THEORY OF NONLINEAR OSCILLATIONS IN A COLLISIONLESS PLASMA

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A general perturbation theory is developed for nonlinear oscillations in a collisionless plasma. The theory is not restricted by any assumptions regarding randomness of the phases. Summation of the secular terms of the perturbation theory yields equations for "slow" processes. In the case of sufficiently broad wave packets, the equations go over into the familiar equations of the quasilinear theory for a weakly-turbulent plasma. The converse limiting case, the evolution of a periodic wave in the quasilinear approximation, is investigated in detail.

1. INTRODUCTION

NONLINEAR oscillations in a collisionless plasma, which are described by the equations [1)]

$$\frac{\partial F_j}{\partial t} + \mathbf{v}\frac{\partial F_j}{\partial \mathbf{r}} + \frac{e_j}{m_j}\mathbf{E}\frac{\partial F_j}{\partial \mathbf{v}} = 0 \qquad (1.1)$$

$$\nabla \mathbf{E} = 4\pi \sum_j e_j N_j \int F_j d\mathbf{v} \qquad (1.2)$$

(j—index indicating the species of the particle), were considered in numerous papers (see the reviews [1,2], where a detailed bibliography is found, and also the more recent papers [3-10]). A characteristic feature of the methods developed in these papers is the use, in one form or another, of the so-called "random phase" approximation, so that the results are applicable only to a turbulent plasma, where the wave packet is sufficiently broad. Yet in many cases (for example in a bounded plasma), it is necessary to investigate the dynamics of nonlinear periodic waves characterized by a discrete set of wave numbers k; naturally, the random-phase approximation is no longer applicable in this case.

In the present paper we are considering a general perturbation theory for plasma oscillations, without being confined to any assumption regarding the randomness of the phases. The formal expansion is in powers of the oscillation field. In the general series of this theory we separate and sum sequences of secular terms, in analogy with the procedure used by Van Hove [11], Prigogine [12], and Balescu [13] to obtain the kinetic equations for weakly-nonideal systems. The summation of the "principal" sequences of the secular terms leads to quasilinear equations, which describe the reaction of the oscillations on the distribution function of the plasma particles. The applicability of these equations, however, is not limited by any conditions whatever with respect to the width of the wave packet. If this width is sufficiently large, then the foregoing equations go over into the known equations of the quasilinear theory for a weakly turbulent plasma [14,15]. In the opposite limiting case we obtain equations for a "monochromatic" wave. The solution of these equations, obtained in the present paper, describes the evolution of the plasma distribution function and the field of the "monochromatic" wave with account of the reaction of the wave on the distribution function.

2. PERTURBATION THEORY. SUMMATION OF DIAGRAMS

Following Landau [16], we seek a solution of Eqs. (1.1) and (1.2) subject to the initial condition [2)]

$$F(0, \mathbf{r}, \mathbf{v}) = \sum_{\mathbf{k}} F_{\mathbf{k}}^0(\mathbf{v}) e^{i\mathbf{k}\mathbf{r}} = f(\mathbf{v}) + \sum_{\mathbf{k}\neq 0} g_{\mathbf{k}}(\mathbf{v}) e^{i\mathbf{k}\mathbf{r}} \qquad (2.1)$$

(the index indicating the species of the particles will henceforth be omitted for brevity). Expanding $F(t, \mathbf{r}, \mathbf{v})$ and $E(t, \mathbf{r})$ in Fourier series

$$F(t, \mathbf{r}, \mathbf{v}) = \sum_{\mathbf{k}} F_{\mathbf{k}}(t, \mathbf{v}) e^{i\mathbf{k}\mathbf{r}}, \quad \mathbf{E}(t, \mathbf{r}) = \sum_{\mathbf{k}} \mathbf{E}_{\mathbf{k}}(t) e^{i\mathbf{k}\mathbf{r}} \qquad (2.2)$$

[1)]For simplicity we consider potential oscillations without a magnetic field, although all the results can be extended without fundamental difficulty to include the general case.

[2)]The normalization volume is assumed equal to unity throughout.

and applying the Laplace transformation to the time-dependent quantities[3)]:

$$F_{\mathbf{k}}(\omega, \mathbf{v}) = \int_0^\infty F_{\mathbf{k}}(t, \mathbf{v}) e^{i\omega t}\, dt, \qquad \mathbf{E}_{\mathbf{k}}(\omega) = \int_0^\infty \mathbf{E}_{\mathbf{k}}(t) e^{i\omega t}\, dt, \tag{2.3}$$

we obtain in place of (1.1) the following integral equation

$$F_{\mathbf{k}}(\omega, \mathbf{v}) = \frac{1}{\omega - \mathbf{kv}} \frac{e}{im} \sum_{\mathbf{k}=\mathbf{k}'+\mathbf{k}''} \int \frac{d\omega'}{2\pi} \mathbf{E}_{\mathbf{k}'}(\omega') \frac{\partial F_{\mathbf{k}''}(\omega - \omega', \mathbf{v})}{\partial \mathbf{v}} + \frac{iF_{\mathbf{k}}^0(\mathbf{v})}{\omega - \mathbf{kv}}, \tag{2.4}$$

where $f_{\mathbf{k}}^0(\mathbf{v})$ is determined by the initial conditions (2.1). In deriving (2.4) we used the convolution theorem (see footnote 3).

We expand the solution of (2.4) in powers of the field $\mathbf{E}$:

$$F_{\mathbf{k}}(\omega, \mathbf{v}) = \sum_{n=0}^{\infty} F_{\mathbf{k}}^{(n)}(\omega, \mathbf{v}), \tag{2.5}$$

where $F_{\mathbf{k}}^{(n)}(\omega, \mathbf{v}) \sim E^n$ and $E_{\mathbf{k}} \sim g_{\mathbf{k}}(\mathbf{v})$. As the zeroth approximation we choose $f(\mathbf{v})$ [see (2.1)]. The first approximation is given by the relation ($\mathbf{k} \neq 0$)

$$F_{\mathbf{k}}^{(1)}(\omega, \mathbf{v}) = \frac{e}{im} \frac{\mathbf{E}_{\mathbf{k}}(\omega)}{\omega - \mathbf{kv}} \frac{df}{d\mathbf{v}} + \frac{ig_{\mathbf{k}}(\mathbf{v})}{\omega - \mathbf{kv}}. \tag{2.6}$$

