TWO-PHOTON EMISSION IN ELECTRON COLLISIONS

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The c.m.s. cross section for double bremsstrahlung emitted as a result of collision of two high-energy electrons, integrated over all final states except the photon frequencies, is calculated under the assumption that one of the emitted photons is soft and the other has arbitrary energy.

In experiments on colliding beams there is considerable interest in the process of double bremsstrahlung (i.e., in the process in which the collision of two electrons or an electron and a positron results in the emission of two photons). This process can be used as a monitor to register collisions in the beams. Clearly the simplest experimental arrangement would be one in which one detects quanta emitted in two opposite directions. In such an experimental arrangement the double bremsstrahlung process competes with the process of two quantum annihilation (in the case of electron-positron collisions). For the case of high-energy electrons and for a sufficiently wide frequency range of the detected photons, the cross section of this process may exceed the cross section for two-photon annihilation. This is connected with the fact that the double bremsstrahlung cross section, as opposed to the two-photon annihilation cross section, does not decrease with increasing energy of the colliding particles.

Use of double bremsstrahlung as a monitor technique requires rather accurate knowledge of the theoretical formulae for the cross section of this process. Since photon emission occurs primarily at angles of the order of $1/\gamma$, and since for large energies this angle is considerably smaller than the angular size of the counters, it is clear that what is of interest is the cross section integrated over the emission angles of the photons. Since the electron is not registered, in the final state, it is also necessary to integrate over the final states of the electrons. The quantity obtained in this way is the differential cross section $d\sigma_{\omega_1\omega_2}$ (with respect to the frequencies of both photons) for double bremsstrahlung, where ω_1 is the frequency of the photon emitted in direction 1 and ω_2 is the frequency of the quantum emitted in direction 2, opposite to direction 1.

Calculation of this cross section for the case of the emission of classical quanta ($\omega_1, \omega_2 \ll \epsilon$) was carried out in an earlier paper [1]. In the present paper we calculate the cross section for double bremsstrahlung in the c.m.s. of the colliding particles, under the assumption that one of the photons is soft but that the other can have arbitrary energy. It is assumed that the energy of the electrons is very large so that one may make an expansion in inverse powers of γ. We calculate the dominant term of this expansion and make estimates of the correction terms. The expressions obtained in this way are useful for calculating the cross-sections for various processes involving photons. As an example we calculate to logarithmic accuracy the cross section for single bremsstrahlung during electron-electron or electron-positron collisions.

2. The problem as stated above determines the choice of diagrams. Since small scattering angles are important and since each electron emits into a narrow cone having an angle of the order $1/\gamma$, the most important contribution to the cross-section is made by diagrams describing the emission of quanta by different particles. Out of a total of 40 diagrams, only 16 belong to the above class. Of these, 8 diagrams govern scattering processes; and 8 are of the annihilation type (in the case of electron-positron scattering) or of the exchange type (in the case of electron-electron scattering). For the emission of photons of not too high energy, the momentum transfer in the annihilation diagrams is of the order of the square of the initial energy and hence the contribution of these diagrams is small. Diagrams of the exchange type give contributions similar to those of the diagrams for direct scattering. Interference terms between the exchange and non-exchange diagrams are small. Because of the indistinguishability of the electrons,

the total contribution of the direct and the exchange diagrams must be divided by two. This means that one need consider only diagrams corresponding to non-exchange scattering and that one does not need to take account of the indistinguishability of the electrons. Hence the results obtained will be valid for either electron-electron or electron-positron scattering.

Of the eight remaining diagrams we need consider only the four shown in Fig. 1. The four remaining diagrams can be obtained from the diagrams of Fig. 1 by permuting the photons. The contributions of the diagrams of the first and second types are equal because of the indistinguishability of the photons; hence their sum must be divided by two. Since the cones into which these photons are emitted do not overlap in practice, the interference terms between the diagrams of the first and second type are small. The role of the discarded diagrams in contributing to correction terms will be discussed later on.

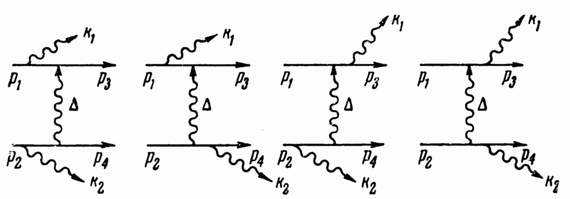

FIG. 1

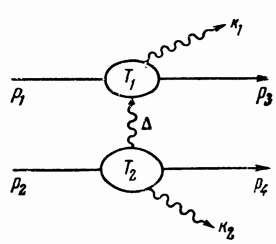

FIG. 2

Thus we will consider in what follows four diagrams, all of which can be conveniently represented in the form shown in Fig. 2. The matrix element corresponding to these four diagrams has the form

$$M=\frac{ie^4}{(2\pi)^5}(2\omega_1 2\omega_2\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4)^{-1/2}\frac{1}{\Delta^2}(\bar{u}(p_3)$$

