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SELF-SIMILAR MOTION OF RAREFIED PLASMA

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A nonlinear kinetic equation is obtained to describe the self-similar motion of an electron-ion plasma in the absence of collisions. The problem of the expansion of a plasma into a vacuum is solved. The density and the velocity distribution of the ions is obtained. It is shown that in the course of filling the rarefied half-space a part of the ions is accelerated by the action of the resulting electric field up to velocities of the order of the thermal velocity of the electrons. At the same time the effective temperature of the ions drops sharply; it turns out to be many times smaller than the electron temperature (in the case of identical temperatures in the initial plasma). Results of a numerical calculation are presented.

AN important place in the hydrodynamics of a compressible gas is occupied by self-similar problems which do not contain any characteristic dimensions in the initial and final conditions. The time t and the coordinate x can appear in the solution of such a problem only in the combination x/t. This enables one to simplify the equations considerably and in the one dimensional case to obtain analytic solutions [1]. Self-similar solutions in hydrodynamics describe a large class of physically interesting problems (the expansion of a gas into a vacuum, a point explosion, the decay of a discontinuity in the initial conditions, etc.). It is of interest to carry out an analogous investigation in the dynamics of a rarefied plasma which is described by a collisionless kinetic equation with a self consistent field. One should keep in mind that the system of equations describing a plasma is so complicated that it is difficult to exhibit a nonstationary nonlinear problem which would have a clear physical meaning and the solution of which could be carried through to the end. One can, therefore, presume that the investigation of self-similar problems with a sharp physical formulation in addition to being of interest in its own right will also turn out to be useful in aiding the understanding of the situation in more complicated cases.

In the present paper we investigate the problem of the expansion of a plasma into a vacuum. We assume that the plasma at the initial time occupies the half space $x < 0$ and at time $t = 0$ begins to expand into a vacuum. The plasma is described by the kinetic equation for the distribution function for the ions

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} v - \frac{\partial f}{\partial v} \frac{e}{M} \frac{\partial \varphi}{\partial x} = 0 \tag{1}$$

(M is the mass of the ion, φ is the electric potential) which is analogous to the equation for the electron distribution function, and by the Poisson equation:

$$\frac{\partial^2 \varphi}{\partial x^2} = -4\pi e (N - N_e), \qquad N = N(x, t) = \int_{-\infty}^{\infty} f dv, \tag{2}$$

where N is the ion density, N_e is the electron density.

In the hydrodynamics of an ideal fluid such a problem is a strictly self-similar one. On our case this is, generally speaking, not so, since in equations (1)—(2) themselves there exists a parameter of the dimension of length—the Debye radius $D = (T/4\pi Ne^2)^{1/2}$, where t is the temperature of the plasma in energy units (for the sake of simplicity the temperature of the ions and of the electrons is assumed to be the same at the initial instant). However, it can be easily seen that with the passage of time the motion of the plasma rapidly approaches a self-similar motion. In order to verify this, we consider the successive stages of the process. At the initial instant during a time of the order $t_1 \sim D(m/2T)^{1/2}$ the electrons on the average will separate from the ions by a distance $\sim D$, so that at the boundary there will be formed a double layer of thickness [1)] D. The electrons

[1)]Numerical calculations for the initial stages of the expansion are given in the paper by Stocker[2].

cannot go further due to the electric field. Therefore, subsequently a relatively slow expansion of the plasma will begin with a velocity of the order of the mean thermal velocity of the ions.

During a time $t_2 \gg D\sqrt{M/2T} \gg t_1$ the boundary between the plasma and the vacuum will be smeared over a distance $\sim t_2\sqrt{2T/M} \gg D$. The characteristic size of the inhomogeneity will become much larger than the Debye radius. At the same time the plasma is quasineutral, so that Eq. (2) reduces simply to the equation

$$N = N_e. \tag{3}$$

Since during this time the motion of the plasma occurs with the velocity of the ions, i.e., relatively slowly, the electrons will, to a high degree of accuracy, have a Boltzmann distribution ($N_e = N_0 e^{e\varphi/T}$) Therefore, it follows from (3) that

$$e\varphi = T \ln (N / N_0). \tag{4}$$

Substituting this expression into (1) we finally obtain

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} v - \frac{\partial f}{\partial v} \frac{T}{M} \frac{\partial}{\partial x} \left(\ln \int_{-\infty} f \, dv \right) = 0. \tag{5}$$