From (2.4) follows the following recurrence formula for $F_{\mathbf{k}}^{(n)}(\omega, \mathbf{v})$ when $n \geq 2$:

$$F_{\mathbf{k}}^{(n)}(\omega, \mathbf{v}) = \frac{1}{\omega - \mathbf{kv}} \sum_{\mathbf{k}=\mathbf{k}'+\mathbf{k}''} \frac{e}{im} \int \frac{d\omega'}{2\pi} \mathbf{E}_{\mathbf{k}'}(\omega') \times \frac{\partial F_{\mathbf{k}''}^{(n-1)}(\omega - \omega', \mathbf{v})}{\partial \mathbf{v}} \tag{2.7}$$

[3)]In the Laplace representation, the corresponding quantities are analytic in the upper half plane of ω, with the possible exception of some vicinity of the real axis. The inverse transformations are

$$\mathbf{E}_{\mathbf{k}}(t) = \frac{1}{2\pi} \int_{-\infty+i\delta}^{\infty+i\delta} \mathbf{E}_{\mathbf{k}}(\omega) e^{-i\omega t}\, d\omega, \qquad \delta > 0.$$

For what follows, it is also useful to bear in mind the convolution formula

$$\int_0^\infty F_1(t) F_2(t) e^{i\omega t}\, dt = \frac{1}{2\pi} \int d\omega' F_1(\omega') F_2(\omega - \omega') = \frac{1}{(2\pi)^2 i} \int \frac{d\omega' d\omega''}{\omega' + \omega'' - \omega} F_1(\omega') F_2(\omega''),$$

where the integration is along lines lying in the upper half-plane and satisfying the condition $\mathrm{Im}\,\omega > \mathrm{Im}\,\omega' + \mathrm{Im}\,\omega''$.

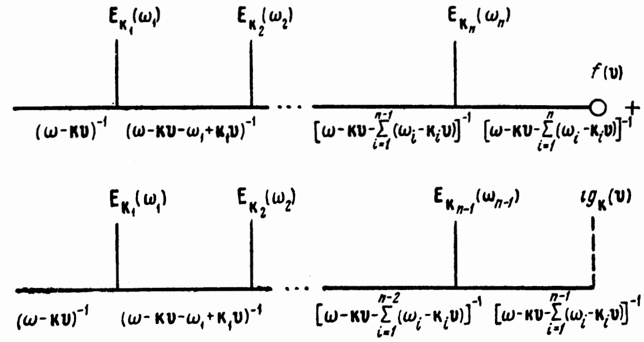

FIG. 1

With the aid of (2.7) and (2.6) we obtain an expression for the general term of the series (2.5):

$$F_{\mathbf{k}}^{(n)}(\omega, \mathbf{v}) = \left(\frac{e}{2\pi i m}\right)^n \sum_{\mathbf{k}=\mathbf{k}_1+\ldots+\mathbf{k}_n} \Bigg\{ \int d\omega_1 d\omega_2 \ldots d\omega_n \frac{\mathbf{E}_{\mathbf{k}_1}(\omega_1)}{\omega - \mathbf{kv}} \times \frac{\partial}{\partial \mathbf{v}} \frac{\mathbf{E}_{\mathbf{k}_2}(\omega_2)}{\omega - \mathbf{kv} - \omega_1 + \mathbf{k}_1\mathbf{v}} \times \frac{\partial}{\partial \mathbf{v}} \cdots \frac{E_{\mathbf{k}_n}(\omega_n)}{\omega - \mathbf{kv} - \sum_{i=1}^{n-1} (\omega_i - \mathbf{k}_i\mathbf{v})} \times \frac{\partial}{\partial \mathbf{v}} \frac{f(\mathbf{v})}{\omega - \mathbf{kv} - \sum_{i=1}^{n} (\omega_i - \mathbf{k}_i\mathbf{v})}$$
$$- \frac{2\pi m}{e} \int d\omega_1 d\omega_2 \ldots d\omega_n \frac{\mathbf{E}_{\mathbf{k}_1}(\omega_1)}{\omega - \mathbf{kv}} \times \frac{\partial}{\partial \mathbf{v}} \frac{\mathbf{E}_{\mathbf{k}_2}(\omega_2)}{\omega - \mathbf{kv} - \omega_1 + \mathbf{k}_1\mathbf{v}} \times \frac{\partial}{\partial \mathbf{v}} \cdots \frac{\mathbf{E}_{\mathbf{k}_{n-1}}(\omega_{n-1})}{\omega - \mathbf{kv} - \sum_{i=1}^{n-2} (\omega_i - \mathbf{k}_i\mathbf{v})} \times \frac{\partial}{\partial \mathbf{v}} \frac{g_{\mathbf{k}_n}(\mathbf{v})}{\omega - \mathbf{kv} - \sum_{i=1}^{n-1} (\omega_i - \mathbf{k}_i\mathbf{v})} \Bigg\}. \tag{2.8}$$

We can visualize the general term in simple fashion by introducing its diagram representation (Fig. 1). The solid vertical lines on Fig. 1 correspond to $\mathbf{E}_{\mathbf{k}_s}(\omega_s)$, and the horizontal ones correspond to the "propagators"

$$\left[\omega - \mathbf{kv} - \sum_{i=1} (\omega_i - \mathbf{k}_i\mathbf{v})\right]^{-1} \qquad (0 \leqslant s \leqslant n);$$

the s-th vertex corresponds to the integro-differential operator

$$\frac{e}{im} \frac{\partial}{\partial \mathbf{v}} \int \frac{d\omega_s}{2\pi};$$

the circle in the first term of Fig. 1 represents the function $f(\mathbf{v})$, and the vertical dashed line in the second term represents the function $ig_k(\mathbf{k})$. The order of the diagram is determined by the number of vertical lines (including the dashed line). It is seen from Fig. 1 that the term distribution-function of each perturbation-theory order consists of two separate parts, one dependent on $f(\mathbf{v})$ (with a circle on the right end in Fig. 1), and one dependent on $g_k(\mathbf{v})$ (with one dashed line in the right end of the diagram).

