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REALIZATION OF THE THREE-DIMENSIONAL UNITARY GROUP BY "SPHERICAL FUNCTIONS"

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By a suitable parametrization the three-dimensional unitary group U_3 is realized in the form of "spherical functions." The supermultiplets are classified in terms of rectangular diagrams in the (-S, 2T) plane. The infinitesimal operators are introduced as linear differential operators and their matrix elements are computed for an arbitrary representation.

RECENTLY the group of unitary unimodular transformations in complex three-dimensional space (SU_3), which was invoked by Gell-Mann[1] and Ne'eman[2] to explain the symmetry of the strongly interacting particles, has become very important for the physics of elementary particles. The Gell-Mann–Okubo mass formulas,[1,3] obtained on the basis of SU_3, are in brilliant agreement with experiment. In the present paper the mathematical formalism for the three-dimensional unitary group U_3 is presented on the basis of a parametrization and realization of the representations of this group by "spherical functions"[1)] which, for certain purposes, have definite advantages over the abstract operator approach.[4]

1. REALIZATION OF THE THREE-DIMENSIONAL UNITARY GROUP BY "SPHERICAL FUNCTIONS."

To explain the method of realization by "spherical functions,"[2)] we shall first apply it to the three-dimensional rotation group. We consider the set of unit vectors **n**:

$$\mathbf{n} = (\sin\vartheta\cos\varphi, \sin\vartheta\cos\varphi, \cos\vartheta).$$

On this set we define a metric invariant under the rotation group:

$$dl^2 = d\mathbf{n}^2 = d\vartheta^2 + \sin^2\vartheta d\varphi^2.$$

In the present case the invariance of the metric can be seen both formally (the square of the length of the vector $d\mathbf{n}$ is by definition invariant under the group of rotations) and geometrically: dl is the distance between two infinitesimally separated points on the unit sphere.

To the invariant metric we can always associate an invariant operator—the Laplacian Δ:

$$\Delta = \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial}{\partial\vartheta} + \frac{1}{\sin\vartheta}\frac{\partial^2}{\partial\varphi^2}.$$

The wave functions Ψ on the sphere, which are single-valued and nonsingular[3)] solutions of the wave equation

$$(\Delta + k^2)\Psi = 0, \quad (1)$$

will, by virtue of the invariance of the Laplacian Δ, transform linearly, and thus realize a representation of the rotation group. In fact, for $k^2 = l(l+1)$, there are $2l+1$ independent nonsingular solutions, and after an arbitrary rotation each of them remains nonsingular and continues to satisfy Eq. (1), i.e., is transformed into a definite linear combination of these same solutions. The matrix of the transformation will represent the particular rotation.

Thus the method reduces to 1) parametrization of the manifold of unit vectors, 2) determination of the invariant metric, 3) finding the invariant Laplacian, 4) solving the wave equation.

First we consider the two-dimensional unitary group U_2. It is natural to parametrize the two-dimensional complex unit vectors $\mathbf{b} = (b_1, b_2)$ as follows:

$$\mathbf{b} = (\cos\varphi e^{i\alpha_1}, \sin\varphi e^{i\alpha_2}), \quad \mathbf{b}^*\mathbf{b} = |b_1|^2 + |b_2|^2 = 1.$$

If we set $|b_1| = \cos\varphi$, $|b_2| = \sin\varphi$, then

$$0 \leqslant \varphi \leqslant \pi/2.$$

[1)]A brief presentation of the first part of this paper was given in a lecture by one of the authors (V.V.S.) at the Nor-Amberd school in April, 1964.[5] Later, results coinciding in part with the results of the first part of this paper were obtained by Beg and Ruegg.[6]

[2)]A similar method was applied by Vilenkin and Smorodinskiĭ to the Lorentz group.[7]

[3)]For brevity we shall combine both requirements under the term "nonsingular."

In order for the phases of b_1 and b_2 to run through all values, α_1 and α_2 must vary over the range

$$0 \leqslant \alpha_1, \quad \alpha_2 \leqslant 2\pi.$$

The metric is defined as follows:

$$dl_2^2 = db^* db = d\varphi^2 + \cos^2\varphi d\alpha_1^2 + \sin^2\varphi d\alpha_2^2. \qquad (2)$$

Its invariance follows from the invariance of the square of the length of any vector under an arbitrary unitary transformation.

For the metric $dl^2 = \Sigma h_i^2 dx_i^2$ the invariant Laplacian is defined as

$$\Delta = \sum_i \frac{1}{h}\frac{\partial}{\partial x_i}\frac{h}{h_i^2}\frac{\partial}{\partial x_i}, \quad h = \prod_i h_i. \qquad (3)$$

The Laplacian Δ_2 corresponding to the metric (2) is

$$\Delta_2 = \frac{1}{\sin\varphi\cos\varphi}\frac{\partial}{\partial\varphi}\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi} + \frac{1}{\cos^2\varphi}\frac{\partial^2}{\partial\alpha_1^2} + \frac{1}{\sin^2\varphi}\frac{\partial^2}{\partial\alpha_2^2}.$$

By introducing the variable $z = \cos 2\varphi$, $-1 \leq z \leq 1$, we bring it to the form

$$\Delta_2 = 4\frac{\partial}{\partial z}(1-z^2)\frac{\partial}{\partial z} + \frac{2}{1+z}\frac{\partial^2}{\partial\alpha_1^2} + \frac{2}{1-z}\frac{\partial^2}{\partial\alpha_2^2}.$$

Another obvious invariant of the representation is the operator corresponding to the rotation of the phase of the vector b as a whole. Since this transformation commutes with any unitary transformation, the corresponding infinitesimal operator

$$m_{12} = \frac{1}{i}\left(\frac{\partial}{\partial\alpha_1} + \frac{\partial}{\partial\alpha_2}\right)$$

will be a second invariant of the representation.

