

NONLINEAR MODE INTERACTION IN LASERS

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We consider a solid-state laser oscillating in two modes within a radiation linewidth. If the spacing between interacting modes is large enough the rate equations apply and the stationary state is stable. If the modes are closely spaced the stationary state can be unstable, leading to undamped oscillatory variations in the radiated power.

INTRODUCTION

A characteristic feature of the operation of solid-state lasers is the oscillatory variation in radiated power.^[1-5] While the operation of gas lasers has been interpreted in reasonable fashion^[6,7] the explanation of the experimental results obtained with solid-state lasers is still an open question. Investigation of the rate equations, which follow from energy balance considerations, has shown that the process by which the amplitudes are established is of the nature of a damped oscillation. In a number of initial experiments the length of the pulse from the pumping lamp was equal to or less than the characteristic damping time of the oscillations as calculated by the theory.

It was reasonable to assume that the pulsations of radiated laser power observed in the early experiments were due to the damped oscillatory nature of the transient process. However, in later experiments it was established that lasers exhibit undamped "auto-modulation" of the output power.^[3,4] An investigation of the rate equations has shown that these equations do not have periodic solutions.^[8,9] The complete system of equations for a single-mode laser, which takes account of the phase relations between the field and the polarization of the material, does admit auto-modulating solutions^[10-12] but the conditions under which these solutions apply do not obtain in existing lasers.^[12] Attempts have been made to circumvent this difficulty by introducing into the rate equations a damping time which depends on the pumping power.^[13,14]

The dependence of the operation of a laser on the properties of the resonator^[3,15] indicates that a rational description of the self-oscillatory processes must take account of the interaction between a number of resonator modes. From the point of view

of generalized oscillators it is clear that the analysis of the mode interactions in a resonator filled with a nonlinear active medium does not require calculation of the actual field configuration: expansion of the field in eigenmodes of the ideal resonator appears to be a reasonable method of approach.

To understand the operation of a laser in which many modes are excited within a radiative linewidth of the working material we must first study possible regimes of oscillation when only two modes are excited. In the present work we present an analysis of a laser oscillating in two modes within a radiative linewidth and investigate the limiting cases in which the detuning of the individual modes (due to coupling) is either very large or very small.

DERIVATION OF THE EQUATIONS

Consider a high-Q resonator completely filled with an active medium. The medium consists of noninteracting fixed atoms, molecules, or ions which have two energy levels. The pumping mechanism transfers electrons from the ground state to the excited state. The electric field $\mathbf{E}(\mathbf{r}, t)$ interacting with the medium produces a macroscopic polarization density $\mathbf{P}(\mathbf{r}, t)$. For simplicity we assume that the electric field is a scalar quantity and that all the molecules are oriented in the same direction in space.

The basic difficulty in solving the problem lies in the description of a polarization due to the effect of several modes. The starting equations comprise the system (1), which has been used earlier for the description of a single-mode laser:^[16,17]

$$\ddot{P} + \frac{2}{T_2} \dot{P} + \omega_l^2 P = - \frac{2d_{12}^2 \omega_{12}}{\hbar} ER, \quad (1a)$$

$$\dot{R} + \frac{1}{T_1}(R - R_0) = \frac{2}{\hbar\omega_{12}} \left(\dot{P} + \frac{1}{T_2}P \right) E,$$

$$R_0 = N \frac{W - \tau^{-1}}{W + \tau^{-1}}. \quad (1b)$$

Here, ω_l is the frequency of the radiating line of the material, R is the density of the inverted population, R_0 characterizes the pumping intensity, N is the density of atoms in the working material, W is the probability of excitation under the effect of the pumping radiation, τ is the lifetime of the excited state, $T_1 = \tau/(1 + W\tau)$, T_2^{-1} is the width of the radiating line of the material, $\hbar\omega_{12}$ is the energy difference between the upper and lower levels and d_{12} is the dipole matrix element.

A nontrivial point is the separation in (1) of the component of the polarization which excites a corresponding mode in the resonator.