We now consider Eq. (1.2) for the wave field. Taking (2.2), (2.3), and (2.5) into account, we rewrite (1.2) in the form

$$\mathbf{k}\mathbf{E}_{\mathbf{k}}(\omega) = -4\pi ieN \sum_{n=1}^{\infty} \int F_{\mathbf{k}}^{(n)}(\omega, \mathbf{v})\, d\mathbf{v} \qquad (2.9)$$

(to abbreviate the notation, we leave out here and throughout the symbols denoting summations over the particle species; the restoration of the corresponding summation in the final formulas entails no difficulty). Substituting $F_k^{(1)}(\omega, \mathbf{v})$ from (2.6) in the first-approximation term in the right side of (2.9) and combining the terms that are linear in E, we can rewrite (2.9) in the form

$$\varepsilon_{\mathbf{k}}(\omega)\mathbf{E}_{\mathbf{k}}(\omega) = \frac{4\pi eN}{k^2}\mathbf{k}\int \frac{g_{\mathbf{k}}(\mathbf{v})\, d\mathbf{v}}{\omega - \mathbf{k}\mathbf{v}} - \frac{4\pi ieN}{k^2}\mathbf{k}\sum_{n=2}^{\infty}\int F_{\mathbf{k}}^{(n)}(\omega, \mathbf{v})\, d\mathbf{v}, \qquad (2.10)$$

where $\epsilon_k(\omega)$ is the dielectric constant of the plasma:

$$\varepsilon_{\mathbf{k}}(\omega) = 1 + \frac{\omega_0^2}{k^2}\int \frac{d\mathbf{v}}{\omega - \mathbf{k}\mathbf{v}}\mathbf{k}\frac{df}{d\mathbf{v}}, \qquad \omega_0^2 = \frac{4\pi e^2 N}{m}. \qquad (2.11)$$

The second term in the right side of (2.10) describes nonlinear effects. If we neglect them, we obtain the well known equation obtained by Landau[16] for the oscillation field in the linear approximation,

$$\mathbf{E}_{\mathbf{k}}(t) = \int \frac{\mathbf{r}_{\mathbf{k}}(\omega)}{\varepsilon_{\mathbf{k}}(\omega)} e^{-i\omega t}\frac{d\omega}{2\pi} \approx \mathbf{E}_{\mathbf{k}}{}^0 e^{-i\omega_{\mathbf{k}} t}; \qquad (2.12)$$

$$\mathbf{r}_{\mathbf{k}}(\omega) = \frac{4\pi eN}{k^2}\mathbf{k}\int \frac{g_{\mathbf{k}}(\mathbf{v})\, d\mathbf{v}}{\omega - \mathbf{k}\mathbf{v}}, \qquad \mathbf{E}_{\mathbf{k}}{}^0 = \frac{\mathbf{r}_{\mathbf{k}}(\omega_{\mathbf{k}})}{\varepsilon_{\mathbf{k}}{}'(\omega_{\mathbf{k}})},$$

$$\varepsilon_{\mathbf{k}}{}'(\omega_{\mathbf{k}}) = \frac{d\varepsilon_{\mathbf{k}}(\omega)}{d\omega}\Big|_{\omega=\omega_{\mathbf{k}}}; \qquad (2.13)$$

$$\omega_{\mathbf{k}} = \omega_{\mathbf{k}}{}^0 + i\gamma_{\mathbf{k}}, \qquad \gamma_{\mathbf{k}} = \pi\frac{\omega_0^2}{\varepsilon_{\mathbf{k}}{}'k^2}\frac{df}{dv}\Big|_{v=\omega_{\mathbf{k}}^0/k}, \qquad (2.14)$$

where $\mathbf{E}_k^0$ is the amplitude of the oscillations, which, as noted above, has the same order of magnitude as the initial perturbation $g_k(\mathbf{v})$, while ω_k is the complex frequency of the oscillations, which is the root of the dispersion equation $\epsilon_k(\omega) = 0$ having the largest imaginary part. In the right side of (2.12) we have left out terms corresponding to other roots of the dispersion equation; these terms are exponentially small as $t \to \infty$. We shall assume henceforth that the following condition is satisfied

$$\gamma_{\mathbf{k}} / \omega_{\mathbf{k}} \ll 1, \qquad (2.15)$$

and that without this condition the results given below are not valid.

In the formal expansion of (2.5), not all the nonlinear terms are actually small (in the sense that after going through the t-representation they can become large for sufficiently large t). To obtain the correct asymptotic expression for large t, such terms must be separated and summed. To separate the "large" terms we first replace, in the expression for the general term (2.8), the exact field components $\mathbf{E}_{k_s}(\omega_s)$ [which satisfy Eq. (2.10)] by their values in the linear approximation (2.12), neglecting at the same time the imaginary parts of the frequencies ω_{k_s}. In the Laplace representation, the corresponding expressions will take the form

$$\mathbf{E}_{\mathbf{k}_s}(\omega_s) = \frac{i\mathbf{E}_{\mathbf{k}_s}^0}{\omega_s - \omega_{k_s}^0}, \qquad (2.16)$$

where $\mathbf{E}_{k_s}^0$ is the oscillation amplitude obtained from Eq. (2.13) in the linear approximation. Because of the simple form of $\mathbf{E}_{k_s}(\omega_s)$ in (2.16), we can readily integrate over all the ω_s in the general term (2.8), as a result of which the ω_s are replaced by $\omega_{k_s}^0$ and $\mathbf{E}_{k_s}(\omega_s)$ by $\mathbf{E}_{k_s}^0$.

We now note that in some of the different diagrams of the sum over k_s in Fig. 1 neighboring lines will correspond to conjugate components of the field ($\mathbf{k}_s = -\mathbf{k}_{s+1}$, $\omega_{k_s}^0 = -\omega_{k_{s+1}}^0$). Such lines will be called paired. Paired lines are closed on the diagrams into loops (see Fig. 2). The propagators on the two sides of a loop turn out to be identical, thus leading to the appearance of multiple poles with respect to ω in $F_k^{(n)}(\omega, \mathbf{v})$. In the t-representation, the corresponding terms will be secular, i.e., proportional to t^r, where $r+1$

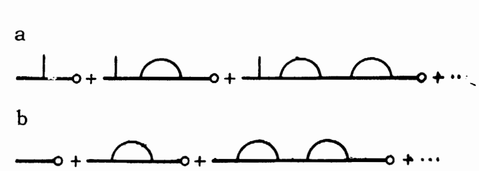

FIG. 2

is the multiplicity of the pole in the Laplace representation.

