# ELECTRON-POSITRON PAIR PRODUCTION BY A STRONG ELECTROMAGNETIC WAVE IN THE FIELD OF A NUCLEUS

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An exact expression is obtained for the differential cross section for electron-positron production by a strong electromagnetic wave in the field of a nucleus. The partial and total cross sections for a circularly polarized wave are derived taking into account nonlinearity. It is shown that the energy of the particles produced depends on the angles, but not uniquely.

1. Several recent papers [1-3] considered the influence of a strong electromagnetic wave on different quantum processes. In interactions between particles and a strong electromagnetic wave containing a large number of identical quanta, an appreciable role is played by nonlinear effects connected with the absorption of several quanta simultaneously from the wave, or emission of several quanta simultaneously into the wave. The presence of nonlinear effects causes the cross section to start to depend on the intensity of the wave, and, in addition, the angular and the spectral distributions in the various physical processes change.

We consider in this paper the production of an electron-positron pair by a strong electromagnetic wave in the Coulomb field of a nucleus. The wave is regarded as a classical electromagnetic field whose action on the electron and the positron is taken into account exactly, since there exists an exact solution of the Dirac equation in the field of a plane wave. The interaction with the Coulomb field is considered by perturbation theory. An exact expression is obtained for the differential cross section for pair production for an elliptically polarized wave. The nonlinear effects cause the differential cross section to depend on the "quasi-momenta" of the particles. In addition, the energies of the produced particles turn out at infinity to be functions of the angles, which, generally speaking, are not unique.

For a circularly polarized wave, the cross section is investigated in limiting cases of small and large wave intensity:

$$\xi = eA_0/m \ll 1, \quad \xi \gg 1.$$

The partial cross sections were obtained by integration over the angles The total cross section was obtained for $\xi \gg 1$.

2. A solution of Dirac's equation for an electron in the field of a plane electromagnetic wave, obtained by Volkov,[4] can be written in the following form ($\hbar = c = 1$):

$$\Psi_{\mathbf{f},r}^{(-)} = \left(1 + \frac{e}{2fk}\hat{k}\hat{A}\right)u^{(r)}(\mathbf{f})\exp\{iS^{(-)}\}. \tag{1}$$

The vector potential $A_\mu$ of a wave with four-momentum $k_\mu$ satisfies the conditions

$$A_4 = 0, \quad \mathbf{Ak} = 0$$

and is a periodic function of the variable $s = (x_\mu k_\mu)/|\mathbf{k}|$.

If there is no interaction, then (1) goes over into the solution of the Dirac equation for the free electron with four-momentum $f_\mu = \{\mathbf{f}, i\epsilon\}$. The state of the particle in the wave is characterized by a "quasi-momentum" $p_\mu = \{\mathbf{p}, iE\}$:

$$p_\mu = f_\mu - \frac{1}{2}\xi^2\frac{m^2}{fk}k_\mu, \quad \xi^2 = \frac{e^2\overline{\mathbf{A}^2}}{m^2}, \tag{2}$$

which satisfies the conditions

$$p^2 = -m^{*2}, \quad pk = fk, \quad \mathbf{pA} = \mathbf{fA}, \quad [\mathbf{pk}] = [\mathbf{fk}], \tag{3}$$*

where $m^* = m(1+\xi^2)^{1/2}$ is the effective mass of the electron in the wave.

$$S^{(-)} = fx + \frac{e\omega}{fk}\int \mathbf{fA}\,ds - \frac{e^2\omega}{2fk}\int \mathbf{A}^2\,ds \tag{4}$$

represents (accurate to $\hbar$) the classical action for the motion of an electron in the field of a plane electromagnetic wave.[5] From the classical point of view, the occurrence of the effective mass is connected with the fact that the wave causes the electron oscillations to build up continuously, and the lowest energy state of the particle in the wave corresponds to motion along a certain closed trajectory with zero average momentum $\mathbf{p}$.

*$[\mathbf{pk}] = \mathbf{p}\times\mathbf{k}$.

The solution of the Dirac equation, corresponding to positron states, is written in the form

$$\Psi^{(+)}_{\mathbf{f},r} = \left(1 - \frac{e}{2fk}\hat{k}\hat{A}\right) v^{(r)}(-\mathbf{f}) \exp\{-iS^{(+)}\}, \qquad (5)$$

where $S^{(+)}$ is determined by expression (4) with the substitution $e \to -e$.

3. Let us consider the process of production of an electron-positron pair by a plane monochromatic electromagnetic wave, elliptically polarized in the xy plane:

$$A_x = A_1 \cos \omega s, \quad A_y = A_2 \sin \omega s \qquad (6)$$

in the Coulomb field of the nucleus

$$\varphi(\mathbf{r}) = -Ze / 4\pi r, \qquad (7)$$

where Ze is the charge of the nucleus in Heavyside units. Using the wave functions (1) and (5), we can write for the differential cross section of pair production in first-order perturbation theory

$$d\sigma = \sum_n d\sigma_n, \quad d\sigma_n = \frac{Z^2\alpha^3}{(2\pi)^2} 4nG \frac{\delta(E_+ + E_- - \omega_n)\, d\mathbf{f}_+ d\mathbf{f}_-}{\omega_n{}^3 q^4 \varepsilon_+ \varepsilon_-};$$