Thus the nonsingular solutions of the system of equations

$$(\Delta_2 + k_2^2)\Psi = 0, \quad (\hat{m}_{12} - m_{12})\Psi = 0 \qquad (4)$$

will give an irreducible representation of the group U_2. We shall look for solutions of the system (4) in the form

$$\Psi = y(z)\exp[i(\alpha_1 m_1 + \alpha_2 m_2)].$$

For Ψ to be single-valued, the quantities m_1 and m_2 must be integers. The second equation of (4) establishes a relation between these numbers:

$$m_1 + m_2 = m_{12}.$$

For $y(z)$ we have the equation

$$\left(4\frac{\partial}{\partial z}(1-z^2)\frac{\partial}{\partial z} - \frac{2m_1^2}{1+z} - \frac{2m_2^2}{1-z} + k_2^2\right)y(z) = 0. \qquad (5)$$

It has three singular points $z = \pm 1, -\infty$, at which the solution has power singularities, and consequently reduces to the hypergeometric equation. We shall not make this reduction, but rather give a direct solution.

At the points $z = \pm 1$ the function y behaves like

$$(1+z)^{\pm|m_1|/2}, \quad (1-z)^{\pm|m_2|/2}.$$

A nonsingular solution has both powers positive. Then $y(z)$ must have the form

$$y(z) = (1+z)^{|m_1|/2}(1-z)^{|m_2|/2}P_{n_2}(z), \qquad (6)$$

where $P_{n_2}(z)$ is a polynomial of degree n_2. For large z

$$y \rightarrow z^{(|m_1|+|m_2|)/2+n_2} \equiv z^t.$$

Calculating k_2^2 for large z, we have

$$k_2^2 z^t = 4\frac{\partial}{\partial z}z^2\frac{\partial}{\partial z}z^t = 4t(t+1)z^t.$$

From the definition of t it follows that

$$t = {}^1/_2(|m_1| + |m_2|) + n_2 \qquad (7)$$

takes on integral and half-integral nonnegative values. Physically t is identified with the magnitude of the isotopic spin, while $t_z = (m_1 - m_2)/2$ is its projection.

We map the region of admissible value of m_1 and m_2 for a given t. We obtain the boundaries of the region by setting $|m_1| + |m_2| = 2t$. In Fig. 1 we have taken $t = {}^3/_2$. The boundaries form a square with vertices at the points $(\pm 2t, 0)$ and $(0, \pm 2t)$. If we join the points on the boundary having integer coordinates (m_1, m_2) by straight lines parallel to the boundaries, the nodes of the resulting net lying on the boundary and in the interior of the square give the permissible states. To choose a definite irreducible representation of U_2 we must fix m_{12}. The straight line $m_{12} = \text{const}$ makes an angle of $-45°$ with the m_1 axis, and it follows from formula (7) that m_{12} has the same parity as 2t and satisfies the inequality $|m_{12}| \leq 2t$. Altogether m_{12} takes on $2t + 1$ different values.

As we see from Fig. 1, $2t_z$ runs through the same set of values as m_{12}, i.e., $|2t_z| \leq 2t$, and has the same parity, i.e., we get the usual rule that t_z takes on values differing by unity: from $-t$ to t.

From the point of view of the two-dimensional unitary unimodular group SU_2, all the representations with different m_{12} are equivalent: its representations are characterized only by the value of the isospin t and are self-adjoint. It is also easy to see that we have not enumerated all the representations of U_2. We have found only one scalar relative to SU_2, with $t = 0$ and $m_{12} = 0$. From the

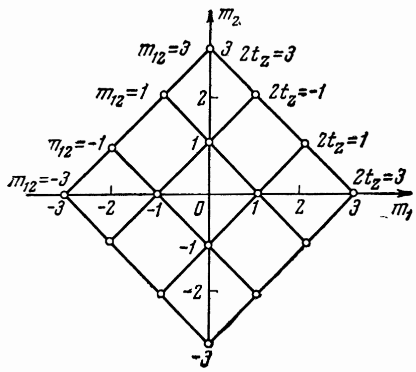

FIG. 1.

two independent isotopic spinors with $m_{12} = \pm 1$, $t = 1/2$, we can easily construct another scalar: the determinant of the matrix of their product. If $m_{12} = 1(-1)$ for each of the spinors, the sum $m_{12} = 2(-2)$, i.e., we get a representation with $t = 0$ and $m_{12} = \pm 2$. Multiplying any of these representations by any power of the scalars we have found enables us to change m_{12} by any integer. Thus we shall enumerate all the representations of U_2 if to an arbitrary integer or half-integer $t \geq 0$ we associate an arbitrary m_{12} of the same parity as 2t.

One may get the impression that, as in the case of the rotation group, this method does not give us the multivalued representations, since the spherical functions give single-valued representations of the group. But the presence or absence of multiple-valued representations is related to the topological structure of the group, to its connectivity. Thus the presence of two-valued representations for the rotation group reflects the fact that it is doubly connected, i.e., that the group contains two types of closed curves that are not reducible to one another by continuous deformation. The unitary groups of arbitrary dimension are simply connected, so that we can be sure that the single-valued representations we have found give all the representations of $U_2(SU_2)$.

Now let us go on to the three-dimensional unitary group U_3.