By way of a typical example, let us consider the expression represented by the second-order diagram in Fig. 2b. After substituting the approximation (2.16) for the field $\mathbf{E}_{k_s}(\omega_s)$ and integrating with respect to ω_s, we obtain for it

$$\left(\frac{e}{im}\right)^2 \sum_{\mathbf{q}} \frac{\mathbf{E}_{\mathbf{q}}^0}{\omega} \frac{\partial}{\partial \mathbf{v}} \frac{\mathbf{E}_{-\mathbf{q}}^0}{\omega - \omega_{\mathbf{q}}^0 + \mathbf{qv}} \frac{\partial}{\partial \mathbf{v}} \frac{f(\mathbf{v})}{\omega} \tag{2.17}$$

This expression, which is one of the contributions to the Fourier component of the distribution function with $\mathbf{k} = 0$, has a second-order pole at $\omega = 0$; accordingly, it yields in the t-representation a secular term proportional to t. Analogously, for the expression shown by the third-order diagram in Fig. 2a (which makes a contribution to the Fourier k-component of the distribution function), we obtain

$$\left(\frac{e}{im}\right)^3 \sum_{\mathbf{q}} \frac{\mathbf{E}_k^0}{\omega - \mathbf{kv}} \frac{\partial}{\partial \mathbf{v}} \frac{\mathbf{E}_{\mathbf{q}}^0}{\omega - \omega_k^0} \frac{\partial}{\partial \mathbf{v}} \frac{\mathbf{E}_{-\mathbf{q}}^0}{\omega - \omega_k^0 - \omega_{\mathbf{q}}^0 + \mathbf{qv}} \times \frac{\partial}{\partial \mathbf{v}} \frac{f(\mathbf{v})}{\omega - \omega_k^0}. \tag{2.18}$$

Expression (2.18) has a second order pole at $\omega = \omega_k^0$ (it is proportional to $t \exp(-i\omega_k^0 t)$ in the t-representation).

It has been assumed above that expressions (2.16) are substituted for $\mathbf{E}_k(\omega)$. If we substitute the exact $\mathbf{E}_k(\omega)$, or else (2.16) with complex frequency $\omega_k = \omega_k^0 + i\gamma_k$, then no secular term will appear after going over to the t-representation, but if $|\mathbf{E}_k(t)|$ depends slowly on the time, these terms containing paired $\mathbf{E}_k$ will be large for sufficiently large t, even if they are not secular[4)].

Thus, the nonlinear terms in Eq. (2.9) for the field will, for sufficiently large times, not merely lead to small corrections to the solution of the corresponding linearized equation, but may radically change this solution. To obtain the correct asymptotic value of the field for large t, the "large" terms indicated above must be summed. The n-th ordered term in (2.9) contains, generally speaking, several secular terms, which differ in their secularity indices (i.e., in the power of t contained in this term after the transition to the t-representation). The terms having the maximum secularity index for a specified order n will be called principal. In this paper we confine ourselves to summation of only the principal secular terms (the allowance for terms having a smaller degree of secularity is considered in [18]). It can be shown[18] that the principal secular terms on the right side of (2.9) are represented by the diagrams of Fig. 2a. We denote the sum of the diagrams in Fig. 2a by $\Phi_k(\omega, \mathbf{v})$. It is readily seen that this quantity is simply expressed in terms of the function $\Phi(\omega, \mathbf{v})$, which is the sum of all the diagrams shown in Fig. 2b, namely

$$\Phi_{\mathbf{k}}(\omega, \mathbf{v}) = \frac{e}{im} \int \frac{d\omega'}{2\pi} \frac{\mathbf{E}_{\mathbf{k}}(\omega')}{\omega - \mathbf{kv}} \frac{\partial \Phi(\omega - \omega', \mathbf{v})}{\partial \mathbf{v}}. \tag{2.19}$$

Equation (2.9) with only the principal secular terms takes the form

$$\mathbf{kE}_{\mathbf{k}}(\omega) = 4\pi eN \int \frac{g_{\mathbf{k}}(\mathbf{v})}{\omega - \mathbf{kv}} d\mathbf{v} - 4\pi ieN \int \Phi_{\mathbf{k}}(\omega, \mathbf{v})\, d\mathbf{v}$$

$$= 4\pi eN \int \frac{g_{\mathbf{k}}(\mathbf{v})}{\omega - \mathbf{kv}} d\mathbf{v} - \omega_0^2 \int \frac{d\omega'}{2\pi} \int \frac{d\mathbf{v} E_{\mathbf{k}}(\omega')}{\omega - \mathbf{kv}} \frac{\partial \Phi(\omega - \omega', \mathbf{v})}{\partial \mathbf{v}}. \tag{2.20}$$

From this we obtain the fundamental equation for the field in the form

$$\int \frac{d\omega'}{2\pi} \varepsilon_{\mathbf{k}}(\omega, \omega') \mathbf{E}_{\mathbf{k}}(\omega') = 4\pi eN \frac{\mathbf{k}}{k^2} \int \frac{g_{\mathbf{k}}(\mathbf{v})}{\omega - \mathbf{kv}} d\mathbf{v}, \tag{2.21}$$

$$\varepsilon_{\mathbf{k}}(\omega, \omega') = \frac{i}{\omega - \omega'} + \frac{\omega_0^2}{k^2} \int \frac{d\mathbf{v}}{\omega - \mathbf{kv}} \mathbf{k} \frac{\partial \Phi(\omega - \omega', \mathbf{v})}{\partial \mathbf{v}}. \tag{2.22}$$

Equation (2.21) differs from the linear equation for the field in that the dielectric constant $\epsilon_k(\omega)$ is replaced by an integral operator with a kernel $\epsilon_k(\omega, \omega')$, expressed in terms of the function $\Phi(\omega, \mathbf{v})$. If we replace $\Phi(\omega, \mathbf{v})$ by the zeroth approximation of the distribution function (the first term in Fig. 2b), i.e., if we put $\Phi(\omega, \mathbf{v}) = -if(\mathbf{v})/\omega$, then (2.21) goes over into the linear equation for the field.

To obtain a complete system of equations describing the evolution of the oscillations, we must also obtain an equation for the function $\Phi(\omega, \mathbf{v})$. This quantity has the meaning of a distribution function averaged over the spatial pulsations. We shall henceforth call it the background distribution function. From the form of the diagrams in Fig. 2b it follows that $\Phi(\omega, \mathbf{v})$ satisfies the equation

4)We replace $E_k(\omega)$ by (2.16) with complex ω_k then, as can be readily verified, we obtain in place of the secular multiplier t^r the factor $(\omega_k^0/\gamma_k)^r \gg 1$, so that the corresponding terms must again be regarded as large. The main difference between our results and those of Montgomery[17] is that these large terms are not summed in the perturbation theory developed in[17].

$$-i\omega\Phi(\omega, \mathbf{v}) = f(\mathbf{v}) + \frac{ie^2}{m^2}\sum_{\mathbf{q}}\int \mathbf{E}_{-\mathbf{q}}(\omega')\frac{\partial}{\partial \mathbf{v}}\frac{\mathbf{E}_{\mathbf{q}}(\omega'')}{\omega - \mathbf{qv} - \omega'} \times \frac{\partial\Phi(\omega - \omega' - \omega'', \mathbf{v})}{\partial \mathbf{v}}\frac{d\omega'}{2\pi}\frac{d\omega''}{2\pi}, \tag{2.23}$$

which is an analog of Dyson's equation in quantum field theory.