If one expands the variables E , P , and R in (1) in eigenmodes of the field it is found that the expansion for the particle populations does not provide separate classes of particles, each of which radiates in only one mode. This complication results from the nonlinear field dependence of the populations of the energy levels. Substituting R from (1b) in (1a) we obtain (2), which contains only the polarization and the field:

$$\ddot{P} + \frac{2}{T_2}\dot{P} + \omega_l^2 P = -\frac{2d_{12}^2\omega_{12}}{\hbar}$$

$$\times E \left[R_0 + \frac{2}{\hbar\omega_{12}} e^{-t/T_1} \int e^{t/T_1} \left(\dot{P} + \frac{1}{T_2}P \right) E dt \right]. \quad (2)$$

We expand the field and polarization in eigenmodes of the lossless resonator φ_i :

$$E(\mathbf{r}, t) = \sum_i E_i(t) \varphi_i(\mathbf{r}), \quad P(\mathbf{r}, t) = \sum_i P_i(t) \varphi_i(\mathbf{r}). \quad (3)$$

Now Maxwell's equations assume the form

$$\ddot{E}_i + \omega_i \dot{E}_i / Q_i + \omega_i^2 E_i = -4\pi \ddot{P}_i, \quad (4)$$

where ω_i is the characteristic frequency and Q_i is the quality factor of the i -th mode.

We shall limit ourselves to the case in which two high- Q modes fall within the radiative linewidth; in the expansion in (3) we retain only two terms.¹⁾ The material equations for the two-mode case being considered can be obtained if we use the equation for the material (2) and the orthogonality of the eigenmodes. Equation (2) is equivalent to the follow-

ing system:

$$\ddot{P}_i + \frac{2}{T_2}\dot{P}_i + \omega_i^2 P_i = -\frac{2d_{12}^2\omega_{12}}{\hbar} [(R_i^0 - \mu_i R_i - \mu_{12} R_j) E_i + (R_{12}^0 - \mu_{12} R_{12}) E_j], \quad (5a)$$

$$\dot{R}_i + \frac{1}{T_1} R_i = -\frac{2}{\hbar\omega_{12}} \left(\dot{P}_i + \frac{1}{T_2} P_i \right) E_i, \quad (5b)$$

$$\dot{R}_{12} + \frac{1}{T_1} R_{12} = -\frac{2}{\hbar\omega_{12}} \sum_{1,2} \left(\dot{P}_i + \frac{1}{T_2} P_i \right) E_j, \quad (5c)$$

where $i, j = 1, 2, i \neq j$ while

$$\mu_i = \int \varphi_i^4 dV, \quad \mu_{12} = \int \varphi_1^2 \varphi_2^2 dV,$$

$$R_i^0 = \int R_0 \varphi_i^2 dV, \quad R_{12}^0 = \int R_0 \varphi_2 \varphi_1 dV,$$

where R_i denotes particles which "radiate" in the i -th mode, R_{12} is an interference term which depends on the fields in both modes, and it is assumed that

$$\mu_{13} = \int \varphi_1^3 \varphi_2 dV = \int \varphi_2^3 \varphi_1 dV = 0.$$

In this work we consider the case in which the pumping power is uniform over the entire volume of the active material so that $R_i^0 = R_0$ and $R_{12}^0 = 0$. This approximation holds for low-order modes if the transverse dimensions of the region in which the exciting field is strong are much smaller than the transverse dimensions of the volume occupied by the medium with the inverted population.

In (5) we can distinguish two different forms of interaction between the modes. The first term on the right side of (5a) arises by virtue of the fact that the mode fields have different spatial distributions; consequently, as has already been noted,^[18] it is possible to excite two or more modes for homogeneous broadening of the radiative line, because of "hole-burning" in the active material. The coefficient μ_{12} depends on the degree of overlap of the fields φ_1 and φ_2 . Using the eigenmodes of a plane-parallel resonator

$$\varphi_{mnp} = 2^{3/2} \sin m\pi(x/L_1) \sin n\pi(y/L_2) \sin p\pi(z/L_3)$$

we have $\mu_1 = \mu_2 = \mu$ and the quantity $\mu_{12}\mu^{-1}$ assumes the values 8/27, 4/9, and 2/3, depending on the number of doubly repeated indices characterizing the field distributions $\varphi_{m_1 n_1 p_1}$ and $\varphi_{m_2 n_2 p_2}$.

The next term in R_{12} is due to an interaction which depends on the phase relations between the modes. This interaction becomes important when the characteristic resonator frequencies ω_1 and ω_2 approach each other. Terms of this kind do not appear if one carries out the analysis starting with

¹⁾In operation at levels far above threshold in general it would be necessary to include a number of terms in the polarization expansion.

the rate equations, as has been done in^[18].