For this purpose we consider the set of complex unit vectors **c**:

$$\mathbf{c} = (\sin\vartheta\cos\varphi e^{i\alpha_1}, \sin\vartheta\sin\varphi e^{i\alpha_2}, \cos\vartheta e^{i\alpha_3});$$

$$0 \leqslant \vartheta, \varphi \leqslant \pi/2, \; 0 \leqslant \alpha_1, \alpha_2, \alpha_3 \leqslant 2\pi.$$

The invariant metric has the form

$$dl_3^2 = d\vartheta^2 + \cos^2\vartheta d\alpha_3^2 + \sin^2\vartheta dl_2^2, \tag{8}$$

where dl_2^2 is the metric for U_2. From (3) we find the invariant Laplacian Δ_3 for U_3:

$$\Delta_3 = \frac{1}{\sin^3\vartheta\cos\vartheta}\frac{\partial}{\partial\vartheta}\sin^3\vartheta\cos\vartheta\frac{\partial}{\partial\vartheta} + \frac{1}{\cos^2\vartheta}\frac{\partial^2}{\partial\alpha_3^2} + \frac{1}{\sin^2\vartheta}\Delta_2. \tag{9}$$

Changing to the variables

$$z = \cos 2\varphi, \; w = \cos 2\vartheta, \; -1 \leqslant z, \; w \leqslant 1,$$

we rewrite the Laplacian in the form

$$\Delta_3 = \frac{4}{1-w}\frac{\partial}{\partial w}(1-w)^2(1+w)\frac{\partial}{\partial w} + \frac{2}{1+w}\frac{\partial^2}{\partial\alpha_3^2} + \frac{2}{1-w}\Delta_2(z, \alpha_1, \alpha_2).$$

As for the case of U_2, there is another invariant operator

$$\hat{m} = \frac{1}{i}\left(\frac{\partial}{\partial\alpha_1} + \frac{\partial}{\partial\alpha_2} + \frac{\partial}{\partial\alpha_3}\right).$$

Thus a representation of the group U_3 is realized by the set of nonsingular solutions of the equations

$$(\Delta_3 + k_3^2)\Psi = 0, \quad (\hat{m} - m)\Psi = 0 \tag{10}$$

and is given by the invariants k_3^2 and m.

We shall look for a solution of Eqs. (10) in the form

$$x(w)y(z)\exp\left\{i\sum \alpha_k m_k\right\},$$

which, when substituted in (10), gives

$$\left(\frac{4}{1-w}\frac{\partial}{\partial w}(1-w)^2(1+w)\frac{\partial}{\partial w} + \frac{2m_3^2}{1+w} - \frac{2k_2^2}{1-w} + k_3^2\right)x = 0, \tag{11}$$

$$\left(4\frac{\partial}{\partial z}(1-z^2)\frac{\partial}{\partial z} - \frac{2m_1^2}{1+z} - \frac{2m_2^2}{1-z} + k_2^2\right)y = 0, \tag{12}$$

where the integers m_i are connected by the relation

$$m_1 + m_2 + m_3 = m. \tag{13}$$

Substituting in (11) the value $k_2^2 = 4t(t+1)$, we get

$$\left(\frac{4}{1-w}\frac{\partial}{\partial w}(1-w)^2(1+w)\frac{\partial}{\partial w} - \frac{2m_3^2}{1+w} - \frac{8t(t+1)}{1-w} + k_3^2\right)x = 0. \tag{14}$$

In the neighborhood of the singular points $w = \mp 1$, the solution behaves like

$$x \sim (1+w)^{-|m_3|/2}, \; (1-w)^t, \; (1-w)^{-t-1}.$$

Thus the nonsingular solution must have the form

$$x \sim (1+w)^{|m_3|/2}(1-w)^t Q_{n_3}(w),$$

where $Q_{n_3}(w)$ is a polynomial of degree n_3. When $w \to \infty$,

$$x \to w^{|m_3|/2+t+n_3} \equiv w^u.$$

Calculating $k_3^2 = -\Delta_3$ for $w \to \infty$, we have: $k_3^2 = 4u(u+2)$. Instead of using k_3^2 and m, we shall characterize the representation by the numbers u and m. The basis functions will then have the form

$$\Psi(\mathbf{c}) \sim x_{u,\,t,\,m_3}(w)\, y_{t,\,m_1,\,m_2}(z) \exp(i\Sigma\alpha_k m_k). \tag{15}$$

Thus we have the relations

$$u = {}^1/_2|m_3| + t + n_3, \quad t = {}^1/_2(|m_1| + |m_2|) + n_2, \tag{16}$$

from which we see that $u, t \geq 0$ and take on integral and half-integral values. From these equalities it follows that

$$2u \geqslant |m|, \quad 2u - |m_3| \geqslant 2t \geqslant |m - m_3|, \tag{17}$$

where the parities of all terms in the inequalities are the same.

If we mark off in the $m_3, 2t$ plane the region of admissible values of m_3 and 2T for a given value of 2u, we get the diagram shown in Fig. 2. The construction is similar to the one given earlier and proceeds as follows: we mark off the value of 2u on the 2t axis and draw two half-lines downward at angles of 45° to the 2t axis; we mark the value of m on the m_3 axis and draw two half-lines upward at 45° to the 2t axis. We join the integral points lying on the boundaries by lines parallel to the boundaries. The nodes of the resulting net, both on the boundary and in the interior, give the values of 2t and m_3 that appear in the representation. Each point determines the quantum numbers of a function x(w).

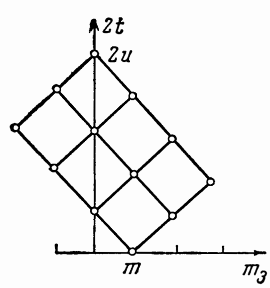

FIG. 2.