Equations (2.21) and (2.23) can be called generalized quasilinear equations [5]. Under certain supplementary assumptions they go over into the well known equations of the quasilinear theory for a weakly turbulent plasma [14,15]. As will be seen from what follows, the condition under which this takes place is that the width of the spectrum of the plasma oscillations be sufficiently large. In the other limiting case, when the spectrum is very narrow, for example when a "monochromatic" (more accurately—periodic) wave is excited in the plasma, Eq. (2.23), while retaining the same meaning as for a weakly turbulent plasma (account of the reaction of the wave on the distribution function), has entirely different properties and correspondingly different solutions.

We now consider in greater detail the manner in which (2.23) goes over into the quasilinear equation for a weakly-turbulent plasma. We substitute first in the right side $\mathbf{E}_q(\omega)$ in the form (2.16) (this means that we neglect the time dependence of the field amplitudes, $|\mathbf{E}_q(t)| = \mathbf{E}_q^0$), and integrate with respect to ω' and ω''. As a result we obtain

$$-i\omega\Phi(\omega, \mathbf{v}) = f(\mathbf{v}) + i\left(\frac{e}{m}\right)^2\sum_{\mathbf{q}}\frac{|\mathbf{E}_{\mathbf{q}}^0|^2}{\mathbf{q}^2}\mathbf{q}\frac{\partial}{\partial \mathbf{v}}\frac{1}{\omega + \omega_{\mathbf{q}} - \mathbf{qv}} \times \mathbf{q}\frac{\partial\Phi(\omega, \mathbf{v})}{\partial \mathbf{v}}. \tag{2.24}$$

The range of values of ω in which $\Phi(\omega, \mathbf{v})$ is large, is connected with the characteristic time τ of the variation of the function $\Phi(t, \mathbf{v})$ by the relation $|\omega| \sim \tau^{-1}$. We assume that the following condition is satisfied

$$\tau^{-1} \ll \overline{|\omega_{\mathbf{q}}^0 - \mathbf{qv}|}, \tag{2.25}$$

where the averaging is over $\mathbf{q}$. Then we can neglect ω in the denominator of the right side of (2.24); however, inasmuch as ω lies in the upper half-plane, we must put $\omega \to i0$. We can then go over directly to the t-representation in (2.24), and obtain the quasilinear equation for a weakly turbulent plasma:

$$\frac{\partial\Phi(t, \mathbf{v})}{\partial t} = i\frac{e^2}{m^2}\int\frac{d\mathbf{q}}{\mathbf{q}^2}|\mathbf{E}^0|^2\mathbf{q}\frac{\partial}{\partial \mathbf{v}}\frac{1}{\omega_{\mathbf{q}}^0 - \mathbf{qv} + i0} \times \mathbf{q}\frac{\partial\Phi(t, \mathbf{v})}{\partial \mathbf{v}} = \pi\frac{e^2}{m^2}\int\frac{d\mathbf{q}}{\mathbf{q}^2}|\mathbf{E}_{\mathbf{q}}^0|^2\mathbf{q}\frac{\partial}{\partial \mathbf{v}}\delta(\omega_{\mathbf{q}}^0 - \mathbf{qv})\mathbf{q}\frac{\partial\Phi(t, \mathbf{v})}{\partial \mathbf{v}}. \tag{2.26}$$

Let us explain the meaning of the condition (2.25). Inasmuch as the right side of (2.25) will be minimal when $\mathbf{v}$ lies in the resonant region of velocities, we put $v \sim \omega_q/q$. Considering, for simplicity, the case of Langmuir oscillations, for which $\omega_q \sim \omega_0$, and putting $v = \omega_0/q_0$, where q_0 is the average wave number, we obtain in place of (2.25)

$$\tau^{-1} \ll \omega_0\Delta q / q_0 \tag{2.27}$$

(Δq —width of the wave packet). The quantity τ has here the meaning of the characteristic relaxation time of the distribution function in the resonant region of velocities, and is determined by the condition

$$\tau \sim D/(\Delta v)^2, \quad D \sim e^2q^2\varphi^2/m^2\Delta qv, \quad \Delta v = \Delta(\omega_q/q) \sim \omega_0\Delta q/q^2, \tag{2.28}$$

where D is the coefficient of quasilinear diffusion in velocity space, and φ is the average potential of the electric field of the wave:

$$|\varphi|^2 = \int|\varphi_{\mathbf{q}}|^2 d\mathbf{q} \sim |\varphi_{\mathbf{q}_0}|^2\Delta q.$$

Substituting (2.28) in (2.27) we find that the condition under which (2.24) goes over into the quasilinear equation takes the form

$$\Delta v = \Delta(\omega_{\mathbf{q}}/q) \gg (e\varphi/m)^{1/2}, \tag{2.29}$$

which coincides with the condition for the applicability of the quasilinear equation, obtained by Vedenov et al. [14] from other considerations.

The physical meaning of (2.29) consists in the fact that the scatter of the phase velocities of the wave should be considerably larger than the velocity of the oscillations of the particles in the potential well of the wave field, with amplitude φ. On the other hand, (2.29) can be regarded as a criterion of plasma turbulence. Equation (2.26) contains no terms describing an "adiabatic" change of the distribution function in the nonresonant region. To obtain these terms, we turn to the integral in the right side of (2.23) and take into account effects connected with a weak change of the field ampli-

[5] We emphasize that when summing the diagrams that lead to (2.21) and (2.23) we have not used the approximation (2.16) for the field. The latter was needed only to disclose the "large" terms.

tude. Inasmuch as $E_q(\omega)$ becomes especially large when $\omega = \omega_q^0$, the main contribution to the integral in (2.23) is made by the regions in which $\omega'' \sim \omega_q^0$ and $\omega' \sim \omega_{-q}^0 = -\omega_q^0$. Therefore we represent the denominator of the integrand in the form $\omega - \omega' - \omega_q^0 + \omega_q^0 - \mathbf{q}\cdot\mathbf{v}$ and expand $(\omega - \mathbf{q}\cdot\mathbf{v} - \omega')^{-1}$ in powers of $\omega - \omega' - \omega_q^0$, confining ourselves to two terms of the expansion:

$$\frac{1}{\omega - \mathbf{qv} - \omega'} \approx \frac{1}{\omega_q{}^0 - \mathbf{qv} + i0} - \frac{\omega - \omega' - \omega_q{}^0}{(\omega_q{}^0 - \mathbf{qv} + i0)^2}. \tag{2.30}$$