$$\times T_{1\mu}u(p_1)(\bar{u}(p_4)T_2^{\mu}u(p_2)), \quad (1)$$

$$T_{1\mu}=\gamma_\mu\frac{i(\hat{p}_1-\hat{k}_1)-1}{2\varkappa_1}\hat{e}_1-\hat{e}_1\frac{i(\hat{p}_3+\hat{k}_1)-1}{2\varkappa_3}\gamma_\mu, \quad (2)$$

$$T_{2\mu}=\gamma_\mu\frac{i(\hat{p}_2-\hat{k}_2)-1}{2\varkappa_2}\hat{e}_2-\hat{e}_2\frac{i(\hat{p}_4+\hat{k}_2)-1}{2\varkappa_4}\gamma_\mu,$$

$$\varkappa_1=-(k_1p_1),\ \varkappa_2=-(k_2p_2),$$

$$\varkappa_3=-(k_1p_3),\ \varkappa_4=-(k_2p_4),$$

$$\Delta=p_3+k_1-p_1=p_2-k_2-p_4. \quad (3)$$

Here and in what follows we make use of the metric $(ab)=\mathbf{a}\cdot\mathbf{b}-a_0b_0$ and the system of units $\hbar=c=m=1$.

Averaging over the spins of the initial electrons and summing over the spins of the final electrons and over the polarizations of the photons, we obtain

$$d\sigma=\frac{e^8}{4(2\pi)^8[(p_1p_2)^2-1]^{1/2}}$$

$$\times\int\frac{N_{1\mu\nu}N_2^{\mu\nu}}{\Delta^4}\delta(p_3+p_4+k_1+k_2-p_1-p_2)$$

$$\times\frac{d^3p_3}{\varepsilon_3}\frac{d^3p_4}{\varepsilon_4}\frac{d^3k_1}{\omega_1}\frac{d^3k_2}{\omega_2} \quad (4)$$

Here

$$N_{1\mu\nu}=-\frac{1}{2}\left[\frac{1}{\varkappa_3^2}Q_{\mu\nu}+\frac{1}{\varkappa_1^2}R_{\mu\nu}-\frac{1}{\varkappa_1\varkappa_3}S_{\mu\nu}\right], \quad (5)$$

$$Q_{\mu\nu}=g_{\mu\nu}[1+\varkappa_3-\varkappa_1+(p_1p_3)+\varkappa_1\varkappa_3]-(p_{1\mu}p_{3\nu}+p_{1\nu}p_{3\mu})$$

$$+(\varkappa_3-1)(k_{1\mu}p_{1\nu}+k_{1\nu}p_{1\mu}), \quad (6)$$

$$R_{\mu\nu}=Q_{\mu\nu}(k_1\leftrightarrow -k_1,\ p_1\leftrightarrow p_3), \quad (7)$$

$$S_{\mu\nu}=2g_{\mu\nu}(p_1p_3)[\varkappa_1-\varkappa_3-(p_1p_3)-1]+2\varkappa_3p_{1\mu}p_{1\nu}$$

$$-2\varkappa_1p_{3\mu}p_{3\nu}+2k_{1\mu}k_{1\nu}+(p_{1\mu}p_{3\nu}+p_{1\nu}p_{3\mu})$$

$$\times[2(p_1p_3)+\varkappa_3-\varkappa_1]+(p_{1\mu}k_{1\nu}+p_{1\nu}k_{1\mu})(p_1p_3)$$

$$-(p_{3\mu}k_{1\nu}+p_{3\nu}k_{1\mu})(p_1p_3), \quad (8)$$

where the quantity $N_{2\mu\nu}$ is obtained from $N_{1\mu\nu}$ by the substitutions

$$p_1\to p_2,\quad p_3\to p_4,\quad k_1\to k_2,\quad \varkappa_1\to\varkappa_2,\quad \varkappa_3\to\varkappa_4.$$

As mentioned previously, we are interested in the double bremsstrahlung cross section integrated over the final electron states and over the emission angles of the quanta. Direct integration of expression (4) is awkward because the conservation law (3) connects the variables on which the quantities $N_{1\mu\nu}$ and $N_{2\mu\nu}$ depend. Hence it is useful to separate out the integrations over the final states of each electron and its emitted photon by introducing an additional delta function. The cross section (4) then takes the form

$$d\sigma_{\omega_1\omega_2}=\frac{4\alpha^4}{(2\pi)^4[(p_1p_2)^2-1]^{1/2}}\frac{d\omega_1}{\omega_1}\frac{d\omega_2}{\omega_2}\int\frac{d^4\Delta}{\Delta^4}K_{1\mu\nu}K_2^{\mu\nu}. \quad (9)$$

Here

$$K_{1\mu\nu}=\int N_{1\mu\nu}\delta(\Delta+p_1-p_3-k_1)\frac{d^3p_3}{\varepsilon}\omega_1^2\,d\Omega_1,$$

$$K_{2\mu\nu}=\int N_{2\mu\nu}\delta(\Delta+p_4-p_2+k_2)\frac{d^3p_4}{\varepsilon}\omega_2^2\,d\Omega_2. \quad (10)$$