It was shown earlier that to each value of t there correspond $\gamma_2 = 2t + 1$ different functions y(z); γ_2 is the multiplicity (weight) of each point of Fig. 2. The average multiplicity $\bar{\gamma}_2 = 2\bar{t} + 1$, where $2\bar{t}$, the ordinate of the center of gravity of the rectangle, is obviously equal to u. Thus $\bar{\gamma}_2 = u + 1$. Now there is no difficulty in computing the supermultiplicity. The numbers of points on the boundaries of the rectangle are equal, respectively, to $u + m/2 + 1$ and $u - m/2 + 1$. The total number of nodes is equal to the product of these numbers:

$$(u + 1)^2 - m^2/4.$$

We get the supermultiplicity if we multiply this number by the average multiplicity; thus

$$\gamma_3 = (u + 1)\,[(u + 1)^2 - m^2/4]. \tag{18}$$

Earlier we identified t with the magnitude of the isospin, and t_z with its projection. We should also set $m_3 = -S + \text{const}$, where S is the strangeness. We can, however, set the constant in this relation equal to zero. To see this we note that the representations we have found for U_3, just as for U_2, do not give all the representations of the group, although they do give all the representations of SU_3. The situation is completely identical with that of the group of orthogonal transformations, which splits into the product of the rotation group and the inversion group. Spherical functions give us all the (single-valued) representations of the rotation group, but not all the representations of the orthogonal group, since the parity of the representation is $p = (-1)^l$. To get all the representations of the orthogonal group we must admit the opposite parity $p = (-1)^{l+1}$, i.e., we must multiply the representations of the rotation group by the two possible representations (even and odd) of the inversion group. In our case the group corresponding to the inversion group is the group of the overall phase rotation of the three-dimensional vector **c**.

We thus get all the representations of U_3 if we multiply the representations we have found by arbitrary representations of the group of the overall phase. These representations are given by the integral nonnegative powers of the two scalars of the SU_3 group:

$$\varepsilon_{ikl} c_i^{(1)} c_k^{(2)} c_l^{(3)}, \quad (\varepsilon_{ikl} c_i^{(1)} c_k^{(2)} c_l^{(3)})^*.$$

Multiplication by the first factor adds unity to all the m_i, and multiplication by the second subtracts unity. The rectangle in Fig. 2 then is shifted to the right or left by one unit. If we consider integral positive powers of the factors, the shift occurs through an integral number corresponding to the power of the factor. Then the number m introduced earlier has the significance of a difference $m_{30} - m_{3u}$, where 0 and u characterize the isospins of the lower and upper nodes of the rectangle, while the actual value of m is $3m_{3u} + (m_{30} - m_{3u}) = 2m_{3u} + m_{30}$. The supermultiplicity is now expressed as

$$\gamma_3 = (u + 1)\,[(u + 1)^2 - {}^1/_4(m_{30} - m_{3u})^2]. \tag{19}$$

Thus, if we shift the rectangle as required we can set $m_3 = -S$. As examples we give the diagrams for the meson and baryon octets (Figs. 3 and 4) and the decuplet (Fig. 5). If we use the

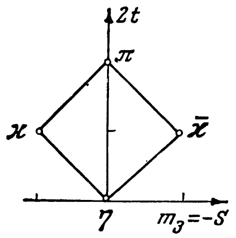

FIG. 3.

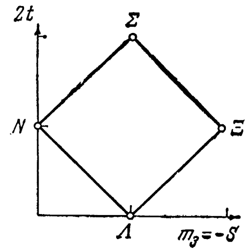

FIG. 4.

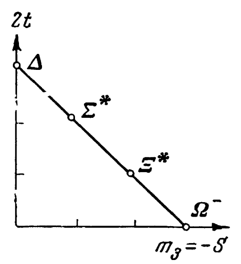

FIG. 5.

quark model, we can identify the vector **c** with the quarks and obtain the meson octet as a quark-antiquark combination, and the baryon octet and the decuplet as combinations of three quarks. In this case Figs. 3–5 and the correspondence $m_3 = -S$ are gotten automatically.

We give the normalized nonsingular solutions of (11) and (12). They have the form

$$\Psi(\mathbf{c}) = \sqrt{2}C(t, \mu_1, \mu_2)C(\gamma, \tau, \mu_3)y(z)x(w)(2\pi)^{-3/2} \times \exp\{i(\alpha_1 m_1 + \alpha_2 m_2 + \alpha_3 m_3)\}, \quad (20)$$

where

$$C(z_1, z_2, z_3) = \frac{(2z_1+1)^{1/2}2^{1/2-z_1}(z_1+z_2+z_3)!}{[\Pi(z_1 \pm z_2 \pm z_3)!]^{1/2}},$$

$$y(z) = (1+z)^{-\mu_1}(1-z)^{-\mu_2}\left(\frac{d}{dz}\right)^{t-\mu_1-\mu_2}[(1+z)^{t+\mu_1-\mu_2} \times (1-z)^{t-\mu_1+\mu_2}],$$

$$x(w) = (1+w)^{-\mu_3}(1-w)^{-\tau-1/2}\left(\frac{d}{dw}\right)^{\gamma-\tau-\mu_3}[(1+w)^{\gamma-\tau+\mu_3} \times (1-w)^{\gamma+\tau-\mu_3}],$$

$$\gamma = u + 1/2, \ \tau = t + 1/2, \ 2\mu_k = |m_k|, \ k = 1, 2, 3,$$

$$\Pi(z_1 \pm z_2 \pm z_3)! = (z_1+z_2+z_3)!(z_1-z_2-z_3)! \times (z_1-z_2+z_3)!\,(z_1+z_2-z_3)!$$

2. INFINITESIMAL OPERATORS OF U_3 AND THEIR MATRIX ELEMENTS

We now proceed to determine the infinitesimal operators of U_3 as linear differential operators and to calculate their matrix elements in an arbitrary representation. The infinitesimal operators of the group are defined as follows:[4)]

$$\mathscr{E}_k{}^i c_l = \delta_l{}^i c_k, \quad i, k, l = 1, 2, 3, \quad (21)$$

where c_l are the components of the vector **c**. From this definition of the operators $\mathscr{E}_k^i$ it is clear that $\mathscr{E}_k^k = -i\partial/\partial\alpha_k$.