Equation (5) no longer contains any parameters of the dimension of length, and, therefore, beginning with a time $\sim t_2$ the motion is self-similar, so that

$$f = f(x / t, v).$$

We introduce the dimensionless self-similar variable

$$\tau = \sqrt{M / 2T} x / t$$

and the dimensionless quantities g and u:

$$g = (2\pi T / M)^{1/2} f / N_0, \quad u = v\sqrt{M / 2T}. \tag{6}$$

After this the desired equation for the dimensionless ion distribution function $g(\tau, u)$ will assume the form

$$(u - \tau) \frac{\partial g}{\partial \tau} - \frac{1}{2} \frac{\partial g}{\partial u} \frac{d}{d\tau} \left(\ln \int_{-\infty}^{\infty} g \, du \right) = 0. \tag{7}$$

(This equation was first obtained in [9].)

For $x \to -\infty$ the plasma is not perturbed, while for $x \to +\infty$ there is no plasma present. Therefore, assuming that the ion distribution in the unperturbed plasma is Maxwellian, we write the boundary conditions for (7) in the form

$$\tau \to -\infty, \quad g \to e^{-u^2}, \quad \tau \to +\infty, \quad g \to 0. \tag{8}$$

We investigate first of all the asymptotic properties of the solution of (7) for $\tau \to +\infty$. Large τ correspond to large distances x. The ions having traversed such a large distance have been strongly accelerated by the electric field and their own thermal motion is of little significance. Therefore, neglecting the thermal motion of the ions we seek the solution of (5) for large values of x/t in the form

$$f = N(x, t) \delta[v - \bar{v}(x, t)],$$

where the ion density N and their directed velocity $\bar{v}$ are determined by the equations

$$\frac{\partial N}{\partial t} + \frac{\partial}{\partial x} (N\bar{v}) = 0, \tag{9}$$

$$M \left[\frac{\partial \bar{v}}{\partial t} + \bar{v} \frac{\partial \bar{v}}{\partial x} \right] = -e \frac{\partial \varphi}{\partial x} = -T \frac{\partial \ln N}{\partial x}. \tag{10}$$

These equations, as they must when the thermal motion of the ions is neglected, do formally coincide with the equations of the hydrodynamics for an isothermal gas (cf., [3], p. 109). In going over in these equations to dimensionless variables we obtain[2)]

$$(\bar{u} - \tau) \frac{d\bar{u}}{d\tau} + \frac{1}{2} \frac{d \ln N}{d\tau} = 0,$$

$$(\bar{u} - \tau) \frac{d \ln N}{d\tau} + \frac{d\bar{u}}{d\tau} = 0, \tag{11}$$

from which it follows that

$$(\bar{u} - \tau)^2 = 1/2, \quad \bar{u} = \tau + 1/\sqrt{2}. \tag{12}$$

The choice of the root in (12) is determined by the fact that, as will be shown below, we have $\bar{u} > \tau$ for all the ions. Substituting into the second equation of (11) we obtain

$$N / N_0 = C \exp(-\tau\sqrt{2}), \quad \tau \to +\infty. \tag{13}$$

The constant C is determined by joining the hydrodynamic equation (13) smoothly with the exact solution in the region $\tau \gg 1$.[3)]

[2)]Naturally Eqs. (10) and (11), and formulas (12) and (13), which follow from them, are known not to hold for very large values of $\tau \gg \sqrt{M/m}$ when the ratio x/t becomes equal to the mean thermal velocity of the electrons. Under these conditions we can no longer consider the electron distribution to be a Boltzmann distribution. We also note that since for large values of τ the particle density diminishes and the Debye radius grows exponentially, D may become greater than the characteristic distance over which N changes. However, this does not lead to a violation of the quasineutrality of the plasma and of the self-similar character of the motion. The point is that for large values of τ, as can be seen from (4) and (13), φ is a linear function of x. Therefore, the principal term in the asymptotic expansion of $\partial^2\varphi / \partial x^2$ disappears, and this guarantees the preservation of quasineutrality.

[3)]If the initial ion temperature were zero, the hydrodynamic equations (11) would be applicable for all values of τ. In this case $N / N_0 = 1$ for $\tau < -1/\sqrt{2}$ and $N / N_0 = \exp[-\sqrt{2}(\tau + 1/\sqrt{2})]$ for $\tau > -1/\sqrt{2}$.

It should be noted that in order for the asymptotic behavior (12)—(13) to be valid it is necessary that at the initial instant the Maxwellian distribution would hold not only for the great majority of the electrons, but also for fast electrons with energies of the order $T\tau$, since only such electrons can penetrate into the region of large values of τ. Therefore, any distortion of the "tail" of the Maxwellian distribution of the electrons can lead to a change in the asymptotic behavior of N. This fact must be taken into account in comparing theoretical results with experimental ones.