Although the volume element $\omega^2 d\Omega$ is an invariant, the integration over the photon emission angles is not covariant because of the non-invariance of the integration volume. Hence the quantities $K_{\mu\nu}$ depend on the system of units in which the energy of the quantum ω is specified. In order to write these quantities in covariant form we introduce the four-vector n_μ, defined so that in the reference frame of interest to us its components are $n_\mu = (0, 0, 0, 1)$. Then the tensor $K_{1\mu\nu}$ can be expressed in terms of the three vectors $p_{1\mu}$, Δ_μ, and n_μ. The most general form of this tensor is

$$K_{1\mu\nu} = c_1 g_{\mu\nu} + c_2 p_{1\mu}p_{1\nu} + c_3\Delta_\mu\Delta_\nu + c_4(p_{1\mu}\Delta_\nu + p_{1\nu}\Delta_\mu) + c_5 n_\mu n_\nu + c_6(n_\mu\Delta_\nu + n_\nu\Delta_\mu) + c_7(n_\mu p_{1\nu} + n_\nu p_{1\mu}). \quad (11)$$

An analogous expression can be given for the tensor $K_2^{\mu\nu}$.

These tensors are gauge invariant, so that we must have

$$K_{1\mu\nu}\Delta^\mu = K_{1\mu\nu}\Delta^\nu = 0, \quad (12)$$

and hence only four of the functions c_i are independent. To determine these four functions it is necessary to calculate four covariant integrals, which we define as follows:

$$J_1 = g^{\mu\nu}K_{1\mu\nu}, \quad J_2 = p_1^\mu p_1^\nu K_{1\mu\nu}, \quad J_3 = n^\mu n^\nu K_{1\mu\nu},$$
$$J_4 = (n^\mu p_1^\nu + n^\nu p_1^\mu)K_{1\mu\nu}. \quad (13)$$

$K_{1\mu\nu}$ is proportional to the cross section for scattering of an arbitrarily polarized photon by an electron, integrated over all final states except the frequencies; the square of the mass of the initial photon is $-\Delta^2$. This quantity may be used in calculating the cross sections of a number of processes: the Compton effect of a polarized photon in an arbitrary system of units, bremsstrahlung during the scattering of an electron by a charged particle, etc.

3. We limit ourselves to the case in which one of the photons is soft, i.e., its energy is much less than the energy of the initial electron. Without loss of generality we can take the soft photon to be one with momentum k_2, i.e.,

$$\omega_2 \ll \varepsilon. \quad (14)$$

As is well known, in this case one can neglect the recoil of the electron during emission of photon number 2; thus the kinematics of the process are the same as for single bremsstrahlung. We now write down certain relations which will be useful in what follows:

$$(\Delta p_2) = \Delta^2/2, \quad \varkappa_3 = -(\Delta p_1) - \Delta^2/2. \quad (15)$$

The probability for the emission of a classical photon enters multiplicatively in the cross section for the process, so that when condition (14) is fulfilled the tensor $N_2^{\mu\nu}$ takes the form

$$\frac{\alpha}{2\pi^2} N_2^{\mu\nu}\frac{d^3k_2}{2\omega_2} = \Gamma^{\mu\nu}\, dW(k_2). \quad (16)$$

Here $dW(k_2)$ is the probability for the emission of a classical photon, given by

$$dW(k_2) = \frac{e^2}{(2\pi)^3}\frac{d\omega_2}{2\omega_2}\left[\frac{p_2}{(p_2k_2)} - \frac{p_4}{(p_4k_2)}\right]^2 \omega_2^2\, d\Omega_2, \quad (17)$$

where $\Gamma^{\mu\nu}$ is the usual tensor current for scattering of an electron

$$\Gamma^{\mu\nu} = {}^1/_2\{g^{\mu\nu}[1 + (p_2p_4)] - (p^{2\mu}p^{4\nu} + p^{2\nu}p^{4\mu})\}. \quad (18)$$

Putting $N_2^{\mu\nu}$ in (10) and carrying out the integration we obtain the following expression for $K_2^{\mu\nu}$

$$K_2^{\mu\nu} = -\left[g^{\mu\nu}\frac{\Delta^2}{2} + 2p_2^\mu p_2^\nu - p_2^\mu\Delta^\nu - p_2^\nu\Delta^\mu\right] \times 4\pi\Phi\left(\frac{\Delta^2}{4}\right)\frac{\delta(\Delta_0 + \varepsilon_4 - \varepsilon_2)}{\varepsilon_4}, \quad (19)$$

where $\Phi(x^2)$ is a function which occurs frequently in the theory of the emission of classical photons[1],

$$\Phi(x^2) = \frac{1 + 2x^2}{x(1 + x^2)^{1/2}}\ln(x + \sqrt{1 + x^2}) - 1. \quad (20)$$