To determine the $\mathscr{E}_k^i$ for $i \neq k$ we write these operators as linear forms in the five differential operators:

$$\mathscr{E}_k{}^i = e^{i(\alpha_k-\alpha_i)}\left[a_{k1}{}^i\frac{\partial}{\partial\vartheta} + a_{k2}{}^i\frac{\partial}{\partial\varphi} + a_{k3}{}^i\frac{1}{i}\frac{\partial}{\partial\alpha_1} + a_{k4}{}^i\frac{1}{i}\frac{\partial}{\partial\alpha_2} + a_{k5}{}^i\frac{1}{i}\frac{\partial}{\partial\alpha_3}\right], \quad (22)$$

where the a_{kj}^i are functions only of ϑ and φ and are determined by the conditions (21). After some simple operations we get the following expressions for the operators $\mathscr{E}_k^i$:

$$\mathscr{E}_k{}^k = \frac{1}{i}\frac{\partial}{\partial\alpha_k}, \qquad k = 1, 2, 3,$$

$$\mathscr{E}_2{}^1 = \frac{1}{2}e^{i(\alpha_2-\alpha_1)}\left[-\frac{\partial}{\partial\varphi} + \operatorname{tg}\varphi\frac{1}{i}\frac{\partial}{\partial\alpha_1} + \operatorname{ctg}\varphi\frac{1}{i}\frac{\partial}{\partial\alpha_2}\right],$$

$$\mathscr{E}_1{}^2 = \frac{1}{2}e^{i(\alpha_1-\alpha_2)}\left[\frac{\partial}{\partial\varphi} + \operatorname{tg}\varphi\frac{1}{i}\frac{\partial}{\partial\alpha_1} + \operatorname{ctg}\varphi\frac{1}{i}\frac{\partial}{\partial\alpha_2}\right],$$

$$\mathscr{E}_1{}^3 = \frac{1}{2}e^{i(\alpha_1-\alpha_3)}\left[\cos\varphi\left(-\frac{\partial}{\partial\vartheta} + \operatorname{tg}\vartheta\frac{1}{i}\frac{\partial}{\partial\alpha_3}\right) + \operatorname{ctg}\vartheta\left(\sin\varphi\frac{\partial}{\partial\varphi} + \frac{1}{\cos\varphi}\frac{1}{i}\frac{\partial}{\partial\alpha_1}\right)\right],$$

$$\mathscr{E}_3{}^1 = \frac{1}{2}e^{i(\alpha_3-\alpha_1)}\left[\cos\varphi\left(\frac{\partial}{\partial\vartheta} + \operatorname{tg}\vartheta\frac{1}{i}\frac{\partial}{\partial\alpha_3}\right) + \operatorname{ctg}\vartheta\left(-\sin\varphi\frac{\partial}{\partial\varphi} + \frac{1}{\cos\varphi}\frac{1}{i}\frac{\partial}{\partial\alpha_1}\right)\right],$$

$$\mathscr{E}_3{}^2 = \frac{1}{2}e^{i(\alpha_3-\alpha_2)}\left[\sin\varphi\left(\frac{\partial}{\partial\vartheta} + \operatorname{tg}\vartheta\frac{1}{i}\frac{\partial}{\partial\alpha_3}\right) + \operatorname{ctg}\vartheta\left(\cos\varphi\frac{\partial}{\partial\varphi} + \frac{1}{\sin\varphi}\frac{1}{i}\frac{\partial}{\partial\alpha_2}\right)\right],$$

$$\mathscr{E}_2{}^3 = \frac{1}{2}e^{i(\alpha_2-\alpha_3)}\left[\sin\varphi\left(-\frac{\partial}{\partial\vartheta} + \operatorname{tg}\vartheta\frac{1}{i}\frac{\partial}{\partial\alpha_3}\right) + \operatorname{ctg}\vartheta\left(-\cos\varphi\frac{\partial}{\partial\varphi} + \frac{1}{\sin\varphi}\frac{1}{i}\frac{\partial}{\partial\alpha_2}\right)\right]. \quad (23)^*$$

It is easily verified that these operators satisfy the following commutation relations:

$$[\mathscr{E}_m{}^n, \mathscr{E}_p{}^q] = \delta_p{}^n\mathscr{E}_m{}^q - \delta_m{}^q\mathscr{E}_p{}^n. \quad (24)$$

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4)We keep the same notation as in the book of Lyubarskii.[8] In Okubo's paper[3] these operators with a somewhat different definition of the diagonal elements are denoted by $A_k{}^i$.

*tg ≡ tan, ctg ≡ cot.

3. MATRIX ELEMENTS OF THE OPERATORS $\mathscr{E}_k^i$

To simplify the computation of matrix elements of the operators $\mathscr{E}_k^i$, we change somewhat the form of the solution of Eqs. (11) and (12). Since only the squares of m_1 and m_2 appear in (12), we can make the substitutions $|m_1| \to \pm|m_1|$, $|m_2| \to \pm|m_2|$ in the solution, and we will again get a nonsingular solution, i.e., this substitution amounts at most to introducing a factor which remains when we compare the solutions for $z \to \infty$. If we choose the normalization factor so that when $z \to \infty$ the function $y(z)$ is unchanged, we get a solution that is invariant under this substitution. If we deal with such an invariant solution we can write simply m_1 and m_2 in place of $|m_1|$ and $|m_2|$. The possibility of writing the solution in these four different forms is extremely useful for computing the matrix elements.

