dt} = \frac{1}{r} - \frac{t}{r^2}\dot{r} = \frac{1}{pr^2}(pr + qt), \quad (13)$$

when $pr + qt$ changes sign, so does dx/dt, i.e., x now varies in the opposite direction.

In obtaining the equations of motion, Leibnitz' formula for differentiating an integral with respect to a parameter was used. This formula is applicable if all the integrals appearing in the equation are convergent. For this condition to be satisfied, along the path of integration there must be no roots α of the equation

$$a_2 + (1-x^2)(a_1-\Phi)^2 = 0, \quad (14)$$

in whose neighborhood the left side becomes an infinitesimal quantity equivalent to $(x-a)^\gamma$, where $\gamma \geq 2$.

We consider the case where equation (14) has such a root. According to (13),

$$\frac{dx}{dt} = \frac{P(x)}{pr^2}(x-\alpha),$$

where

$$P(x)(x-\alpha) = pr + qt = \sigma_2[a_2 + (1-x^2)(a_1-\Phi)^2]^{1/2};$$

we note that the function $P(x)$ is bounded at $x = \alpha$. Integrating the equation over an interval not containing $x = \alpha$ and $r = 0$, we get

$$x - \alpha = (x_1 - \alpha)\exp\left[\int_{t_1}^{t} \frac{P(x)}{pr^2}\,dt\right].$$

Here t_1 is the time corresponding to $x = x_1$. From the finiteness of the integral in this case we see that for $x_1 \neq \alpha$ the value of x does not reach the value α, so that the equations of motion derived earlier are valid. If, however, $x_1 = \alpha$, the correct equation of motion is

$$r = \dot{r}_0 t. \tag{15}$$

The expressions for φ and p are found from (1)–(3) and (15) as functions of the time:

$$\varphi = \varphi_0 + \frac{(1 - \dot{r}_0^2)^{1/2}}{\dot{r}_0}\ln\frac{t}{t_0}\frac{a_1 - \Phi + [\dot{r}_0^2 t_0^2 + (a_1 - \Phi)^2]^{1/2}}{a_1 - \Phi + [\dot{r}_0^2 t^2 + (a_1 - \Phi)^2]^{1/2}}$$

$$p = -\frac{1}{1 - \dot{r}_0^2}\left(1 - \dot{r}_0^2 + \frac{a_2}{t^2}\right)^{1/2}.$$

[1] W. F. G. Swann, Proceedings of the Moscow Cosmic Ray Conference, 1960, Vol. III, p. 167.
[2] A. A. Kolomenskiĭ and A. N. Lebedev, JETP **44**, 261 (1963), Soviet Phys. JETP **17**, 179 (1963).
[3] V. N. Smirnov, Kurs Vyssheĭ Matematiki (Course of Higher Mathematics) Vol. 4, p. 362.

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