In what follows it will be necessary to specify the coordinate system. As the problem was defined we are interested in the double bremsstrahlung cross section in the c.m.s. of the colliding particles. In this case the vector n_μ is determined by

$$n_\mu = (p_{1\mu} + p_{2\mu})/2\varepsilon, \quad (21)$$

where $\epsilon = \epsilon_1 = \epsilon_2$ are the energies of the initial particles. Using (21) to express the vector $p_{2\mu}$ in terms of the vectors $p_{1\mu}$ and n_μ and putting these in (19) we obtain

$$K_2^{\mu\nu} = -\Big[g^{\mu\nu}\frac{\Delta^2}{2} + 2p_1^\mu p_1^\nu + 8\varepsilon^2 n^\mu n^\nu - 4\varepsilon(n^\mu p_1^\nu + n^\nu p_1^\mu) - (p_2^\mu\Delta^\nu + p_2^\nu\Delta^\mu)\Big] \times 4\pi\Phi\left(\frac{\Delta^2}{4}\right)\frac{\delta(\Delta_0 + \varepsilon_4 - \varepsilon_2)}{\varepsilon_4}. \quad (22)$$

Replacing this tensor with the tensor $K_{1\mu\nu}$ we obtain the following expression for the cross section

$$d\sigma_{\omega_1\omega_2} = \frac{4\alpha^4}{(2\pi)^3 p\varepsilon}\frac{d\omega_1}{\omega_1}\frac{d\omega_2}{\omega_2} \times \int\frac{d^4\Delta}{\Delta^4}\left[-\frac{\Delta^2}{2}J_1 - 2J_2 - 8\varepsilon^2 J_3 + 4\varepsilon J_4\right] \times \frac{\delta(\Delta_0 + \varepsilon_4 - \varepsilon_2)}{\varepsilon_4}\Phi\left(\frac{\Delta^2}{4}\right), \quad (23)$$

where the J_m are given by (13).

Because of the azimuthal symmetry of the problem, the quantities J_m are functions of Δ^2, Δ_z, and Δ_0, where the z axis is directed along the velocity of the incident electron p_1. We can carry out in closed form both the integration over the azimuthal angle φ and the integration over the z component of the vector Δ_μ. As a result we obtain

$$d\sigma_{\omega_1\omega_2} = \frac{\alpha^4}{(2\pi)^2 p^2 \varepsilon^2}\frac{d\omega_1}{\omega_1}\frac{d\omega_2}{\omega_2} \times \int \frac{d\varkappa\, d\Delta^2}{\Delta^4}\left[-\frac{\Delta^2}{2}J_1 - 2J_2 - 8\varepsilon^2 J_3 + 4\varepsilon J_4\right]\Phi\left(\frac{\Delta^2}{4}\right). \tag{24}$$

Here $\kappa = 2\epsilon\Delta_0$. It is not difficult to see that $\kappa = \kappa_3$; in fact

$$\varkappa_3 = -(k_1 p_3) = -(\Delta p_1) - (\Delta p_2) = 2\varepsilon\Delta_0. \tag{25}$$

We now determine the region of integration of the variables κ and Δ^2. From the first equation of (15) it follows directly that

$$\Delta^2 + \varkappa \leqslant 2p|\mathbf{\Delta}|, \qquad p = |\mathbf{p}_1| = |\mathbf{p}_2|, \tag{26}$$

and from this, squaring, we find

$$\Delta^4 + 2(\varkappa - 2p^2)\Delta^2 + \varkappa^2/\varepsilon^2 \leqslant 0. \tag{27}$$

Similarly from the second expression of (15) we obtain

$$\varkappa^2 - 2\varkappa\omega_1\varepsilon_3 + \omega_1^2 \geqslant 0. \tag{28}$$

This means that κ varies over the interval

$$\varkappa_l \leqslant \varkappa \leqslant \varkappa_h, \tag{29}$$

where

$$\varkappa_l = \varepsilon^2\xi[1 - (1 - 1/\varepsilon^2(1-\xi))^{1/2}],$$
$$\varkappa_h = \varepsilon^2\xi[1 + (1 - 1/\varepsilon^2(1-\xi))^{1/2}]. \tag{30}$$

Here we have introduced the notation $\xi = \omega_1/\epsilon$, which will be used frequently in what follows.

The region of integration is shown in Fig. 3.

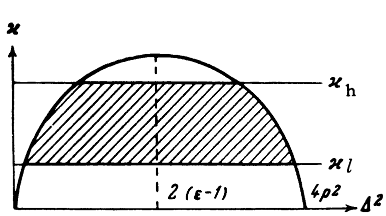

FIG. 3

It is bounded by a hyperbola (27) and the straight lines $\kappa = \kappa_l$ and $\kappa = \kappa_h$. The dependence of the limits κ_l and κ_h on the quantity ξ is shown in Fig. 4. The quantity $\xi_{max} = 1 - 1/\epsilon^2$ determines the maximum energy of the quantum k_1, for which case the final particles move with equal energies in the same direction, opposite to the direction of the photon emission. When $\xi = \xi_0 = 2(\epsilon - 1)/(2\epsilon - 1)$ the maximum energy transfer Δ_0 occurs ($\kappa_l = \kappa_{max} = 2\epsilon(\epsilon - 1)$) so that the electron emitting the hard quantum comes to rest in its final state.