We now introduce dimensionless variables in accordance with the following notation ($i = 1, 2$):

$$\begin{aligned} t' &= \omega_l t, & u_i &= \frac{d_{12} E_i}{\hbar} (T_2 T_1)^{1/2}, \\ v_i &= \frac{P_i}{d_{12} R_0} \left(\frac{T_1}{T_2} \right)^{1/2}, & w_i &= \frac{R_i}{R_0}, \\ w_{12} &= \frac{R_{12}}{R_0}, & 2h_i &= \frac{\omega_i}{\omega_l Q_i}, & \epsilon_i^2 &= \frac{\omega_i^2}{\omega_l^2}, \\ h_l^{-1} &= \omega_l T_2, & h^{-1} &= \omega_l T_1, & 2k &= 4\pi R_0 d_{12}^2 h^{-1} T_2. \end{aligned} \quad (6)$$

The equations describing a laser oscillating in two modes within a radiative linewidth then become^[19] ($i, j = 1, 2; i \neq j$):

$$\begin{aligned} \dot{w}_i + h w_i &= -2h(\dot{v}_i + 2h_l v_i) u_i, & \dot{v}_i + 2h_l \dot{v}_i + v_i \\ &= -2h_l [(1 - \mu_i w_i - \mu_{12} w_j) u_i - \mu_{12} w_{12} u_j], \\ \dot{w}_{12} + h w_{12} &= -2h(\dot{v}_i + 2h_l v_i) u_i, \\ \dot{w}_{12} + h w_{12} &= -2h \sum_{1,2} (v_i + 2h_l v_i) u_j. \end{aligned} \quad (7)$$

In the notation of (6) the variables u_i , v_i , and w_i are of order unity. If the working material is ruby typical values of the parameters appearing in (7) are as follows:

$$h_i \sim 10^{-7}, \quad h_l \sim 10^{-4}, \quad h \sim 10^{-14}, \quad k \sim h_i.$$

The nature of the solution of (7) depends on the mutual effects of the modes on each other. If the distance between the characteristic frequencies is appreciably greater than the width of each resonance line, simultaneous oscillation at both frequencies is possible; however, if the resonance curves overlap to a considerable extent one expects that the oscillations will be essentially harmonic (single-mode). We consider the two cases separately.

1. APPROXIMATELY BIHARMONIC OSCILLATIONS

Assume that the characteristic frequencies of the modes are widely spaced so that the resonance curves do not overlap, that is to say, the system losses exhibit separate minima at frequencies ω_2 and ω_1 . In a plane-parallel resonator this situation obtains for modes with different axial indices. Depending on the magnitude of the negative loss at frequencies ω_2 and ω_1 it is clear that oscillation is possible either at both frequencies, corresponding to simultaneous excitation of the modes, or at one of the frequencies, in which case only one mode is excited.

If the oscillations are approximately biharmonic (5) can be simplified in cases of practical interest, in which T_1 is large. The term R_{12} varies at the difference frequency $\omega_2 - \omega_1$ and is negligibly small compared with R_1 and R_2 when $(\omega_2 - \omega_1) T_1 \gg 1$. In ruby, for example, $T_1 \sim 10^{-3} - 10^{-4}$ sec and for close modes with different axial indices $\omega_2 - \omega_1 \sim c/L$ (L is the length of the resonator) and $R_{12} \ll R_i$ if L is not too large. Hence, the interaction of two modes characterized by different axial indices can be described with a high degree of accuracy by (5) with $R_{12} = 0$.

A. Rate equations. In lasers the radiation linewidth of the material is generally much smaller than the radiated frequency while the width of the resonance curve for each mode is much smaller than the spectral linewidth, i.e.,

$$h_i \ll h_l \ll 1. \quad (8)$$

The width of the radiating line is always larger than the actual linewidth and $h < h_l$. Then the system in (7) approximates a system whose general solution is of the form ($i = 1, 2$):

$$\begin{aligned} u_i &= x_i \cos \Omega_i t' + y_i \sin \Omega_i t', \\ v_i &= p_i \cos \Omega_i t' + g_i \sin \Omega_i t', \\ \omega_i &= \text{const.} \end{aligned} \quad (9)$$

To solve this problem we use the Van der Pol method, in which the constant terms in (9) are regarded as slowly varying quantities.