$$m^2\xi^2 G = -4\left|\mathcal{R}\left(\frac{E_+\mathbf{f}_-}{f_-k} + \frac{E_-\mathbf{f}_+}{f_+k}\right) - \frac{1}{2}e^2(A_1^2 - A_2^2)\,\omega R_3\left(\frac{E_+}{f_-k} - \frac{E_-}{f_+k}\right)\right|^2$$

$$+ \mathbf{q}^2\left|\mathcal{R}\left(\frac{\mathbf{f}_-}{f_-k} - \frac{\mathbf{f}_+}{f_+k}\right) - \frac{1}{2}e^2(A_1^2 - A_2^2) \times \omega R_3\left(\frac{1}{f_-k} + \frac{1}{f_+k}\right)\right|^2 + \frac{\omega_n^2}{(f_-k)(f_+k)}$$

$$\times \left[\frac{1}{\omega^2}|\mathcal{R}|^2 - e^2\overline{\mathbf{A}}^2|R_0|^2 - \frac{1}{2}e^2(A_1^2 - A_2^2) \times (R_0^* R_3 + R_0 R_3^*)\right][\mathbf{k}, \mathbf{f}_+ + \mathbf{f}_-]^2, \qquad (8)$$

$$\mathcal{R} = \{e\omega A_1 R_1,\ e\omega A_2 R_2,\ 0\}; \quad \mathcal{R}^2 = \mathcal{R}\mathcal{R}^*, \quad \omega_n = n\omega. \qquad (9)$$

Here $\alpha = e^2/4\pi = 1/137$, the + and − signs refer respectively to the electron and the positron; the asterisk denotes complex conjugation;

$$\{R_0, R_1, R_2, R_3\} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{1, \cos x, \sin x, \frac{1}{2}\cos 2x\right\} e^{ig(x)}\, dx;$$

$$g(x) = nx + a\sin x + b\cos x + c\sin 2x;$$

$$a = eA_1\left(\frac{f_{-x}}{f_-k} - \frac{f_{+x}}{f_+k}\right); \quad (10) \quad b = eA_2\left(\frac{f_{-y}}{f_-k} - \frac{f_{+y}}{f_+k}\right);$$

$$c = -\frac{1}{8}e^2(A_1^2 - A_2^2)\left(\frac{1}{f_-k} + \frac{1}{f_+k}\right).$$

In expression (9), summation has been carried out with respect to the polarizations [6] of the electron and the positron. The amplitude of the wave is normalized in such a way that

$$\overline{\mathbf{A}}^2 = {}^1/_2(A_1^2 + A_2^2) = \rho / \omega,$$

where $\rho$ is the photon density.

4. The momentum transferred to the nucleus

$$\mathbf{q} = n\mathbf{k} - \mathbf{p}_+ - \mathbf{p}_- \qquad (11)$$

and the law of energy conservation

$$E_+ + E_- = n\omega \qquad (12)$$

have been expressed in terms of the "quasimomenta" of the electron and positron in the wave, $p_-$ and $p_+$. Taking into account relations (3), it is easy to verify that the function G (9) actually also depends on the "quasimomenta." This is a direct consequence of the fact that the state of the particle in the wave is characterized by the quantity $p_\mu$.

The law of energy conservation (12) shows that the n quanta captured from the wave are consumed in the production of an electron and positron, the state of which is characterized by the quantities $E_-$ and $E_+$. The pair production process begins here with a certain $n_0$, such that the energy $n_0\omega$ is sufficient for the production of two effective masses:

$$n_0 \geqslant 2m^* / \omega = 2m\omega^{-1}(1 + \xi^2)^{1/2}. \qquad (13)$$

With increasing wave intensity $\xi^2$, the production threshold $n_0$ increases, so that the field causes the particle oscillation to build up more and more and increases the gap between the electronic and positronic states.

The particle characterized in the wave by a wave function (1) or (5), does not have any definite energy. Let us expand the wave function (1) in terms of states with definite energy. For simplicity we assume that the wave is circularly polarized ($A_1 = A_2 = A_0$) and omit the factor in front of the exponential. Then the time-dependent part of the wave function can be written in the form

$$\Psi^{(-)} \sim \sum_{n_-=-\infty} A_n \exp\{-i(E_- + n_-\omega)t\}, \qquad (14)$$

where the expansion coefficients $A_n$ are expressed in terms of Bessel functions:

$$A_n = e^{inx_0} J_n(z), \qquad z = \frac{eA_0}{\omega}\left|\left[\mathbf{k}, \frac{\mathbf{f}_-}{f_-k}\right]\right|$$

and satisfy the following conditions:

$$\sum_n |A_n|^2 = \sum_{n=-\infty} J_n^2 = 1, \qquad |A_n|^2 = |A_{-n}|^2.$$

The quantity $E_- + n_-\omega$ is the instantaneous energy of the particle in the wave and can assume, generally speaking, either positive or negative values. The mean value of the energy is $E_-$. From this point of view, the conservation law (12) takes the form of the vanishing of the sum of the instantaneous energies of the electron and positron in the wave:

$$E_+ + n_+\omega + E_- + n_-\omega = 0; \quad n = -(n_+ + n_-) \geqslant n_0. \quad (15)$$

The quantities $E_+$ and $E_-$ are expressed in terms of the momenta and energies of the particles at infinity, when there is no interaction with the wave.