Substituting this expression in (2.23) and using the convolution theorem (see note 3), we obtain in the t-representation

$$\frac{\partial \Phi(t, \mathbf{v})}{\partial t} = \pi\left(\frac{e}{m}\right)^2 \int \frac{d\mathbf{q}}{\mathbf{q}^2} |\mathbf{E}_\mathbf{q}(t)|^2 \mathbf{q}\frac{\partial}{\partial \mathbf{v}} \times \delta(\omega_q{}^0 - \mathbf{qv})\, \mathbf{q}\frac{\partial}{\partial \mathbf{v}}\Phi(t, \mathbf{v}) + \frac{1}{2}\left(\frac{e}{m}\right)^2 \int \frac{d\mathbf{q}}{\mathbf{q}^2}\mathbf{q}\frac{\partial}{\partial \mathbf{v}} \times \left\{\frac{1}{(\omega_q{}^0 - \mathbf{qv})^2}\left[\frac{d|\mathbf{E}_\mathbf{q}(t)|^2}{dt}\mathbf{q}\frac{\partial}{\partial \mathbf{v}}\Phi(t, \mathbf{v}) + 2|\mathbf{E}_\mathbf{q}(t)|^2 \times \mathbf{q}\frac{\partial}{\partial \mathbf{v}}\frac{\partial}{\partial t}\Phi(t, \mathbf{v})\right]\right\}, \tag{2.31}$$

where the second integral is taken in the sense of principal value. The term with $\partial\Phi/\partial t$ in the right side can obviously be neglected. Once this is done, (2.31) will coincide fully with the quasilinear equation, which takes into account the adiabatic variation of the distribution function in the nonresonant region. The conditions for the applicability of this equation are determined as before by (2.23).

We can show analogously (for more details see [19], pp. 14 and 15), that the equation for the field (2.21) goes over when condition (2.29) is satisfied, into the equation of the quasilinear theory for a weakly turbulent plasma:

$$\frac{d\mathbf{E}_\mathbf{k}(t)}{dt} = [-i\omega_\mathbf{k}{}^0 + \gamma_k(t)]\,\mathbf{E}_\mathbf{k}(t), \tag{2.32}$$

$$\gamma_\mathbf{k}(t) = \pi\frac{\omega_0{}^2}{\varepsilon_\mathbf{k}{}'k^2}\frac{\partial \Phi(t, v)}{\partial v}\Big|_{v=\omega_\mathbf{k}{}^0/k}, \tag{2.33}$$

where ω_k^0 is the real part of the frequency in the linear approximation.

3. QUASILINEAR THEORY OF "MONOCHROMATIC" WAVES

In this section we examine in detail the application of the generalized quasilinear equations (2.21) and (2.23) to an investigation of the evolution of a nonlinear "monochromatic" wave. The sum in (2.23) pertains now to a discrete set of wave vectors $\mathbf{q} = n\mathbf{k}$, $n = \pm 1, 2, \ldots$, where $2\pi/k$ is the wavelength, and the amplitudes of the multiple harmonics have a higher order of smallness compared with the amplitudes of the first harmonic, so that they can be neglected. Neglecting also the dependence of the wave amplitude on the time in (2.23)[6)], we arrive at Eq. (2.24), in which the sum consists of two terms corresponding to $\mathbf{q} = \pm\mathbf{k}$. Introducing a notation which is more convenient for this case

$$\alpha^2 = 2^{1/2}keE_k{}^0/m, \qquad u = v - \omega_k/k, \qquad x = ku/\alpha \tag{3.1}$$

(the quantity α is of the order of the frequency of the oscillation frequency of the particles captured by the potential well of the wave; α/k is accordingly of the order of the velocity of the captured particles), we obtain the fundamental equation in the form

$$\Phi(\omega, x) = \frac{i}{\omega}f(x) - \alpha^2\frac{\partial}{\partial x}\frac{1}{\omega^2 - \alpha^2x^2}\frac{\partial}{\partial x}\Phi(\omega, x). \tag{3.2}$$

To solve this equation we introduce the function

$$\Psi(\omega, x) = \frac{\alpha^2}{\omega^2 - \alpha^2x^2}\frac{\partial}{\partial x}\Phi(\omega, x). \tag{3.3}$$

Differentiating both halves of (3.2) with respect to x and substituting (3.3), we obtain

$$\frac{\partial^2}{\partial x^2}\Psi(\omega, x) + \left(\frac{\omega^2}{\alpha^2} - x^2\right)\Psi(\omega, x) = \frac{i}{\omega}\frac{df}{dx}. \tag{3.4}$$

The solution of (3.4) can be represented in the form of an expansion in normalized parabolic-cylinder functions $\psi_n(x)$:

$$\psi_n(x) = (2^n n!\,\pi^{1/2})^{-1/2} e^{-x^2/2} H_n(x), \qquad \int_{-\infty}^{\infty} \psi_n{}^2(x)\,dx = 1,$$

where $H_n(x)$ are Hermite polynomials. The functions $\psi_n(x)$ satisfy the equation

$$\frac{d^2\psi_n(x)}{dx^2} + (2n + 1 - x^2)\,\psi_n(x) = 0. \tag{3.5}$$

Putting

$$\frac{df}{dx} = \sum_{n=0}^{\infty}\beta_n\psi_n(x), \qquad \beta_n = \int_{-\infty}^{\infty}\psi_n(x)\frac{df}{dx}dx, \tag{3.6}$$

we get from (3.3) and (3.4)

$$\frac{\partial\Phi(\omega, x)}{\partial x} = i\frac{\omega^2 - \alpha^2x^2}{\omega}\sum_n\frac{\beta_n\psi_n(x)}{\omega^2 - (2n+1)\alpha^2} = \frac{i}{\omega}\frac{df}{dx} + \frac{i\alpha^2}{\omega}\sum\frac{2n + 1 - x^2}{\omega^2 - (2n+1)\alpha^2}\beta_n\psi_n(x). \tag{3.7}$$

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6)It will be shown below that this is valid if condition (3.28) is satisfied.