We now investigate the behavior of the characteristics of Eq. (7), i.e., of curves along which g has a constant value in the plane of τ and u. The equation of the characteristics has the form

$$\frac{du}{d\tau}=\frac{1}{2}\frac{F(\tau)}{u-\tau}, \tag{14}$$

where F is a dimensionless force:

$$F=-\frac{d}{d\tau}\ln\left(\int_{-\infty}^{\infty} g\,du\right). \tag{15}$$

The intensity of the field in the plasma E is expressed in terms of F by the formula

$$E=F\frac{1}{e}\sqrt{\frac{MT}{2}}\frac{1}{t}$$

for $\tau \to -\infty$, $F \to 0$. From (13) it can be seen that for $\tau \to +\infty$ the quantity $F \to \sqrt{2} = F_\infty$.

A more detailed investigation shows that everywhere $F > 0$. This means that the value of $du/d\tau$ along all the characteristics which begin in the region of the unperturbed plasma ($\tau \to -\infty$) is greater than zero, i.e., the velocity u along a characteristic increases monotonically with increasing τ. Moreover, none of such characteristics can intersect the straight line $u = \tau$. Indeed, near the point $\tau = \tau_0$ at which $u = \tau$, the solution of (14) has the form

$$(u-\tau)^2=F(\tau_0)(\tau-\tau_0). \tag{16}$$

Both branches of the curve (16) are directed towards large values of τ. Therefore, taking into account the fact that $u(\tau)$ varies monotonically along the characteristics we conclude that the curve intersecting the straight line $u = \tau$ cannot belong to the family of characteristics emerging from the region $\tau = -\infty$.

The investigation carried out above shows that in our problem the distribution function is equal to zero for $u < \tau$. Indeed, for large negative values of τ the distribution function is Maxwellian and there are no particles with $u < \tau$ due to the exponential decrease of this function for $u \to \pm\infty$. But subsequently the characteristics will not intersect the straight line $u = \tau$ so that no particles with $u < \tau$ will appear. For $\tau \to \infty$ all the characteristics must in accordance with (12) bunch near the straight line $u = \tau + 1/\sqrt{2}$. The nature of this bunching can be determined directly from (14). Indeed, for $\tau \to \infty$, $F \to \sqrt{2}$. The solution of (14) for $F = \sqrt{2}$ has the form

$$\tau=u-1/\sqrt{2}+A\exp(-u\sqrt{2}),$$

or for $\tau \to +\infty$

$$u-\tau-1/\sqrt{2}\approx A'\exp(-\tau\sqrt{2}). \tag{17}$$

We see that the bunching of the characteristics occurs very rapidly (exponentially). Correspondingly, the distribution function will just as rapidly degenerate into a δ-function, which confirms the assumption made in the derivation of (9) and (10).

The properties of Eq. (7) noted above were utilized to integrate it numerically with an electronic computer. The solution proceeded in steps with respect to τ starting with the negative value $\tau_0 = -3$. The value of the function $g(\tau, u)$ for $\tau = \tau_k + \Delta\tau$ was evaluated in terms of the value of $g(\tau_k, u)$ by means of the formula

$$g(\tau_k+\Delta\tau, u)=g\left(\tau_k,\ u-\frac{1}{2}\frac{F(\tau_k)}{u-\tau}\Delta\tau\right),$$

which is an obvious consequence of the equation for the characteristics (14). At the same time we have

$$F(\tau_k)=-\left[\ln\int_{\tau_k}^{\infty} g(\tau_k, u)\,du-\ln\int_{\tau_k-\Delta\tau}^{\infty} g(\tau_k-\Delta_z, u)\,du\right]\Big/\Delta_z.$$

Over the first two steps the function g was assumed to be Maxwellian for $u > \tau$.

The results of this calculation are shown in the diagrams. In Fig. 1 are plotted the values of the ion density (curve 1).

$$\frac{N}{N_0}=\frac{1}{\sqrt{\pi}}\int_{\tau}^{\infty} g(\tau, u)\,du.$$

For comparison the dotted curve shows

$$\frac{1}{\sqrt{\pi}}\int_{\tau}^{\infty} e^{-u^2}\,du$$

which describes collisionless expansion into a vacuum of a neutral gas. We see that the effect of the electric field for $\tau \lesssim 2$ is not very great. On the contrary, for large values of τ the ion density increases greatly under the influence of the field.