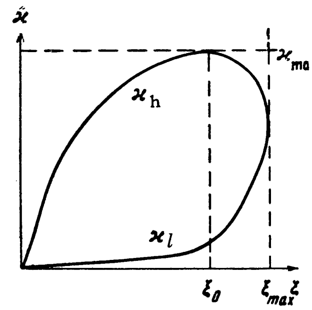

FIG. 4

4. We now calculate the integrals J_m in (13). Using the definition (10) of the tensor $K_{1\mu\nu}$ and finding the corresponding contractions with the tensor $N_{1\mu\nu}$ it is easy to see that the J_m depend only on the quantities κ_1, κ_3, Δ^2, ω_1, and ϵ. Hence in calculating the integrals over the final states (10) the only variable is the quantity κ_1, which occurs raised to various powers. Hence the integration reduces to evaluation of the following integrals

$$I_n = \int \varkappa_1^n \delta(p_1 + \Delta - p_3 - k_1)\frac{d^3p_3}{\varepsilon_3}\omega_1^2 d\Omega_1, \quad n = 1, 0, -1, -2. \tag{31}$$

The calculation of these integrals is simplest in the center of mass system of electron p_3 and photon k_1. This system is specified by the four-vector $Q_\mu = p_{1\mu} + \Delta_\mu$. All quantities necessary for transformations to this system are easily expressed in terms of this vector. As a result we obtain for the I_n the following expressions

$$I_1 = 2\pi\xi P Q^{-3/2}, \quad I_0 = 2\pi\xi Q^{-1/2},$$
$$I_{-1} = 2\pi\xi R^{-1/2}, \quad I_{-2} = 2\pi\xi P R^{-3/2}; \tag{32}$$
$$Q = (1 - \varkappa/2\varepsilon^2)^2 - 1/\varepsilon^2, \qquad P = f + g\varkappa + h\varkappa^2, \tag{33}$$
$$R = a + b\varkappa + c\varkappa^2, \tag{34}$$
$$f = \frac{1}{2}\Delta^2\xi, \quad g = 1 - \xi - \frac{1}{\varepsilon^2}\left(1 - \frac{\xi}{2}\right)\left(1 + \frac{\Delta^2}{2}\right),$$
$$h = -\frac{1}{2\varepsilon^2}(1-\xi), \quad a = \xi^2\Delta^2(1 + \Delta^2/4),$$
$$b = -\Delta^2\xi(1-\xi), \quad c = (1-\xi)(1 - \xi - 1/\varepsilon^2).$$

We now give the explicit expression for the J_m in terms of the integrals I_n

$$J_1 = -\frac{1}{\varkappa}I_1 - \frac{I_0}{\varkappa^2}\left[1 + 2\varkappa - \Delta^2\left(\frac{1}{2} + \varkappa\right)\right] + \frac{I_{-1}}{\varkappa}\left[2 + 2\varkappa - \varkappa^2 - \Delta^2\left(\varkappa + \frac{\Delta^2}{2}\right)\right] - \left(1 - \frac{\Delta^2}{2}\right)I_{-2}; \tag{35}$$

$$J_2 = I_1\left(1 + \frac{1}{2\varkappa}\right) + \frac{I_0}{\varkappa^2}\left[1 + 3\varkappa + \varkappa^2 - \varkappa^3\right.$$

$$+\frac{\Delta^2}{4}(1+2\varkappa-2\varkappa^2)\Big]$$
$$-\frac{I_{-1}}{2\varkappa}\left[4+8\varkappa+3\varkappa^2+3\Delta^2(1+\varkappa)+\frac{\Delta^4}{2}\right]$$
$$+I_{-2}\left(1+\varkappa+\frac{\Delta^2}{4}\right); \qquad (36)$$

$$J_3=\frac{1}{2\varkappa}I_1+\frac{I_0}{2\varkappa^2}\left\{2\varepsilon^2+\varkappa\left[1-\varkappa(1-\xi)-\frac{\varkappa^2}{2\varepsilon^2}\right]\right.$$
$$\left.-\Delta^2\left(\frac{1}{2}+\varkappa\right)\right\}-\frac{I_{-1}}{2\varkappa}\left\{4\varepsilon^2(1-\xi)+\varkappa(2-\xi)\right.$$
$$\left.+\Delta^2\left[\varepsilon^2+\varepsilon^2(1-\xi)-1-\frac{\varkappa\xi}{2}-\frac{\Delta^2}{2}\right]\right\}$$
$$+I_{-2}\left[\varepsilon^2(1-\xi)^2+\frac{\varkappa}{2}(1-\xi)-\frac{\Delta^2}{4}\right] \qquad (37)$$

$$J_4=\varepsilon\left\{\frac{I_1}{\varkappa}\left(1+\frac{\varkappa}{2\varepsilon^2}\right)+\frac{I_0}{\varkappa^2}\left[2+3\varkappa-\varkappa^2(1-\xi)\right.\right.$$
$$\left.+\frac{\varkappa}{\varepsilon^2}\left(\frac{1}{2}-\varkappa^2\right)-\frac{\Delta^2\varkappa}{2}\left(1+\xi+\frac{\varkappa}{2\varepsilon^2}\right)\right]$$
$$-\frac{I_{-1}}{\varkappa}\left[2(2-\xi)+\varkappa(4-3\xi)+\frac{\varkappa}{\varepsilon^2}(1+\varkappa)\right.$$
$$\left.+\frac{\Delta^2}{2}\left(4+\varkappa-3\xi-\varkappa\xi+\frac{\varkappa}{\varepsilon^2}\right)-\frac{\xi\Delta^4}{4}\right]$$
$$\left.+I_{-2}\left[(1-\xi)(2+\varkappa)+\frac{\varkappa}{2\varepsilon^2}-\xi\frac{\Delta^2}{2}\right]\right\}. \qquad (38)$$