Going over to the t-representation, we obtain

$$\frac{\partial \Phi(t,x)}{\partial x} = \frac{df}{dx} - \sum_n \frac{2n+1-x^2}{2n+1} \beta_n \psi_n(x)[1-\cos \alpha t\sqrt{2n+1}]. \quad (3.8)$$

To calculate the distribution function $\Phi(t,x)$ we replace $(x^2-2n-1)\psi_n(x)$ in (3.8) by $d^2\psi_n(x)/dx^2$, after which the integration becomes elementary. Returning to the variable u, in accordance with (3.1), we obtain

$$\Phi(t,u) = f(u) + \frac{\alpha}{k}\sum \frac{\beta_n}{2n+1}\frac{d}{du}\psi_n\left(\frac{ku}{\alpha}\right) \times (1-\cos \alpha t\sqrt{2n+1}). \quad (3.9)$$

We see from (3.9) that $\Phi(t,u)$ differs little from its initial value $f(u)$ for $t \ll \alpha^{-1} \sim (keE_0/m)^{-1/2}$, where E_0 is the amplitude of the wave. When $t > \alpha^{-1}$, the second term in (3.9) oscillates with variation of t, and decreases rapidly when

$$u = v - \omega_k/k \gg \alpha/k \sim (eE_0/km)^{1/2},$$

i.e., the quasilinear distribution function is distorted only near the resonant velocity $v = \omega_k/k$ in the velocity interval of the order of the velocity of the oscillation of the particles "captured" by the wave. The frequency of the oscillations of the distribution function near the phase velocity, as can be seen from (3.9), is of the order of α, which coincides with the oscillation frequency of the particles in the potential well of the wave.

Substituting now (3.7) in expression (2.22) for $\epsilon_k(\omega,\omega')$, we obtain

$$\varepsilon_k(\omega,\omega') = \frac{i}{\omega-\omega'}\Bigg[1 + \frac{\omega_0^2}{k^2}\int\frac{dv}{\omega-kv}k\frac{df}{dv} + \frac{\omega_0^2}{\alpha}\sum_n \frac{\beta_n}{(\omega-\omega')^2-(2n+1)\alpha^2} \times \int du \frac{(2n+1)\alpha^2-k^2u^2}{\omega-\omega_k-ku}\psi_n\left(\frac{ku}{\alpha}\right)\Bigg]. \quad (3.10)$$

Using (3.10) and (2.21) and carrying out elementary transformations, we get

$$E_k(\omega) = \frac{4\pi eN}{k\varepsilon_k(\omega)}\int\frac{g_k(v)}{\omega-kv}dv - \frac{i\omega_0^2}{2\pi\alpha\varepsilon_k(\omega)}\sum_n \beta_n \times \int du\psi_n\left(\frac{ku}{\alpha}\right)\frac{(2n+1)\alpha^2-k^2u^2}{\omega-\omega_k-ku} \times \int \frac{d\omega' E_k(\omega')}{(\omega-\omega')[(\omega-\omega')^2-(2n+1)\alpha^2]}, \quad (3.11)$$

where $\epsilon_k(\omega)$ is determined by formula (2.11). Multiplying both halves of (3.11) by $i(\omega-\omega_k)$ (ω_k—oscillation frequency in the linear approximation, see (2.14)), and going over to the t-representation, we get

$$\frac{dE_k(t)}{dt} = -i\omega_k E_k(t) - \frac{\omega_0^2}{\varepsilon_k'\alpha^3}\sum_n\frac{\beta_n}{2n+1}\int du\psi_n\left(\frac{ku}{\alpha}\right) \times [(2n+1)\alpha^2-k^2u^2]\int_0^t dt'(1-\cos\alpha t'\sqrt{2n+1}) \times E_k(t')\exp[i(ku+\omega_k)(t'-t)]. \quad (3.12)$$

In the derivation of (3.12) we have neglected all the singularities of the right side of (3.11), which are located below the point ω_k, since they make an exponentially small contribution for sufficiently large t. In addition, we have left out the term $\int dv g_k(v)\exp(-ikvt)$, which decreases rapidly with increasing t, owing to the factor $\exp(-ikvt)$, which oscillates at large values of t. The characteristic time of this decrease is $(kv_g)^{-1}$, where v_g is the effective "width" of the function $g_k(v)$. We assume that this time is small compared with the other characteristic times, which determine the variation of the field amplitude. Using (3.5) and the relation (see [20])

$$\int_{-\infty}^{\infty}\psi_n(y)e^{iyz}dy = (2\pi)^{1/2}i^n\psi_n(z), \quad (3.13)$$

we carry out the integration with respect to u in the second term of the right side of (3.12). As a result we get

$$\frac{dE_k(t)}{dt} = -i\omega_k E_k(t) - (2\pi)^{1/2}\frac{\omega_0^2\alpha^2}{\varepsilon_k' k}\sum_n\frac{i^n\beta_n}{2n+1} \times \int_0^t dt'(t'-t)^2\psi_n[\alpha(t'-t)] \times (1-\cos\alpha t'\sqrt{2n+1})E_k(t')\exp[i\omega_k(t'-t)]. \quad (3.14)$$

The integrand in (3.14) contains a product of the expression

$$\psi_n[\alpha(t'-t)](1-\cos\alpha t'\sqrt{2n+1})$$

which changes rapidly with variation of t′ (the characteristic time of the variation is $\tau \lesssim \alpha^{-1}$) and the quantity $E_k(t')\exp(i\omega_k t')$, which obviously is equal to the slowly varying amplitude of the field $E_k^0(t')$. Neglecting the variation of $E_k^0(t')$ during the time α^{-1}, we can move $E_k^0(t')$ outside the integral sign. Then we obtain after simple transformations

$$\frac{dE_k(t)}{dt} = [-i\omega_k + \theta_k(t)]E_k(t), \qquad (3.15)$$

$$\theta_k(t) = -(2\pi)^{1/2}\frac{\omega_0^2}{\varepsilon_k' k\alpha}\sum_n(-1)^n\int_0^{\alpha t} d\tau\{\beta_{2n}[\psi_{2n}(\tau) - \psi_{2n}(0)\cos\tau\sqrt{4n+1}] - i\beta_{2n+1}[\psi_{2n+1}(\tau) - (4n+3)^{-1/2}\psi'_{2n+1}(0)\sin\tau\sqrt{4n+3}]\}. \qquad (3.16)$$

Integrating the equation, we get

$$E_k(t) = E_k(0)\exp\left\{-it\left[\omega_k^0 + i\gamma_k + it^{-1}\int_0 dt'\theta_k(t')\right]\right\}. \qquad (3.17)$$

From (3.17) we see that $t^{-1}\,\mathrm{Re}\int_0^t dt'\theta_k(t')$ is the variation of the increment, and $t^{-1}\,\mathrm{Im}\int_0^t dt'\theta_k(t')$ is the real part of the frequency, due to the distortion of the distribution function by the reaction of the wave. By examining the series (3.16), we can easily verify that it converges quite rapidly, so that the order of magnitude of $\theta_k(t)$ does not exceed the order of magnitude of the first term of the series. Calculating the values of β_n from (3.6) under the assumption that $f(v)$ is a Maxwellian distribution, we can easily obtain

$$t^{-1}\int_0^t \theta_k(t')dt' \lesssim \gamma_k, \qquad (3.18)$$

where γ_k is the increment of the linear theory. Inasmuch as $\gamma_k \ll \omega_k^0$, the nonlinear correction to the real part of the frequency can be neglected. On the other hand, the correction to the linear increment is quite appreciable. We shall therefore examine in greater detail the quantity

$$\Gamma_k(t) = \gamma_k + t^{-1}\,\mathrm{Re}\int_0^t \theta_k(t')dt', \qquad (3.19)$$

which has the meaning of a time-dependent increment.