The variation of these quantities is given by the system of so-called abbreviated differential equations ($i = 1, 2; i \neq j$):

$$\begin{aligned} \dot{x}_i &= -k g_i - h_i x_i - \delta_i' y_i, & \dot{y}_i &= k p_i - h_i y_i + \delta_i' x_i, \\ \dot{p}_i &= h_l [(1 - \mu_i w_i - \mu_{12} w_j) y_i - p_i - \Delta_i g_i], \\ \dot{g}_i &= -h_l [(1 - \mu_i w_i - \mu_{12} w_j) x_i + g_i - \Delta_i p_i], \\ \dot{w}_i + h w_i &= h (y_i p_i - x_i g_i). \end{aligned} \quad (10)$$

Here we have used the notation $\delta_i' = (\Omega_i^2 - \epsilon_i^2)/2$ and $\Delta_i = (\Omega_i^2 - 1)/2h_l$.

According to (8) the rate-of-change of the alternating polarization (p_i , g_i) is much faster than the rate-of-change of the alternating fields (x_i , y_i) and the number of particles w_i . In this case the phase space divides into regions of "fast" and "slow" variation.^[20] It is evident from (10) that the fast variation is always stable (in determining the stability of the fast variation the slow variables are regarded as constants); hence, for arbitrary initial

conditions the system executing the fast variation always moves into the region of slow variations and remains there at subsequent times.

In the slow region we have $\dot{p}_i = 0$ and $\dot{g}_i = 0$. Expressing the fast variables in terms of the slow variables we find

$$\begin{aligned} \dot{A}_1 &= 2[k\gamma_1(1 - \mu_1 w_1 - \mu_{12} w_2) - h_1]A_1, \\ \dot{A}_2 &= 2[k\gamma_2(1 - \mu_2 w_2 - \mu_{12} w_1) - h_2]A_2, \\ \dot{w}_1 &= h[\gamma_1(1 - \mu_1 w_1 - \mu_{12} w_2)A_1 - w_1], \\ \dot{w}_2 &= h[\gamma_2(1 - \mu_2 w_2 - \mu_{12} w_1)A_2 - w_2]. \end{aligned} \quad (11)$$

Here, the variables $A_i = x_i^2 + y_i^2$ and w_i ($i = 1, 2$) are proportional to the mode energies and the number of active particles. The coefficient $\gamma_i^{-1} = 1 + [(\Omega_i^2 - 1)/2h_L]^2$ takes account of the line shape. Equations of this kind are frequently called the rate equations. Analogous equations for two-mode oscillations have been given in^[18].

If $1/T_1 \gg \omega_L/Q$ is satisfied, as is the case for gas lasers, (11) can be simplified further. At low saturation levels these equations become those given by Lamb.^[7]

B. Stationary states and stability. By setting the time derivatives equal to zero we obtain a system of algebraic equations which determine the amplitudes of the stationary states. To obtain a nontrivial solution of (10) we must satisfy the following condition, which yields the oscillation frequencies:

$$\Omega_i = \varepsilon_i + \frac{h_i}{h_i + h_l} \delta_i, \quad i = 1, 2,$$

where $\delta_i = 1 - \varepsilon_i$ measures the detuning of the i -th mode from the center of the line. The deviation of the mode frequency from the characteristic resonator frequency is of the same form as in single-mode oscillators. Obviously, in (11) we must take account of the fact that the oscillations occur at a frequency $\Omega_i = \varepsilon_i$, since these equations are asymptotically valid for $h_i \ll h_L$.

The system in (11) has four equilibrium states:

a) $A_i = w_i = 0$. Linearizing about the origin we find that the oscillations will grow if the following excitation condition is satisfied:

$$k > h_i(1 + (\delta_i/h_l)^2).$$

The excitation condition for each mode is independent of the other.

b) $A_1 = 0$ and $A_2 > 0$, or $A_2 = 0$ and $A_1 > 0$. The modes are orthogonal and each mode can be excited independently. Let $A_2 = 0$ in which case

$$\begin{aligned} A_1 &= (k - h_1\gamma_1^{-1})/h_1\mu_1, \\ w_1 &= (k - h_1\gamma_1^{-1})/k\mu_1. \end{aligned} \quad (12)$$

The condition for excitation of the second mode is of the form

$$k\left(1 - \frac{\mu_{12}}{\mu_1}\right) > h_2\gamma_2^{-1} - \frac{\mu_{12}}{\mu_1}h_1\gamma_1^{-1}. \quad (13)$$