5. The expressions which we have written down are exact. In the subsequent calculations we will make use of the ultrarelativistic nature of the colliding particles, i.e., of the smallness of the parameter ϵ^{-2}. This approximation is a consistent one, since the discarded diagrams make a contribution of just this order (cf. Sec. 2). In the analysis to follow we also assume that ξ and $1-\xi$ are not small in comparison to the parameter ϵ^{-1}. Then it is not difficult to see that J_3 makes the largest contribution to the cross-section. This quantity contains terms of the order of ϵ^2 (37) and occurs in expression (24) with a factor ϵ^2. We shall show below that the integration over the variables Δ^2 and κ does not introduce additional powers of ϵ and hence the double bremsstrahlung cross section is determined by just these terms. This means that the most important components of the tensors $N_{1\mu\nu}$ and $N_2^{\mu\nu}$ are those containing the maximum number of vectors n_μ, i.e., the "least covariant" terms. In fact contraction of the tensor $N_{1\mu\nu}$ with $g^{\mu\nu}$ and with the vector $p_{1\mu}$ gives quantities of the order of unity, whereas the contraction of $N_{1\mu\nu}$ with the vector n_μ can give terms of the form (np_1) and (nk_1), which are of the order of ϵ.

The quantities J_3 and J_1 have a clear physical significance. The quantity J_3 is the 00 component of the tensor $K_{1\mu\nu}$, i.e., the component which occurs in the cross section bremsstrahlung by an electron from a Coulomb center. The quantity J_1 occurs in the cross section for scattering of an unpolarized photon of mass $-\Delta^2$ by an electron. It can be easily verified that there is a simple relationship between these quantities

$$J_3(\Delta^2)=\Delta^2\frac{\varepsilon^2}{2\varkappa^2}J_1(\Delta^2=0)+O(\Delta^4,\varepsilon^{-2}). \qquad (39)$$

Here J_1 ($\Delta^2 = 0$) determines the Compton-effect cross section for a real photon. Expression (39) is the basis of the Weizsacker-Williams method, used to calculate the single bremsstrahlung cross section integrated over all angles. As will be explained in what follows, the Weizsacker-Williams method is not applicable to the calculation of the double bremsstrahlung cross section.

6. The foregoing expressions allow us to obtain fairly easily the differential cross section $d\sigma_\omega$ for single bremsstrahlung. To do so it is sufficient to omit the factor $dW(k_2)$ from the right-hand side of (16). Then the expression for the cross section $d\sigma_\omega$ takes the form

$$d\sigma_\omega=\frac{\alpha^3}{8\pi p^2\varepsilon^2}\frac{d\omega}{\omega}\int\frac{d\varkappa\, d\Delta^2}{\Delta^4}\left[-\frac{\Delta^2}{2}J_1-2J_2-8\varepsilon^2J_3+4\varepsilon J_4\right]. \qquad (40)$$

We are now calculating the cross section $d\sigma_\omega$ to logarithmic accuracy. The lower limit of the integration over Δ^2 is of order ϵ^{-4}, and in the integral in (40) it is low values of Δ^2 which are important. Hence the expressions for J_m can be expanded in powers of Δ^2 and we need retain only the zeroth and first-order terms. The quantity in square brackets of (40) contains terms of order ϵ^4, ϵ^2, 1, etc. When $\Delta^2 = 0$ the terms of order ϵ^4 and ϵ^2 cancel. Integration of the remaining terms gives a contribution

$$\frac{1}{\varepsilon^4}\int\frac{d\Delta^2}{\Delta^4}\approx\frac{1}{\varepsilon^4\Delta^2_{min}}\sim 1, \qquad (41)$$

i.e., a quantity not containing a large logarithm. Such a logarithmic term does arise from integration of the first term of the expansion in powers of Δ^2. In this case it is sufficient to limit oneself to terms of order ϵ^4 in the integral in (40).

It is easy to see that these terms occur only in the expansion of J_3. Using (39) we obtain

$$d\sigma_\omega=\alpha^3\frac{d\omega}{\omega}\int\frac{d\Delta^2}{\Delta^2}\frac{d\varkappa}{\varkappa^2}\xi\left[1-\xi+\frac{1}{1-\xi}-\frac{2\xi}{\varkappa(1-\xi)}\right.$$
$$\left.+\frac{\xi^2}{\varkappa^2(1-\xi)^2}\right]. \qquad (42)$$

The lower limits of the integration over Δ^2 and κ are determined by expressions (27) and (29) and are approximately

$$\Delta_{min}{}^2 = \varkappa^2/4\varepsilon^4, \qquad \varkappa l = \xi/2(1-\xi). \tag{43}$$

Here we have discarded terms which are of higher order in ϵ^{-2}. The upper limit of the integration over Δ^2 is determined by the conditions for the applicability of the expansion and hence we have $\Delta^2_{max} \sim 1$. Since the integrals over κ converge, the upper limit of the integration over κ is not significant. By the same reasoning the quantity κ may be replaced by κl in the expression for Δ^2_{min} in the logarithmically-accurate integral.