To be specific we shall assume that $z \to +\infty + i\delta$; then

$$1 - z = e^{-i\pi}(z-1), \qquad y \to e^{-i\pi(t-\mu_1)}\frac{(2t)!}{(t+\mu_1+\mu_2)!}z^t. \tag{25}$$

Thus if we multiply $y(z)$ by

$$e^{i\pi(t-\mu_1)}(t+\mu_1+\mu_2)!,$$

the asymptotic expression will not depend on μ_1 and μ_2. Also only the squares of m_3 and $\tau = t + 1/2$ appear in (11). Thus the solution $x(w)$ also can be chosen to be invariant under the interchange

$$|m_3| \to \pm|m_3|, \; \tau \to \pm\tau.$$

We give the final form of these invariant solutions:

$$\Psi(\mathbf{c}) = \sqrt{2}\,M(\gamma, \tau, \mu_3)M(t, \mu_1, \mu_2)x(w)y(z)(2\pi)^{-3/2} \times \exp\left(i\sum \alpha_k m_k\right), \tag{26}$$

where

$$M(\gamma, \tau, \mu_3) = \frac{(2\gamma+1)^{1/2}2^{1/2-\gamma}(\gamma-\tau-\mu_3)!}{[\Pi(\gamma\pm\tau\pm\mu_3)!]^{1/2}}e^{i\pi(\gamma+\mu_3-1/2)},$$

$$M(t, \mu_1, \mu_2) = \frac{(2t+1)^{1/2}2^{1/2-t}(t-\mu_1-\mu_2)!}{[\Pi(t\pm\mu_1\pm\mu_2)!]^{1/2}}e^{i\pi(t+\mu_1)},$$

$$x(w) = (1+w)^{\mu_3}(1-w)^{\tau-1/2}\left(\frac{d}{dw}\right)^{\gamma+\tau+\mu_3}[(1+w)^{\gamma+\tau-\mu_3} \times (1-w)^{\gamma-\tau+\mu_3}],$$

$$y(z) = (1+z)^{\mu_1}(1-z)^{\mu_2}\left(\frac{d}{dz}\right)^{t+\mu_1+\mu_2} \times [(1+z)^{t-\mu_1+\mu_2}(1-z)^{t+\mu_1-\mu_2}].$$

These solutions have the following properties:

$$M(\gamma, \tau, \mu_3)x_{\gamma,\tau,\mu_3}(w) = M(\gamma, -\tau, -\mu_3)x_{\gamma,-\tau,-\mu_3}(w) = M(\gamma, -\tau, \mu_2)x_{\gamma,-\tau,\mu_3}(w) = M(\gamma, \tau, -\mu_3)x_{\gamma,\tau,-\mu_3}(w). \tag{27}$$

Similar relations hold for $M(t, \mu_1, \mu_2)y_{t,\mu_1,\mu_2}(z)$. It is clear that these solutions differ from (20) only by phase factors.

Changing to the variables z, w, and replacing the operators $-i\partial/\partial\alpha_k$ by their eigenvalues $2\mu_k$, we write the operators $\mathscr{E}_k^i$ in the form

$$\mathscr{E}_2^1 = e^{i(\alpha_2-\alpha_1)}(1+z)^{-\mu_1+1/2}(1-z)^{\mu_2+1/2}\frac{d}{dz}(1+z)^{\mu_1}(1-z)^{-\mu_2},$$

$$\mathscr{E}_1^2 = e^{i(\alpha_1-\alpha_2)}(1+z)^{\mu_1+1/2}(1-z)^{-\mu_2+1/2}\frac{d}{dz}(1+z)^{-\mu_1} \times (1-z)^{\mu_2},$$

$$\mathscr{E}_3^1 = \frac{1}{\sqrt{2}}e^{i(\alpha_3-\alpha_1)}\left[\left(\frac{1+w}{1-w}\right)^{1/2}\hat{A}(z) - (1+z)^{1/2}\hat{B}(w)\right],$$

$$\mathscr{E}_1^3 = \frac{1}{\sqrt{2}}e^{i(\alpha_1-\alpha_3)}\left[(1+z)^{1/2}\hat{B}_1(w) - \left(\frac{1+w}{1-w}\right)^{1/2}\hat{A}_1(z)\right],$$

$$\mathscr{E}_0^2 = \frac{1}{\sqrt{2}}e^{i(\alpha_3-\alpha_2)}\left[(1-z)^{1/2}\hat{B}(w) + \left(\frac{1+w}{1-w}\right)^{1/2}\hat{A}_2(z)\right],$$

$$\mathscr{E}_2^3 = \frac{1}{\sqrt{2}}e^{i(\alpha_1-\alpha_3)}\left[(1-z)^{1/2}\hat{B}_1(w) + \left(\frac{1+w}{1-w}\right)^{1/2}\hat{A}_3(z)\right], \tag{28}$$

where

$$\hat{A}(z) = (1+z)^{-\mu_1+1/2}(1-z)^{\mu_1+1/2}\frac{d}{dz}(1+z)^{\mu_1}(1-z)^{-\mu_1},$$

$$\hat{B}(w) = (1+w)^{\mu_3+1/2}(1-w)^{1/2}\frac{d}{dw}(1+w)^{-\mu_3},$$

$$\hat{A}_1(z) = (1+z)^{\mu_1+1/2}(1-z)^{-\mu_1+1/2}\frac{d}{dz}(1+z)^{-\mu_1}(1-z)^{\mu_1},$$

$$\hat{A}_2(z) = (1+z)^{\mu_2+1}(1-z)^{-\mu_2+1/2}\frac{d}{dz}(1+z)^{-\mu_2}(1-z)^{\mu_2},$$

$$\hat{B}_1(w) = \hat{B}(w, -\mu_3), \quad \hat{A}_3(z) = \hat{A}_2(z, -\mu_2).$$

The procedure for computing matrix elements of the operators $\mathscr{E}_k^i$ in the general case is as follows: denoting the matrix elements by $\langle\beta'|\mathscr{E}_k^i|\beta\rangle$, where β is the set of numbers t, μ_1, μ_2, μ_3, we write

$$\langle\beta'|\mathscr{E}_k^i|\beta\rangle = \int \Psi^*(\mathbf{c})\mathscr{E}_k^i\Psi(\mathbf{c})\,d\Omega;$$
$$d\Omega = {}^1/_8(1-w)\,dw\,{}^1/_4\,dz\,d\alpha_1 d\alpha_2 d\alpha_3. \tag{29}$$