It follows from (2) that

$$E_\pm = \varepsilon_\pm + \frac{1}{2}\xi^2 \frac{m^2}{\varepsilon_\pm - |\mathbf{f}_\pm|\cos\vartheta_\pm}. \quad (16)$$

If, for example, we fixed the quantities that characterize the positron at infinity, namely $\epsilon_+$ and $\vartheta_+$, then the energy of the electron at infinity $\epsilon_-$, which can be determined from the conservation law (12), will be a function, and furthermore generally speaking a non-unique function, of $\cos\vartheta_-$. The cause of this ambiguity is that one and the same mean particle energy E in the wave can be obtained, generally speaking, for several values of the particle energy at infinity $\epsilon$. Let us consider, for example, the case when the wave is incident on an electron that moves directly towards it. If the particle has a definite value of the momentum prior to entering the wave $[f_0 = -(1/2)m\xi^2(1+\xi^2)^{-1/2}]$, the particle will have in the wave a zero average momentum $p = 0$ and $E = m^*$. If we now take particles with momenta $f_1 = f_0 + \Delta f_1$ and $f_2 = f_0 - \Delta f_2$, which corresponds to different energies $\epsilon_1$ and $\epsilon_2$, then the average momenta can be made equal in absolute magnitude, but opposite in direction, corresponding to identical average energies.

The determination of the quantity $\epsilon(\vartheta)$ for a specified value of E reduces to a solution of the algebraic equation

$$x^4\sin^2\vartheta - 2Cx^3\sin^2\vartheta + (\xi^2 + C^2\sin^2\vartheta + \cos^2\vartheta)x^2 - 2C(1/2\xi^2 + \cos^2\vartheta)x + 1/4\xi^4 + C^2\cos^2\vartheta = 0;$$
$$C = E/m \geqslant (1+\xi^2)^{1/2},$$
$$1 \leqslant x \equiv \varepsilon/m < C, \quad 0 \leqslant \vartheta \leqslant \pi. \quad (17)$$

An analysis of this equation shows that in the region of angles $0 \le \vartheta < \pi/2$ the equation can have either one or three solutions of interest to us. In the angle region $\pi/2 \le \vartheta \le \pi$, there can be either one or two such solutions. The number of possible solutions is connected with the number of zeros of the derivative $dE/d\epsilon$ as a function of $\epsilon$. If $dE/d\epsilon$ has no zeros, then the function $E(\epsilon)$ is monotonic, and consequently the inverse function $\epsilon(E)$ is unique. If $dE/d\epsilon$ vanishes once, then $E(\epsilon)$ has two intervals of monotonicity, and the inverse function $\epsilon(E)$ is, generally speaking, double-valued, etc. In the angle region $\pi/2 \le \vartheta \le \pi$ the function $dE/d\epsilon$ has a single zero corresponding to a minimum on the $E(\epsilon)$ curve.

We note incidentally that the smallest possible value $E = m^*$ corresponds to an angle $\vartheta = \pi$. In the angle region $0 \le \vartheta < \pi/2$ the existence of zeros of the derivative $dE/d\epsilon$ depends on the value of $\xi^2$. If $0 < \xi^2 \le 2$, the function $dE/d\epsilon > 0$; if $2 < \xi^2 < 8$, then in some region of angles $0 \le \vartheta \le \vartheta_0(\xi)$ the derivative $dE/d\epsilon > 0$, and when $\vartheta_0 \le \vartheta \le \pi/2$ the value of $dE/d\epsilon$ vanishes twice. Finally, when $\xi^2 \ge 8$, the function $dE/d\epsilon$ has two zeros in the entire angle interval $0 < \vartheta < \pi/2$.

We present an approximate solution of (17) for $\xi^2 \ll 1$. If $E \ge m(1 + \xi^2/2)$, then in the entire angle interval $0 \le \vartheta \le \pi$ the solution is unique and of the form

$$\varepsilon \approx E - \frac{1}{2}\xi^2 \frac{m^2}{E - (E^2 - m^2)^{1/2}\cos\vartheta}. \quad (18)$$

If $m^* < E < m(1 + \xi^2/2)$ then there are no solutions for the angles $0 \le \vartheta \le \pi/2$, and for $\pi/2 < \vartheta \le \pi$ there exist two solutions with $|\epsilon_1 - \epsilon_2| \sim m\xi^4$.

To conclude this section, we make one remark which is useful for what follows. As indicated above, the functions contained in the cross section depend only on the "quasimomenta" $p_\pm$. This circumstance simplifies the determination of the integral cross sections $\sigma_n$, since it is possible to change over in the integration to the variables $p_+$ and $p_-$, in which the energy conservation law has the simplest form (12). The transition is carried out in the following manner:

$$\int \frac{d\mathbf{f}_+ d\mathbf{f}_-}{\varepsilon_+\varepsilon_-}\ldots = \int \frac{d\mathbf{p}_+ d\mathbf{p}_-}{E_+E_-}\ldots$$