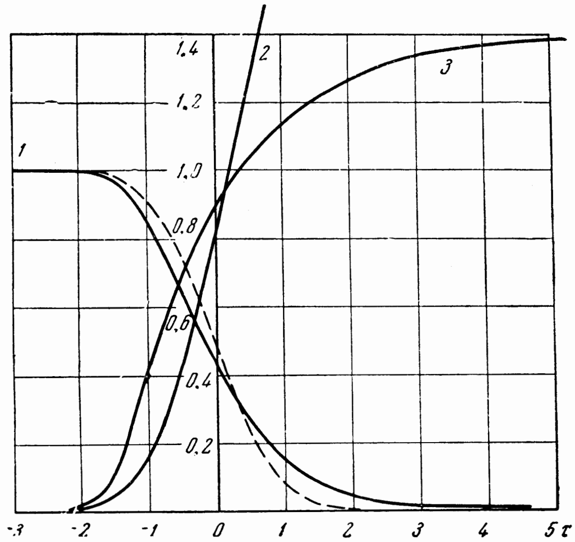

FIG. 1

Comparing the value of the ratio N/N_0 for large values of τ with (13) we find in this asymptotic formula the coefficient $C = 0.70$. Curves 2 and 3 in the same diagram show the variation of the potential $e\varphi/T$ and of the force $F = -d\ln(N/N_0)/d\tau$. For large values of τ the force F approaches the constant $F_\infty = \sqrt{2}$ in agreement with (13). This means that the intensity of the electric field diminishes with increasing t for large values of τ in accordance with $1/t$.

Figure 2 shows the distribution function $g(u)$ for different values of τ (these values are shown by numbers near the corresponding curves). We note that the distribution function for an expanding neutral gas is equal to the Maxwellian value e^{-u^2} for $u > \tau$ and is equal to zero for $u < \tau$. Taking into account the fact that the distribution function shown in Fig. 2 for $\tau = -2$ is in fact close to e^{-u^2}, we can compare the distribution functions for the ions and for neutral particles. It can be seen that for $\tau \le 0$ these functions are sufficiently close to one another. For large positive values of τ the difference between them is, on the contrary, exceedingly great. Consequently, the electric field produces a decisive effect on the distribution of the ions for $\tau \gtrsim 1$. In Fig. 2 one can clearly see the gradual conversion of the Maxwellian distribution function into a δ-like one. We also note that the maximum value of the distribution function for any value of τ is equal to unity. This is clear in advance since, in virtue of the previously established properties of the characteristic curves of our equation, a characteristic along which the distribution function is equal to unity exists in the u, τ plane for any value of τ.

We now consider the fundamental qualitative

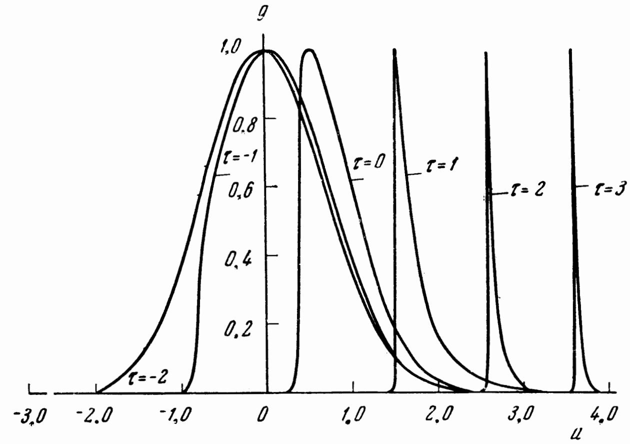

FIG. 2

special features of the solution obtained above. First of all, as can be seen from Fig. 2, the electric field strongly accelerates a portion of the ions which fill the rarefied region. In particular, it follows from formula (12) that for large values of τ the mean energy of the ions is equal to

$$\varepsilon = M\overline{v^2}/2 = T(\tau^2 + \tau\sqrt{2} + 1/2) \approx T\tau^2. \qquad (18)$$

Taking into account the restrictions on the value of τ mentioned in the second footnote we see that the energy of the accelerated ions can exceed by several orders of magnitude the initial thermal energy. The effect of acceleration can be even greater in a plasma in which the electrons and the ions have different temperatures for $T_e/T_i > 1$.