Carrying out the integration we obtain the principal (logarithmic) contribution to the single bremsstrahlung cross section

$$d\sigma_\omega = 4r_0{}^2\alpha\frac{d\omega}{\omega}\frac{\varepsilon-\omega}{\omega}\left[\frac{\varepsilon}{\varepsilon-\omega}+\frac{\varepsilon-\omega}{\varepsilon}-\frac{2}{3}\right] \times \ln\frac{4\varepsilon^2(\varepsilon-\omega)}{\omega}. \tag{44}$$

It must be remembered that the above expression gives the cross section for emission from one of the particles. It is this cross section which is in fact measured if one studies single bremsstrahlung in one direction in colliding-beam experiments. Because of the small interference terms mentioned above, the total cross section for single bremsstrahlung is twice the value given by (44). This result agrees with the logarithmic part of the cross section for single bremsstrahlung obtained by Altarelli and Buccella[2].

We note that the cross section (44) has the same form as the cross section for single bremsstrahlung by an electron at rest, as calculated by the Weizsacker-Williams method, except for the argument of the logarithm, which can be obtained by means of an elementary relativistic recalculation. To obtain the non-logarithmic terms, the expansion made above is not suitable and the integration over Δ^2 must be carried out exactly. Here, as seen from the estimate (41), it is necessary to take into account some of the terms which were omitted previously. This makes the calculation more cumbersome.

7. We now proceed to calculate the differential double bremsstrahlung cross section $d\sigma_{\omega_1\omega_2}$ (24). Compared to the case of single bremsstrahlung, the function in the integrand contains an additional factor $\Phi(\Delta^2/4)$. For small and large values of the argument this function behaves as follows:

$$\Phi(x^2) \approx {}^4/_3x^2, \qquad x^2 \ll 1,$$
$$\Phi(x^2) \approx \ln x^2, \qquad x^2 \gg 1. \tag{45}$$

This means that small values of Δ^2 are not important in the integral (24). In fact, as $\Delta^2 \to 0$ the square bracket in this integral becomes independent of Δ^2 and is of the order of magnitude of unity (cf. the previous section); hence the integral over the region of small Δ^2 is

$$\frac{1}{\varepsilon^4}\int\frac{d\Delta^2}{\Delta^4}\Delta^2 \approx \frac{1}{\varepsilon^4}\ln\frac{1}{\Delta^2_{min}} \tag{46}$$

and hence gives a negligibly small contribution (cf. Eq. (41)).

Inclusion of the next term in the expansion in powers of Δ^2 leads to an integral of the type

$$\int\frac{d\Delta^2}{\Delta^4}\Delta^2\Phi\left(\frac{\Delta^2}{2}\right). \tag{47}$$

This integral converges for small Δ^2 and behaves like $\ln^2\Delta$ for large Δ^2. Thus, unlike the case of single bremsstrahlung, the lower limit of integration with respect to Δ^2 is unimportant here and we cannot use expansion in powers of Δ^2 in calculating the integral (24). This situation always occurs in two-photon emission. It means that the Weizsacker-Williams method is not applicable in such problems.

Thus the integration over Δ^2 must be carried out exactly. Introducing the highest-order terms in ϵ^2 into Eqs. (24) and (37) we obtain

$$d\sigma_{\omega_1\omega_2} = \frac{4\alpha^4}{(2\pi)^2}\frac{d\omega_1}{\omega_1}\frac{d\omega_2}{\omega_2} \times\int\frac{d\varkappa\, d\Delta^2}{\Delta^4}\Big[-\frac{2I_0}{\varkappa^2}+\frac{4(1-\xi)+\Delta^2[1+(1-\xi)^2]}{\varkappa} \times I_{-1}-2(1-\xi)^2 I_{-2}\Big], \tag{48}$$

and in this approximation

$$I_0 = 2\pi\xi; \tag{49}$$

$$I_{-1} = 2\pi\xi[\varkappa^2(1-\xi)^2-\varkappa\Delta^2\xi(1-\xi)+\xi^2\Delta^2(1+\Delta^2/4)]^{-1/2}, \tag{50}$$

$$I_{-2} = 2\pi\xi[\xi\Delta^2/2+(1-\xi)\varkappa][\varkappa^2(1-\xi)^2-\varkappa\Delta^2\xi(1-\xi) + \xi^2\Delta^2(1+\Delta^2/4)]^{-3/2}. \tag{51}$$