5. Let us examine in greater detail the production of a pair by a circularly polarized wave $A_1 = A_2 \equiv A_0$. In this case the functions (10) are calculated in final form and are expressed in terms of Bessel functions.[7] The expression (9) for G can be written in the form

$$G = \frac{n^2}{z^2}J_n^2(z)\left\{-4\frac{(\mathbf{ab})^2}{\mathbf{a}^2} + \mathbf{q}^2\mathbf{a}^2 + \left(1 - \frac{z^2}{n^2}\right)\frac{\omega_n^2}{(f_-k)(f_+k)} \times [\mathbf{k}, \mathbf{f}_+ + \mathbf{f}_-]^2\right\} + J_n'^2\left\{-4\frac{[\mathbf{ab}]^2}{\mathbf{a}^2} + \frac{\omega_n^2}{(f_-k)(f_+k)}[\mathbf{k}, \mathbf{f}_+ + \mathbf{f}_-]^2\right\}, \quad z = \frac{eA_0}{\omega}|\mathbf{a}|;$$

$$\mathbf{a} = \left[\mathbf{k}, \frac{\mathbf{f}_-}{f_-k} - \frac{\mathbf{f}_+}{f_+k}\right]; \quad \mathbf{b} = \left[\mathbf{k}, \frac{E_+\mathbf{f}_-}{f_-k} + \frac{E_-\mathbf{f}_+}{f_+k}\right], \quad (19)$$

where $J_n'$ is the derivative of the Bessel function.

The differential cross section, represented by the formulas (9) and (19), has a very complicated character and can be investigated only in certain limiting cases. In the case of a wave of low intensity $\xi \ll 1$, the arguments z of the Bessel functions satisfy the condition $z < n\xi \ll n$, and therefore the n-th harmonic of the cross section is

$$\sigma_n \sim (\xi^2)^{n-1}. \qquad (20)$$

If the frequency of the field is sufficiently large:

$$\omega / 2m^* \approx \omega / 2m \gtrsim 1, \qquad (21)$$

then the principal role is played by harmonics with $n \sim 1$.

We expand the Bessel functions in powers of $\xi^2$ [7] and, confining ourselves to the first term of the expansion, we obtain from (9) and (19):

$$d\sigma_n = Q_n d\sigma_0,$$
$$Q_n = 4n\left[\frac{n^n}{2^n n!}\right]^2 (\xi^2)^{n-1}\left\{\frac{m^2}{\omega_n^2}\left[\mathbf{k}, \frac{\mathbf{f}_-}{f_-k} - \frac{\mathbf{f}_+}{f_+k}\right]^2\right\}^{n-1}, \qquad (22)$$

where $d\sigma_0$ represents the usual expression for the differential cross section for the production of a pair by a photon with frequency $\omega_n = n\omega$ in the Coulomb field of the nucleus [6]:

$$d\sigma_0 = \frac{Z^2\alpha^3}{(2\pi)^2}\frac{|\mathbf{f}_+||\mathbf{f}_-|}{\omega_n^3}\frac{d\varepsilon_+ do_+ do_-}{q^4}$$
$$\times\left\{-4\left[\mathbf{k}, \frac{\varepsilon_+\mathbf{f}_-}{f_-k} + \frac{\varepsilon_-\mathbf{f}_+}{f_+k}\right]^2 + \mathbf{q}^2\left[\mathbf{k}, \frac{\mathbf{f}_-}{f_-k} - \frac{\mathbf{f}_+}{f_+k}\right]^2\right.$$
$$\left. + 2\frac{\omega_n^2}{(f_-k)(f_+k)}[\mathbf{k}, \mathbf{f}_+ + \mathbf{f}_-]^2\right\}. \qquad (23)$$

We have assumed here that the particle momenta coincide at infinity with the mean particle momenta in the wave, that is, $f_\pm \approx p_\pm$, and the energies of the particles are connected by means of the usual conservation law $\epsilon_+ + \epsilon_- = \omega_n$. This is fully justified, since f differs from p by an amount proportional to $\xi^2$ [cf. (2)].

In particular, for $n = 1$ (if, of course, $\omega > 2m^*$) we have

$$Q_1 = 1, \quad d\sigma_1 = d\sigma_0. \qquad (24)$$

For $n = 2$ (when $\omega_2 = 2\omega > 2m^*$) we get

$$d\sigma_2 = 2\xi^2\frac{m^2}{\omega_2^2}\left[\mathbf{k}, \frac{\mathbf{f}_-}{f_-k} - \frac{\mathbf{f}_+}{f_+k}\right]^2 d\sigma_0. \qquad (25)$$

The differential cross section (25) can be integrated over the angles. However, the resultant expression is very cumbersome and simplifies only in the extremely relativistic region, where all the energies are appreciably larger than m ($\omega_2, \epsilon_\pm \gg m$):

$$d\sigma_2 = \frac{16}{3}\xi^2\overline{\Phi}\frac{d\varepsilon_+}{\omega_2^3}\left\{\left(\varepsilon_+^2 + \varepsilon_-^2 + \frac{4}{5}\varepsilon_+\varepsilon_-\right)\right.$$
$$\times\left(\ln\frac{2\varepsilon_+\varepsilon_-}{m\omega_2} - \frac{1}{2}\right)$$
$$\left. - \frac{11}{5}\cdot\frac{\varepsilon_+\varepsilon_-}{\omega_2^2}\left(\varepsilon_+^2 + \varepsilon_-^2 + \frac{6}{11}\varepsilon_+\varepsilon_-\right)\right\}, \qquad (26)$$

where $\overline{\Phi} = r_0^2 Z^2\alpha$, and $r_0$ is the classical "radius" of the electron.