In the general case the operators $\mathscr{E}_k^i$ have the form

$$\mathscr{E}_k^i = e^{i(\alpha_k-\alpha_i)}(a(z)\hat{b}(w) + \hat{a}_1(z)b_1(w)).$$

Using the invariant eigenfunctions (26), we integrate (29) over α_1, α_2, α_3. This reduces to calculating the integral:

$$\frac{1}{(2\pi)^3}\int_0^{2\pi}\exp\{i[(m_k - m_k' + 1)\alpha_k + (m_i - m_i' - 1)\alpha_i + \alpha_j(m_j - m_j')]\}\,d\alpha_k d\alpha_i d\alpha_j, \tag{30}$$

where $i, j, k = 1, 2, 3$ and take on different values. The integral is equal to unity when

$$m_k - m_k' + 1 = 0,\ m_i - m_i' - 1 = 0,\ m_j - m_j' = 0$$

and is zero otherwise.

We now integrate over z. To do this we must calculate integrals of the form

$$\frac{1}{4} M^*(t', \mu_1', \mu_2') M(t, \mu_1, \mu_2) \int_{-1}^{1} y_{t', \mu_1', \mu_2'}(z) \left\{ \begin{matrix} a(z) \\ \hat{a}_1(z) \end{matrix} \right\} y_{t, \mu_1, \mu_2} \times (z)\, dz.$$

Using the invariance of $M(t, \mu_1, \mu_2) y_{t,\mu_1,\mu_2}(z)$ with respect to the substitution $\mu_1 \to \pm\mu_1$, $\mu_2 \to \pm\mu_2$, we choose the form of writing of this integral which is simplest for the given a(z) and $\hat{a}_1(z)$. A similar procedure is also applied for the w integration.

As an example we give the calculation of the matrix elements of the operator $\mathscr{E}_2^1$. Substituting k = 2, i = 1, j = 3 in the integral in (30),

$$m_2 - m_2' + 1 = 0,\ m_1 - m_1' - 1 = 0,\ m_3' = m_3,$$

i.e.,

$$\mu_1' = \mu_1 - {}^1/_2,\ \mu_2' = \mu_2 + {}^1/_2,\ \mu_3' = \mu_3.$$

We carry out the z integration. We must calculate the integral

$$\frac{1}{4} M^*(t', \mu_1', \mu_2') M(t, \mu_1, \mu_2) \int_{-1}^{1} y_{t', \mu_1', \mu_2'}(z)\, (\mathscr{E}_2^1)'\, y_{t, \mu_1, \mu_2}(z)\, dz$$

$$\equiv \int_{-1}^{1} \langle t', \mu_1', \mu_2' | (\mathscr{E}_2^1)' | t, \mu_1, \mu_2 \rangle\, dz$$

$$= \int_{-1}^{1} \langle t', \mu_1', -\mu_2' | (\mathscr{E}_2^1)' (t, -\mu_1, \mu_2) \rangle\, dz,$$

where

$$(\mathscr{E}_2^1)' = e^{-i(\alpha_2 - \alpha_1)} \mathscr{E}_2^1.$$

Substituting the explicit expressions for the eigenfunctions in the last integral, we have

$$\frac{1}{4} M^*\left(t',\ \mu_1 - \frac{1}{2},\ \mu_2 + \frac{1}{2}\right) M(t, \mu_1, \mu_2) \int_{-1}^{1} \left\{ \left(\frac{d}{dz}\right)^{t'+\mu_1-\mu_2-1} \times [(1+z)^{t'-\mu_1+\mu_2}(1-z)^{t'+\mu_1-\mu_2}] \right\}$$
$$\times \left(\frac{d}{dz}\right) \left\{ \left(\frac{d}{dz}\right)^{t-\mu_1+\mu_2} [(1+z)^{t+\mu_1+\mu_2}(1-z)^{t-\mu_1-\mu_2}] \right\} dz. \qquad (31)$$

This integral can easily be evaluated as follows. It is clear that

$$\left(\frac{d}{dz}\right) \left\{ \left(\frac{d}{dz}\right)^{t-\mu_1+\mu_2} [(1+z)^{t+\mu_1+\mu_2}(1-z)^{t-\mu_1-\mu_2}] \right\}$$
$$= e^{-i\pi(t-\mu_1-\mu_2)} \frac{(2t)!\,(t+\mu_1-\mu_2)}{(t+\mu_1-\mu_2)!} P_{t+\mu_1-\mu_2-1}(z),$$

where the leading coefficient of the polynomial is unity. Thus the integral has the form

$$J = N \int_{-1}^{1} \left\{ \left(\frac{d}{dz}\right)^{t'+\mu_1-\mu_2-1} [(1+z)^{t'-\mu_1-\mu_2}(1-z)^{t'+\mu_1+\mu_2}] \right\} \times P_{t+\mu_1-\mu_2-1}(z)\, dz.$$

Integrating $(t' + \mu_1 - \mu_2 - 1$ times by parts, we find that the integral is equal to

$$J = e^{-i\pi(t'+\mu_1-\mu_2-1)} N \int_{-1}^{1} (1+z)^{t'-\mu_1-\mu_2}(1-z)^{t'+\mu_1+\mu_2} \times \left[\left(\frac{d}{dz}\right)^{t'+\mu_1-\mu_2-1} P_{t+\mu_1-\mu_2-1}(z) \right] dz.$$

From this it is already clear that J = 0 for t′ > t. If we had made the shift to a polynomial in the first factor of the integrand in (31), we would have found that J = 0 for t′ < t. Setting t′ = t, we have

$$J = e^{-i\pi(2t-2\mu_2-1)} \frac{(2t)!\, 2^{2t+1}}{(t+\mu_1-\mu_2-1)!} \times \frac{(t-\mu_1-\mu_2)!\,(t+\mu_1+\mu_2)!}{(2t+1)!}$$

We now carry out the integration over w, setting t′ = t, $\mu_3' = \mu_3$. Since the operator $\mathscr{E}_2^1$ does not depend on w, the integration over w gives unity.