We now carry out the integration over κ. The lower limit in this integration is given by expression (43). The upper limit (cf. Fig. 3) has different dependences on Δ^2 in different regions. However, a rigorous analysis shows that the upper limit of the integration over κ is in general unimportant and the integration may be extended to infinity. As a result we obtain

$$d\sigma_{\omega_1\omega_2} = \frac{8r_0{}^2\alpha^2}{\pi}\frac{d\omega_1}{\omega_1}\frac{d\omega_2}{\omega_2} \times\int\frac{dx}{x^3}\Phi(x^2)\Big[\Phi(x^2)\left(1-\frac{\omega_1}{\varepsilon}\right) + \frac{\omega_1{}^2}{\varepsilon^2}\frac{x}{(1+x^2)^{1/2}}\ln\left(x+\sqrt{1+x^2}\right)\Big]. \tag{52}$$

When $\omega_1 \ll \epsilon$ Eq. (52) takes the form

$$d\sigma_{\omega_1\omega_2} = \frac{8r_0^2\alpha^2}{\pi}\frac{d\omega_1}{\omega_1}\frac{d\omega_2}{\omega_2}\int\frac{dx}{x^3}\Phi^2(x^2). \quad (53)$$

This expression agrees with the result obtained previously by the authors [1] for the cross section for bremsstrahlung of two classical photons in the classical current approximation. We see that the expression in square brackets in (52) can, apart from a constant factor, be considered as a generalization of the probability for photon emission during a collision accompanied by a momentum transfer Δ^2 in the case of the emission of hard photons.

The important region in the integral (52) is $x \sim 1$ and therefore the upper limit of integration over x can be extended to infinity. It can now be seen that the integration over x simply gives a number and hence does not alter the order of magnitude of the terms in powers of ϵ^2 (cf. Sec. 5). Carrying out the integration we obtain the following final expression for the double bremsstrahlung cross section

$$d\sigma_{\omega_1\omega_2} = \frac{8r_0^2\alpha^2}{\pi}\frac{d\omega_1}{\omega_1}\frac{d\omega_2}{\omega_2}\left[\left(1-\frac{\omega_1}{\varepsilon}\right)\left(\frac{5}{4}+\frac{7}{8}\zeta(3)\right) + \frac{\omega_1^2}{\varepsilon^2}\left(\frac{1}{2}+\frac{7}{8}\zeta(3)\right)\right]. \quad (54)$$

Here $\zeta(m)$ is the Riemann zeta function, $7\zeta(3)/8 = 1052$. For $\omega_1 \ll \epsilon$ this expression takes the form of Eq. (22) of [1] (in which the numerical coefficient is in error and should be divided by 4).

8. We now determine the order of magnitude of the neglected terms in (54). These correction terms arise when one takes into account successive terms in ϵ^{-2} in the original formula (24) and when one takes into account the contribution of the neglected diagrams. It is simplest to estimate the magnitude of the discarded terms for the case of low energy photons ($\omega \ll \epsilon$). In this case we can use the classical current approximation and the expression for the cross section takes the form

$$d\sigma = \int \sigma_{e0}\frac{1}{2!}dW_e(k_1)\,dW_e(k_2)\,d\Omega, \quad (55)$$

where σ_{e0} is the cross-section for electron scattering given by the Møller formula. Equation (55) takes the contributions of all diagrams into account.

Using the explicit expression for $dW_e(k)$ (cf. [1]) it is simple to show that the maximum correction terms are contributed by the region of large Δ^2 ($\Delta^2 \sim \epsilon^2$). For these values of Δ^2 the contribution of various diagrams is of the same order of magnitude. The probability for photon emission for large Δ^2 is

$$\frac{4\alpha}{\pi}\ln\Delta\frac{d\omega}{\omega}. \quad (56)$$

The dominant term $d\Delta^2/\Delta^4$ in the Møller formula yielded our main result. The largest of the remaining terms $d\Delta^2/\epsilon^2\Delta^2$ gives the following correction to the cross section

$$\delta\sigma \sim r_0^2\alpha^2\int\frac{d\Delta^2}{\varepsilon^2\Delta^2}\ln^2\Delta\frac{d\omega_1}{\omega_1}\frac{d\omega_2}{\omega_2} \sim r_0^2\alpha^2\frac{\ln^3\varepsilon}{\varepsilon^2}\frac{d\omega_1}{\omega_1}\frac{d\omega_2}{\omega_2}. \quad (57)$$

The remaining terms give expressions differing from (57) by lower powers of $\ln\epsilon$. As far as small values of Δ^2 are concerned, this region gives a correction of order ϵ^{-2}, owing to the fact that the probability for emission accompanied by small angle scattering does not contain logarithms and the fact that the probability for emission in directions opposite to the direction of motion of the particles is of order ϵ^{-2}.

A correction analogous to (57) also arises in the case of emission of hard photons. For electron energies of the order of 50 MeV this correction is of the order of 2–3% and decreases rapidly with increasing energy.

The above considerations do not treat the hardest part of the spectrum where $\epsilon - \omega \sim 1$. This region requires special consideration. Corrections to (54) for this case clearly differ from (57) by a factor $\epsilon/(\epsilon-\omega)$ which is large for the hardest region of the spectrum. In view of the narrowness of this region, the integrated contribution which it makes is obviously small. A more detailed consideration of this question will be given in the future.

[1] V. N. Bayer and V. M. Galitsky, Phys. Lett. **13**, 355 (1964).

[2] G. Altarelli and F. Buccella, Nuovo Cimento **34**, 1337 (1964).

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