Formula (26) does not take into account the screening of the wave of the nucleus. Therefore, according to [6] it is valid if

$$2\varepsilon_+\varepsilon_- / m\omega_2 \ll 137\, Z^{-1/3}. \qquad (27)$$

Integrating expression (26) over the positron energy, we obtain for $\omega_2 \gg m^*$

$$\sigma_2 = \xi^2\overline{\Phi}\left(\frac{64}{15}\ln\frac{2\omega_2}{m^*} - \frac{2792}{225}\right). \qquad (28)$$

If the frequency of the field is small (which corresponds to the actual situation):

$$\omega / m^* \approx \omega / m \ll 1, \qquad (29)$$

then harmonics with large numbers participate in the pair production:

$$n \geqslant n_0 \geqslant 2m^* / \omega \gg 1. \qquad (30)$$

Therefore for the Bessel functions which enter into the cross section we can use the "approximation by means tangents." [7] As a result we obtain for the n-th harmonic of the differential cross section expression (22) in which the quantity $Q_n$ is somewhat modified:

$$d\sigma_n = \overline{Q}_n d\sigma_0,$$
$$\overline{Q}_n = \frac{e^2}{2\pi}(\varepsilon_0^2\xi^2)^{n-1}\left\{\frac{m^2}{\omega_n^2}\left[\mathbf{k}, \frac{\mathbf{f}_-}{f_-k} - \frac{\mathbf{f}_+}{f_+k}\right]^2\right\}^{n-1}, \qquad (31)$$

where $\epsilon_0$ denotes, for convenience, the base of the natural logarithms.

Taking account of the fact that n are large (30), let us estimate the integrals of (31) by the Laplace method. [8] Then the n-th harmonic of the cross section takes the form

$$\sigma_n \cong \overline{\Phi}\frac{1}{8}\varepsilon_0^2\frac{(\varepsilon_0^2\xi^2)^{n-1}}{n^2}\frac{n}{n_0}\left(1 - \frac{n_0^2}{n^2}\right)^{n+2}; \quad n_0 \equiv \frac{2m^*}{\omega}. \qquad (32)$$

For the total cross section we obtain as a result of integration with respect to n

$$\sigma \cong \frac{1}{8}\frac{\overline{\Phi}}{n_0\xi^2}\int_0^1 dx\frac{(1-x^2)^2}{x}\exp\left\{n_0\frac{\ln\varepsilon_0^2\xi^2(1-x^2)}{x}\right\}$$
$$\approx \sqrt{\frac{\pi}{2}}\frac{\Phi}{n_0^{3/2}\xi^2(-\ln\xi^2)^3}\left(\frac{2\varepsilon_0^2\xi^2}{-\ln\xi^2}\right)^{n_0}. \qquad (33)$$

Thus, in this case the pair-production cross section is very small, as expected.

Let us consider the process of the production of a pair by a wave of large intensity, $\xi \gg 1$. In this case the principal role is assumed by harmonics with numbers $n \gg 1$. It will be convenient in what follows to express the arguments of the Bessel functions in terms of the angles $\mathbf{p}_+$ and $\mathbf{p}_-$:

$$z = n\frac{\xi}{(1+\xi^2)^{1/2}}\frac{m^*}{\omega_n}\times\left\{\frac{p_+^2\sin^2\theta_+}{(E_+-p_+\cos\theta_+)^2}+\frac{p_-^2\sin^2\theta_-}{(E_--p_-\cos\theta_-)^2}-2\frac{p_+p_-\sin\theta_+\sin\theta_-\cos\varphi}{(E_+-p_+\cos\theta_+)(E_--p_-\cos\theta_-)}\right\}^{1/2}, \quad (34)$$

where $\theta_\pm$ are the angles between $\mathbf{p}_\pm$ and $\mathbf{k}$, and $\varphi$ is the angle between the planes $(\mathbf{k}, \mathbf{p}_+)$ and $(\mathbf{k}, \mathbf{p}_-)$.

We see from this expression that when $\varphi = \pi$ and $\sin\theta_\pm = m^*/E_\pm$, the variable $z$ assumes a maximum value

$$z_{max} = n\frac{\xi}{(1+\xi^2)^{1/2}}\frac{|\mathbf{p}_+|+|\mathbf{p}_-|}{\omega_n}. \quad (35)$$

If the frequency of the wave is not large, i.e., $m/\omega \gtrsim 1$, harmonics with large numbers participate in the pair production process:

$$n \geqslant n_0 \sim m\xi/\omega \gg 1. \quad (36)$$

Near the threshold, that is, for $n \sim n_0$, the momenta $|\mathbf{p}_\pm| \ll \omega_n$ and therefore, as follows from (35)

$$z \leqslant z_{max} \ll n. \quad (37)$$

In this case, as above, we can use the "tangent approximation"[7] for the Bessel functions. This leads to the following expression for the differential cross section:

$$d\sigma_n = \frac{\varepsilon_0^2}{2\pi}\left(\varepsilon_0^2\frac{\xi^2}{1+\xi^2}\frac{m^{*2}}{\omega_n^2}\mathbf{a}^2\right)^{n-1} d\sigma,$$

$$d\sigma = \frac{Z^2\alpha^3}{(2\pi)^2}\frac{\delta(E_++E_--\omega_n)\,d\mathbf{f}_+d\mathbf{f}_-}{\omega_n^3q^4\varepsilon_+\varepsilon_-}\left\{-4\mathbf{b}^2+\mathbf{q}^2\mathbf{a}^2+\frac{2\omega_n^2}{(f_+k)(f_-k)}[\mathbf{k}, \mathbf{f}_++\mathbf{f}_-]^2\right\}. \quad (38)$$

Here $\omega_n = n\omega$.