Applying this calculation procedure to all the operators, we get

$$\langle \beta' | \mathscr{E}_2^1 | \beta \rangle = -i[(t+t_z)(t-t_z+1)]^{1/2},$$

$$\langle \beta' | \mathscr{E}_2^1 | \beta \rangle = i[(t-t_z)(t+t_z+1)]^{1/2},$$

$$\langle \beta' | \mathscr{E}_3^1 | \beta \rangle = \left\{ \begin{matrix} i(t-t_z+1)^{1/2} A(\mu, \mu_3) \\ i(t+t_z)^{1/2} B(\mu, \mu_3) \end{matrix} \right.$$

$$\langle \beta' | \mathscr{E}_1^3 | \beta \rangle = \left\{ \begin{matrix} -i(t+t_z+1)^{1/2} A(-\mu, -\mu_3) \\ -i(t-t_z)^{1/2} B(-\mu, -\mu_3) \end{matrix} \right.$$

$$\langle \beta' | \mathscr{E}_3^2 | \beta \rangle = \left\{ \begin{matrix} (t+t_z+1)^{1/2} A(\mu, \mu_3) \\ -(t-t_z)^{1/2} B(\mu, \mu_3) \end{matrix} \right.$$

$$\langle \beta' | \mathscr{E}_2^3 | \beta \rangle = \left\{ \begin{matrix} -(t-t_z+1)^{1/2} A(-\mu, -\mu_3) \\ (t+t_z)^{1/2} B(-\mu, -\mu_3) \end{matrix} \right.$$

$$\langle \beta' | \mathscr{E}_k^k | \beta \rangle = 2\mu_k,\ k = 1, 2, 3, \qquad (32)$$

where

$$B(\mu, \mu_3) = \left[\frac{(t+\mu-\mu_3)(u+t-\mu_3+1)(u-t+\mu_3+1)}{2t(2t+1)} \right]^{1/2}$$

$$B(\mu, \mu_3) = \left[\frac{(t+\mu_3-\mu+1)(u+t+\mu_3+2)(u-t-\mu_3)}{2(t+1)(2t+1)} \right]^{1/2} \qquad (33)$$

The upper equalities hold for t′ = t + ½, the lower for t′ = t − ½.

We have obtained this form for the matrix elements by using invariant solutions. It is clear that using any other solutions for computing the matrix elements can change only phase factors in the results. In particular if we choose the solutions in the form (20), all the matrix elements will be positive definite, i.e.,

$$\left\langle t,\ \mu_1-\frac{1}{2},\ \mu_2+\frac{1}{2},\ \mu_3\left|\mathscr{E}_2{}^1\right|\beta\right\rangle=[(t+t_z)(t-t_z+1)]^{1/2},$$

$$\left\langle t,\ \mu_1+\frac{1}{2},\ \mu_2-\frac{1}{2},\ \mu_3\left|\mathscr{E}_1{}^2\right|\beta\right\rangle=[(t-t_z)(t+t_z+1)]^{1/2},$$

$$\left\langle t\pm\frac{1}{2},\ \mu_1-\frac{1}{2},\ \mu_2,\ \mu_3+\frac{1}{2}\left|\mathscr{E}_3{}^1\right|\beta\right\rangle=\begin{cases}(t-t_z+1)^{1/2}A(\mu,\mu_3)\\(t+t_z)^{1/2}B(\mu,\mu_3)\end{cases}$$

$$\left\langle t\pm\frac{1}{2},\ \mu_1+\frac{1}{2},\ \mu_2,\ \mu_3-\frac{1}{2}\left|\mathscr{E}_1{}^3\right|\beta\right\rangle=\begin{cases}(t+t_z+1)^{1/2}A(-\mu,-\mu_3)\\(t-t_z)^{1/2}B(-\mu,-\mu_3)\end{cases},$$

$$\left\langle t\pm\frac{1}{2},\ \mu_1,\ \mu_2-\frac{1}{2},\ \mu_3+\frac{1}{2}\left|\mathscr{E}_3{}^2\right|\beta\right\rangle=\begin{cases}(t+t_z+1)^{1/2}A(\mu,\mu_3)\\(t-t_z)^{1/2}B(\mu,\mu_3)\end{cases},$$

$$\left\langle t\pm\frac{1}{2}\ \mu_1,\ \mu_2+\frac{1}{2},\ \mu_3-\frac{1}{2}\left|\mathscr{E}_2{}^3\right|\beta\right\rangle=\begin{cases}(t-t_z+1)^{1/2}A(-\mu,-\mu_3)\\(t+t_z)^{1/2}B(-\mu,-\mu_3)\end{cases}$$

$$\langle\beta|\mathscr{E}_k{}^k|\beta\rangle=2\mu_k,\ k=1,2,3, \tag{34}$$

$\mu = \mu_1 + \mu_2 + \mu_3$, $\beta \equiv (t, \mu_1, \mu_2, \mu_3)$; A and B are given by (33).

Finally we give the relations between the numbers u, μ and the numbers p, q, which are often used in the literature to characterize the irreducible representations of SU_3:

$$u={}^1/_2(p+q),\ \mu={}^1/_2(p-q). \tag{35}$$

If we shift to the numbers p and q in (18), (34) and (33), and set $2\mu_3 = \frac{1}{3}(p-q) - Y$, where Y is the hypercharge, we get the matrix elements of the infinitesimal operators of SU_3 expressed in terms of p, q, t, t_z, Y.

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