Further, from Fig. 2 it can be seen that as τ increases the thermal velocity spread of the ions diminishes rapidly. Utilizing Eq. (7) and expressions (12) and (13) for $N(\tau)$ and $\bar{u}(\tau)$, it can be easily shown that the quantity $T_{\text{eff}\,i} = 2T(\overline{u^2} - \bar{u}^2)$ which has the meaning of an effective ion temperature varies with increasing τ in accordance with[4)]

$$T_{\text{eff}\ i} \sim T\exp(-2\tau\sqrt{2}).$$

This result can also be obtained directly by taking into account the constancy noted above of the maximum value of the distribution function. Indeed, for a constant height the width of the curve $g(u)$ must vary proportionally to the density $N(\tau)$. The last formula expresses just this relationship. The same result also follows from formula (17) which describes the asymptotic bunching of the

[4)]This decrease in the temperature refers, naturally, only to the thermal motion of the ions along the x axis. Because of the absence of collisions the temperature corresponding to the motion of the ions in the perpendicular direction is not altered in the expansion process.

characteristics of the equation. But the temperature of the electrons does not change (it is just because of this that (9) and (10) coincide with the equations of isothermal hydrodynamics). Consequently, for large values of τ the electron temperature is several times (even by several orders of magnitude) greater than the effective ion temperature. As is well known, the absorption of ion waves in such a plasma characterized by different electron and ion temperatures is small (cf. [3], Sec. 14). This reduces the stability of the plasma under such conditions.

In order to investigate the stability of the self-similar solution $f_a(x/t, v)$ constructed above we must investigate small deviations from this solution by setting $f = f_a + f'$ and linearizing the equation in f'. Naturally, in this discussion the perturbation f' can contain x and t in an arbitrary fashion, and not only in the combination x/t. However, it turns that by means of the substitution $s = \ln t$ one can make the coefficients of the equation for f' independent of s (or t). This means that f' can be represented in the form of linear combinations of solutions of the form

$$f' = f'(u, \tau) e^{-iqs} = f'(u, \tau) e^{-iq \ln t}, \qquad (19)$$

where the function $f'(u, \tau)$ satisfies the equation

$$iqf' + (\tau - u)\frac{\partial f'}{\partial \tau} + \frac{F_a(\tau)}{2}\frac{\partial f'}{\partial u} + \frac{1}{2}\frac{\partial f_a}{\partial u}\frac{d}{d\tau}\left\{\int_{-\infty}^{\infty} f' \, du / N_a(\tau)\right\} = 0. \qquad (20)$$

Here $F_a(\tau)$ and $N_a(\tau)$ are respectively the force and the ion density in the self-similar solution.

The investigation of stability reduces to the determination of the spectrum of the eigenvalues of q. Analysis shows that within the framework of the quasiclassical approximation the dispersion equation in the case under consideration does not have any roots in the upper half-plane. In other words, with respect to perturbations whose wavelength is small compared to the characteristic size of an inhomogeneity the solution constructed above is stable. However, the damping of perturbations in this case is very small for large values of τ. But the investigation carried out above does not exhaust all possible types of instability. We note in connection with this that for large values of τ the velocity distribution of the ions turns out to be sharply anisotropic (cf. footnote [4]). Under such conditions an instability arises [4] in a homogeneous plasma on taking into account the magnetic field of the wave itself. Investigating oscillations of short wavelengths it is also necessary to take into account the term with $\partial^2\varphi/\partial x^2$ in the Poisson equation, particularly since neglect of the leading derivative in the equation always requires careful attention (cf., for example, reference [5]).

There exist experimental papers in which the expansion of a plasma into a vacuum has been observed [6-8]. The published results are, however, insufficient in order to carry out a detailed comparison of theory with experiment. We merely note that experimentally a large number of ions with velocities exceeding thermal velocities is observed.

In conclusion we note that the problem under consideration has a direct relation to the problem of the flow of a rarefied plasma past a rapidly moving object [9]. The curves of Fig. 1 yield directly for this case the distributions of particles near the trailing edge of the object (at distances large compared to D). In this case we must interpret τ to mean $\tau = (MV_0^2/2T)^{1/2} x/|z|$, where V_0 is the velocity of the object (it is assumed that $MV_0^2/2T \gg 1$), z is the coordinate in the direction of motion of the object ($z < 0$), x is the coordinate in the direction of the normal to the edge of the object. The results are applicable in the region $x \ll R_0$, $|z|(2T/MV_0^2)^{1/2} \ll R_0$, R_0 is the characteristic transverse dimension of the object.

It is important to note that in the flow past a finite object the perturbed region turns out to be unstable. Indeed, as can be seen from what has been said above, monochromatic streams of ions leave the edges of the object. Colliding near the axis of the object they give rise to the "bunching" instability of a plasma.

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