To obtain the total cross section of the n-th harmonic it is convenient to go over, as indicated above, to the variables $p_\pm$. Taking into account the fact the $n$ are large, let us estimate the integrals by the Laplace method[8]:

$$\sigma_n \cong \frac{1}{8}\frac{\bar{\Phi}}{n^2\xi^2}\left(\varepsilon_0^2\frac{\xi^2}{1+\xi^2}\right)^n\frac{n}{n_0}\left(1-\frac{n_0^2}{n^2}\right)^{n+2}; \quad n_0 = \frac{2m^*}{\omega} \quad (39)$$

Equations (38) and (39) are valid for $n \sim n_0$. We note that when $\xi \ll 1$ they go over, naturally, into expressions (31) and (32).

Let us consider the integral cross section of the n-th harmonic for the numbers $n$ which exceed the threshold value appreciably:

$$n \gg n_0, \quad (40)$$

assuming here that $m/\omega \gtrsim 1$, i.e.,

$$n_0 \gg 1. \quad (40')$$

This corresponds to the ultrarelativistic "quasimomenta":

$$E_\pm, |\mathbf{p}_\pm| \gg m^*. \quad (41)$$

In this case $z_{max}$ (35) is close to $n$ and, consequently, the principal role is played by the angles $\varphi$ and $\theta_\pm$ close to their optimal values:

$$\varphi_0 = \pi, \quad \theta_{0\pm} \approx \sin\theta_{0\pm} = m^*/E_\pm. \quad (42)$$

In order to estimate the intervals of variation of the angles near $\varphi_0$ and $\theta_{0\pm}$, we use the asymptotic expression for the Bessel function in terms of the Airy function $\Phi$[5,7]:

$$J_n(n\varepsilon) \approx \frac{1}{\pi}\left(\frac{2}{n}\right)^{2/3}\Phi\left[\left(\frac{n}{2}\right)^{2/3}(1-\varepsilon^2)\right], \quad n \gg 1, \ \varepsilon \approx 1. \quad (43)$$

Expanding $1 - z^2/n^2$ near $\varphi_0$ and $\theta_{0\pm}$, we get

$$J_n(z) \approx \frac{1}{\pi}\left(\frac{2}{n}\right)^{2/3}\Phi(u), \quad u = \left(\frac{n}{2}\right)^{2/3}\left(1-\frac{z^2}{n^2}\right) \approx \left(\frac{n}{2}\right)^{2/3}\left\{\frac{1}{\xi^2}+\frac{m^{*2}}{E_+E_-}+\frac{1}{2}\frac{E_+^2}{m^{*2}}(\Delta\theta_+)^2+\frac{1}{2}\frac{E_-^2}{m^{*2}}(\Delta\theta_-)^2+\frac{E_+E_-}{\omega_n^2}(\Delta\varphi)^2\right\},$$

$$\Delta\theta_\pm = |\theta_\pm - \theta_{0\pm}|; \quad \Delta\varphi = |\pi - \varphi|. \quad (44)$$

We see therefore that the important range of angle variation is

$$\Delta\theta_\pm \sim m^*/E_\pm n^{1/3} \sim \theta_{0\pm}/n^{1/3}, \quad \Delta\varphi \sim n^{-1/3}. \quad (45)$$

In addition, a limitation is imposed, generally speaking, on the energy difference

$$n^{2/3}m^{*2}/E_+E_- \lesssim 1 \quad (46)$$

or, if we introduce the quantity $\Delta = |E_+ - E_-|/\omega_n$, then

$$n^{2/3}\frac{m^{*2}}{\omega_n^2}\frac{1}{1-\Delta^2} \lesssim 1. \quad (46')$$

Let us consider the momentum $\mathbf{q}$ (11) transferred to the nucleus. The minimum value of $|\mathbf{q}|$ (when $\mathbf{p}_+$ and $\mathbf{p}_-$ have the same direction as

k) is $q_0 \approx m^{*2}/\omega_n$. For the angles $\varphi = \varphi_0$ and $\theta_\pm = \theta_{0\pm}$ we have

$$q \sim q_0\omega_n^2 / E_+E_- \sim \omega_n m^{*2} / E_+E_- \sim \omega_n\Theta, \qquad (47')$$

where $\Theta$ is the angle between the total momenta $\mathbf{p}_+ + \mathbf{p}_-$ and $\mathbf{k}$. The change in the angle $\Theta$ occurs principally as a result of the variation of the lengths of the vectors $\mathbf{p}_+$ and $\mathbf{p}_-$ (that is, of the energies). At the same time the angles $\theta_{0\pm}$ themselves also change.

From (46) and (47) it follows that $\Theta_{max} \sim n^{-2/3}$. Then $q_{max} \sim \omega_n\Theta_{max} \gg q_0$ if $n \gg n_0^{3/2}$. We shall henceforth consider only such values of n. It will be shown below that the cross section is small when $n_0 \ll n \ll n_0^{3/2}$.