We investigate first $\Gamma_k(t)$ for small t ($\alpha t \ll 1$). We make use of the fact that parabolic-cylinder functions of sufficiently small arguments and high orders have the following asymptotic representation[20]:

$$\psi_{2n}(z) = \psi_{2n}(0)\{\cos z\sqrt{4n+1} + O[z^{5/2}(4n+1)^{-1/4}]\}. \qquad (3.20)$$

Taking (3.20) into account, we obtain

$$t^{-1}\int_0^t dt'\int_0^{\alpha t'} d\tau[\psi_{2n}(\tau) - \psi_{2n}(0)\cos\tau\sqrt{4n+1}] < \psi_{2n}(0)(\alpha t)^{7/2}(4n+1)^{-1/4}, \qquad (3.21)$$

from which we see that when $\alpha t \ll 1$ the difference between $\Gamma_k(t)$ and the linear increment is a small quantity of the order of $(\alpha t)^{7/2}$. We note that when $\alpha t \sim 1$ the left side of (3.21) decreases quite rapidly with increasing n, so that when $t \sim \alpha^{-1}$ we can confine ourselves in the expression (3.19) for $\Gamma_k(t)$ to the first few terms of the series. Inasmuch as the latter are of the same order as the linear increment γ_k, the difference between $\Gamma_k(t)$ and γ_k becomes appreciable when

$$t \sim \alpha^{-1} \sim (keE_k^0/m)^{1/2}.$$

We now investigate $\Gamma_k(t)$ with $t \gtrsim \alpha^{-1}$. We rewrite formula (3.19) in the form

$$\Gamma_k(t) = \gamma_k - (2\pi)^{1/2}t^{-1}\frac{\omega_0^2}{\varepsilon_k' k\alpha}\int_0^t dt'\left\{\sum_n(-1)^n\beta_{2n}\int_0 d\tau\psi_{2n}(\tau) - \sum_n(-1)^n\beta_{2n}\int_{\alpha t'} d\tau\psi_{2n}(\tau)\right\} + (2\pi)^{1/2}t^{-1}\frac{\omega_0^2}{\varepsilon_k' k\alpha^2} \times \sum_n\frac{(-1)^n\beta_{2n}}{4n+1}\psi_{2n}(0)(1-\cos\alpha t\sqrt{4n+1}). \qquad (3.22)$$

Using (3.13) and (3.6), we can write for the first term in the curly brackets

$$\sum_n(-1)^n\beta_{2n}\int_0^\infty\psi_{2n}(\tau)d\tau = \left(\frac{\pi}{2}\right)^{1/2}\sum\beta_{2n}\psi_{2n}(0) = \left(\frac{\pi}{2}\right)^{1/2}\frac{k}{\alpha}\frac{df}{dv}\bigg|_{v=\omega_k/k}. \qquad (3.23)$$

The substitution of (3.23) in (3.22) leads to exact concellation of the first term of γ_k in (3.22). The second term in the curly brackets of (3.22) takes the form

$$\sum_n(-1)^n\beta_{2n}\int_{\alpha t'}^\infty d\tau\psi_{2n}(\tau) = (2\pi)^{-1/2}\sum\beta_{2n}\int_{\alpha t'}^\infty d\tau\int_{-\infty}^\infty dx\psi_{2n}(x)e^{itx} = \alpha(2\pi)^{-1/2}\int_{t'}^\infty dt''\int_{-\infty}^\infty du\frac{df}{du}\cos(kut''). \qquad (3.24)$$

This expression vanishes when $t \to \infty$ (taking into account the fact that $u = z - \omega_k/k$ and $f(v) \propto \exp(-v^2/v_t^2)$, it is easy to verify that the term (3.24) begins to decrease rapidly when $t > (kv_t)^{-1}$). Thus, for sufficiently large t, the value of $\Gamma_k(t)$ becomes

$$\Gamma_k(t) = \frac{2(2\pi)^{1/2}\omega_0^2}{\varepsilon_k' k\alpha^2 t}\sum_n\frac{(-1)^n\beta_{2n}}{4n+1}\psi_{2n}(0)\sin^2\alpha t\sqrt{n+1/4}. \qquad (3.25)$$

Consequently, at times that are short compared with the period of particle oscillation in the potential well of the wave ($\alpha t \ll 1$), the nonlinear increment $\Gamma_k(t)$ is close to the linear increment γ_k. $\Gamma_k(t)$ attenuates in oscillatory fashion like t^{-1} at large values of t. However, the amplitude of the wave, given by the expression

$$E_k^0(t) = E_k^0(0) \exp[t\Gamma_k(t)], \qquad (3.26)$$

does not tend to a stationary value as $t \to \infty$, since the oscillations of $t\Gamma_k(t)$ are not damped. The characteristic period of these oscillations is of the same order of magnitude as α^{-1}—the period of oscillations of the particles in the potential well of the wave. The amplitude of the oscillations of $t\Gamma_k(t)$ is of the order of

$$\overline{t\Gamma_k(t)} = \frac{(2\pi)^{1/2}\omega_0^2}{\varepsilon_k' k\alpha^2} \sum (-1)^n \frac{\beta_{2n}\psi_{2n}(0)}{4n+1} \sim \frac{\gamma_k}{\alpha}. \qquad (3.27)$$

We have continuously assumed above that the field amplitude changes quite little during the time of the nonlinear evolution of the wave. It follows from (3.26) and (3.27) that this takes place under the condition

$$\gamma_k / \alpha \leqslant 1. \qquad (3.28)$$

Under this condition, the exponential in (3.26) can be expanded in a series, and the average amplitude of the wave is equal to

$$\overline{E_k^0(t)} \approx E_k^0(0)\left[1 + \frac{(2\pi)^{1/2}\omega_0^2}{\varepsilon_k' k\alpha} \sum_n \frac{(-1)^n\beta_{2n}}{4n+1}\psi_{2n}(0)\right]. \qquad (3.29)$$

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