Thus, in the situation in question, we can expect in the integration, as a result of $1/q^4$, the appearance of a logarithm

$$\ln\frac{q_{max}}{q_0} \sim \ln\frac{\omega_n}{m^* n^{1/3}}. \qquad (48)$$

We change over in the integration to the variables $p_\pm$. It will be further more convenient to separate the integration with respect to the argument of the Bessel functions. To this end we introduce the additional integration

$$\frac{\omega_n^2}{m^{*2}}\int_0^{x_0^2} dx^2\delta\left(\frac{\omega_n^2}{m^{*2}}x^2 - \mathbf{a}^2\right)\ldots, \qquad (49)$$

$$\mathbf{a} = \left[\mathbf{k}, \frac{\mathbf{p}_-}{p_-k} - \frac{\mathbf{p}_+}{p_+k}\right], \quad x_0^2 = 1 - \frac{n_0^2}{n^2}. \qquad (50)$$

The $\delta$-function which we have introduced can be used for the integration with respect to the angle $\varphi$ between the planes $(\mathbf{k}, \mathbf{p}_+)$ and $(\mathbf{k}, \mathbf{p}_-)$. As a result we obtain

$$\sigma_n = \frac{Z^2\alpha^3}{m^{*2}}\frac{4n}{(2\pi)^2\omega_n}\int_0^{x_0^2} dx^2 \int\frac{|\mathbf{p}_+||\mathbf{p}_-| dE_+ \sin\theta_+ d\theta_+ \sin\theta_- d\theta_-}{q^4}$$
$$\times G\left\{\left(\frac{p_+\sin\theta_+}{E_+ - p_+\cos\theta_+} + \frac{p_-\sin\theta_-}{E_- - p_-\cos\theta_-}\right)^2 - \frac{\omega_n^2}{m^{*2}}x^2\right\}^{-1/2}$$
$$\times\left\{\frac{\omega_n^2}{m^{*2}}x^2 - \left(\frac{p_+\sin\theta_+}{E_+ - p_+\cos\theta_+} - \frac{p_-\sin\theta_-}{E_- - p_-\cos\theta_-}\right)^2\right\}^{-1/2} \qquad (51)$$

where G is determined by formula (19) in which it is necessary to make the substitution $f_\pm \to p_\pm$ and substitute the value of the angle $\varphi$ determined from the $\delta$-function (49). Then

$$\mathbf{q}^2 = \frac{\omega_n^2}{m^{*2}}(E_+ - p_+\cos\theta_+)(E_- - p_-\cos\theta_-)$$
$$\times\left\{1 - x^2 - \left[1 - \frac{m^{*2}(\omega_n - p_+\cos\theta_+ - p_-\cos\theta_-)}{\omega_n(E_+ - p_+\cos\theta_+)(E_- - p_-\cos\theta_-)}\right]^2\right\};$$

$$\frac{[\mathbf{k}, \mathbf{p}_+ - \mathbf{p}_-]^2}{(p_+k)(p_-k)} = \frac{\omega_n^2}{m^{*2}}\left\{1 - x^2\right.$$
$$\left. - \left[1 - \frac{m^{*2}(\omega_n - p_+\cos\theta_+ - p_-\cos\theta_-)}{\omega_n(E_+ - p_+\cos\theta_+)(E_- - p_-\cos\theta_-)}\right]^2\right\}$$
$$- \frac{(\omega_n - p_+\cos\theta_+ - p_-\cos\theta_-)^2}{(E_+ - p_+\cos\theta_+)(E_- - p_-\cos\theta_-)}; \quad 4\frac{[\mathbf{ab}]^2}{\mathbf{a}^2} = \frac{m^{*2}}{x^2}$$
$$\times\left\{\left(\frac{p_+\sin\theta_+}{E_+ - p_+\cos\theta_+} + \frac{p_-\sin\theta_-}{E_- - p_-\cos\theta_-}\right)^2 - \frac{\omega_n^2}{m^{*2}}x^2\right\}$$
$$\times\left\{\frac{\omega_n^2}{m^{*2}}x^2 - \left(\frac{p_+\sin\theta_+}{E_+ - p_+\cos\theta_+} - \frac{p_-\sin\theta_-}{E_- - p_-\cos\theta_-}\right)^2\right\}.$$
$$4\frac{(\mathbf{ab})^2}{\mathbf{a}^2} = \frac{1}{x^2}\frac{m^{*2}}{\omega_n^2}\left\{\omega_n\frac{p_-^2\sin^2\theta_-}{(E_- - p_-\cos\theta_-)^2}\right.$$
$$\left. - \omega_n\frac{p_+^2\sin^2\theta_+}{(E_+ - p_+\cos\theta_+)^2} + (E_+ - E_-)\frac{\omega_n^2}{m^{*2}}x^2\right\}^2;$$
$$z = n\xi(1 + \xi^2)^{-1/2}x. \qquad (52)$$

The region of integration with respect to $dE_+ d\theta_+ d\theta_-$ is bounded by the following conditions:

$$\left|\frac{p_+\sin\theta_+}{E_+ - p_+\cos\theta_+} - \frac{p_-\sin\theta_-}{E_- - p_-\cos\theta_-}\right| \leqslant \frac{\omega_n}{m^*}x$$
$$\leqslant \frac{p_+\sin\theta_+}{E_+ - p_+\cos\theta_+} + \frac{p_-\sin\theta_-}{E_- - p_-\cos\theta_-}. \qquad (53)$$

In view of the fact that the principal role is played by $x \sim x_0 \sim 1$ and by angles $\theta_\pm$ close to $\theta_{0\pm}$, we can carry out an expansion near these angles, assuming where possible that $\theta_\pm = \theta_{0\pm}$ and $x = 1$. We then obtain as a result of the integration the following expression for $\sigma_n$:

$$\sigma_n = \frac{Z^2\alpha^3}{m^{*2}}2n\int_0^{x_0^2}\frac{dx^2}{(1 - x^2)^{1/2}}$$
$$\times\left\{\left[\frac{2}{3}J_n'^2(z) + \left(\frac{1}{\xi^2} + 1 - x^2\right)J_n^2(z)\right]\right.$$
$$\times\ln\frac{\omega_n(1 - x^2)^{1/2}}{m^*} - \frac{37 - 30\ln 2}{18}J_n'^2(z)$$
$$\left. - \frac{21 - 16\ln 2}{8}\left(\frac{1}{\xi^2} + 1 - x^2\right)J_n^2(z)\right\}. \qquad (54)$$

We go over from Bessel functions to their asymptotic expression (44) in terms of the Airy function:

$$\sigma_n = \frac{4\cdot 2^{2/3}}{\pi}\frac{\overline{\Phi}}{n^{2/3}\xi^2}\int_{z_0}^{\infty}\frac{dz}{z^{1/2}}\left\{\left[\frac{2}{3}\Phi'^2(u) + u\Phi^2(u)\right]\right.$$
$$\times\ln\frac{\omega_n}{m^*}\left(\frac{2}{n}\right)^{1/3}\sqrt{z}$$
$$\left. - \frac{37 - 30\ln 2}{18}\Phi'^2(u) - \frac{21 - 16\ln 2}{8}u\Phi^2(u)\right\}, \qquad (55)$$

where $\overline{\Phi} = r_0^2 Z^2\alpha$, $\Phi$ and $\Phi'$ are the Airy function and its derivative

$$u = u_0 + z, \qquad u_0 = (n/2)^{2/3}\xi^{-2},$$
$$z_0 = (n_0/n)^2(n/2)^{2/3}, \quad n_0 = 2m^*/\omega. \tag{56}$$

The value of the integral in (55) depends on the two parameters $u_0$ and $z_0$. We can therefore consider several limiting cases. Let

$$\eta = \omega\xi/m \gg 1. \tag{57}$$

Then the harmonics of importance are those with the numbers

$$n_0^{3/2} \ll n \ll \xi^3. \tag{58}$$

In this case $z_0 \ll 1$, and the lower limit of integration in (55) can be replaced by 0. In addition, $u_0 \ll 1$ and therefore it can be neglected in the arguments of the Airy functions. Taking the logarithm outside the integral sign at the point $z \sim 1$, we obtain, integrating the Airy functions,

$$\sigma_n \cong \frac{3^{7/6}\Gamma(2/3)}{2\pi}\frac{\Phi}{n^{2/3}\xi^2}\ln\beta\frac{\omega n^{2/3}}{m^*}, \tag{59}$$

where $\beta$ is a constant of the order of unity.

For $n_0 \ll n \ll n_0^{3/2}$, the lower limit of integration in (55) is $z_0 \gg 1$. In this case we can use the asymptotic form of the Airy functions [5,7]:

$$\Phi(t) \approx {}^1/_2 t^{-1/4}\exp\{-{}^2/_3 t^{3/2}\}, \quad t \gg 1, \tag{60}$$

which leads to an exponentially small value of the cross section:

$$\sigma_n \sim \exp\{-2n_0^3/3n^2\}. \tag{61}$$

Analogously, when $n \gg \xi^3$, that is, $u_0 \gg 1$, we have

$$\sigma_n \sim \exp\{-{}^2/_3 n/\xi^3\}. \tag{62}$$

Thus, in the limiting case of $\eta \gg 1$ [cf. (57)] the effective number of harmonics, which make the main contribution to the pair-production cross section, is determined by inequalities (58) and is large. Let us estimate in this case the pair-production cross section summed over all harmonics:

$$\sigma = \sum_n \sigma_n \sim \int^{\xi^3} \sigma_n\, dn \sim \overline{\Phi}\frac{1}{\xi}\ln\beta\frac{\omega\xi}{m}. \tag{63}$$

In the opposite limiting case $\eta \ll 1$, the cross section is exponentially small for all $n \gg n_0$:

$$\sigma_n \sim \exp\left\{-\frac{2}{3}\left[\frac{n^{2/3}}{\xi^2} + \frac{n_0^2}{n^{4/3}}\right]^{3/2}\right\}. \tag{64}$$

Finally, if $\eta \sim 1$, there are several non-exponentially small harmonics with numbers $n \sim n_0\xi$.

The parameter $\eta$ can be represented in the following form:

$$\eta^2 = 4\pi r_0 \lambda\!\!\!^-{}_k^2 \rho\hbar\omega/mc^2, \tag{65}$$

where $\rho$ is the density of the quanta, $r_0$ the classical "radius" of the electron, and $\lambda\!\!\!^-_k = \hbar/mc$ is the Compton wavelength of the electron.

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