If the modes are symmetrically displaced with respect to the center of the radiation line ($\gamma_1 = \gamma_2$) and if both quality factors are the same ($h_1 = h_2$) the oscillations in the second mode will always grow; however, when $h_1 < h_2$ they may not be excited. Similar considerations hold for $A_1 = 0$, $A_2 > 0$. Thus, a state is possible in which the excitation conditions are satisfied for both modes but in which auto-modulation occurs only in the mode characterized by the higher quality factor.

c) $A_1 > 0$, $A_2 > 0$. Both modes are excited

$$A_i = \frac{\mu_j[k - h_i\gamma_i^{-1}] - \mu_{12}[k - h_j\gamma_j^{-1}]}{h_i(\mu_1\mu_2 - \mu_{12})}, \quad i, j = 1, 2; \quad i \neq j.$$

Investigation of the characteristic equation shows that the stationary point c) is a stable focus. Depending on the ratios $1/T_1$, the natural line width, and ω_L/Q , the width of the resonance line, the transient process can be either aperiodic or oscillatory. In solid-state lasers the condition $\omega_L/Q \gg 1/T_1$ is usually satisfied²⁾ and this situation is evidently responsible for the pulsations in radiated power under steady-state conditions.

Let $h_1 = h_2$, $\gamma_1 = \gamma_2$ and introduce the quantity $\eta = k/h_i\gamma_i^{-1} - 1$, which is the excess pump power beyond threshold; we now consider motion in the invariant space $A_1 = A_2$, $w_1 = w_2$ and find that close to the stationary state the frequency of these pulsations is $\omega_k = (\omega_L\eta/T_1Q)^{1/2}$ while the damping time is $\tau_k = T_1/2(\eta + 1)$. In ruby $\tau_k \sim 10^{-4}$ sec and $\omega_k \sim 10^{-6}$ sec⁻¹.

Using Fig. 1 we can determine the stationary state of a system for a given pumping power and mode quality factor. The coordinates in this figure

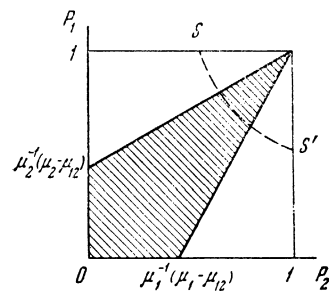


FIG. 1. Diagram for determining the stationary state of a two-mode laser for the case of large detuning.

²⁾In gas lasers the stationary state is established without pulsations.

are the ratio of the loss for each mode to the gain $P_i = h_i \gamma_i^{-1}/k$. When $P_i < 1$ the state $A_i = 0$ is unstable. If the point (P_1, P_2) lies within the cross-hatched region, both modes are excited and both are stable. However, if the point lies outside the cross-hatched region the stable regime is one in which the mode with the lower loss is excited and the field in the other mode is zero.

If the center of the radiation line moves adiabatically during lasing because of heating, the point (P_1, P_2) in Fig. 1 is displaced. For example, when the point moves along SS' it intersects the boundary of the cross-hatched region, and the oscillation frequency changes abruptly to $|\omega_2 - \omega_1|$. Frequency jumps connected with transient generation of one set of axial modes and then another have been observed experimentally in^[21].

2. ALMOST PERIODIC OSCILLATIONS

A. Abbreviated equations. The frequency spectrum of an open resonator is usually degenerate to a considerable degree. In a confocal resonator each characteristic frequency is multiply degenerate and in a plane-parallel resonator at large Fresnel numbers mode resonances with the same axial index but different transverse indices are usually unresolved.

Let us now consider the interaction of two closely-spaced modes. In this case the solution of (7) is almost periodic

$$u_i = x_i \cos \Omega t' + y_i \sin \Omega t', \quad v_i = p_i \cos \Omega t' + g_i \sin \Omega t',$$

$$w_i = \text{const}, \quad w_{12} = \text{const}. \quad (14)$$

Here, Ω is the oscillation frequency. Converting to the van der Pol variables and averaging over the explicit time we obtain a system of abbreviated equations for the case of harmonic oscillations. Since we have assumed that the width of the radiating line in the material is much greater than the width of the resonance curve for each mode we can obtain a system for the slow motion from the Van der Pol equations (7).

In the interest of simplicity we consider the case $h_1 = h_2$, $\delta_1 = -\delta_2$. This means that the modes have the same quality factor and that they are symmetrically displaced with respect to the center of the radiation line. Furthermore, if

$$\int \varphi_1^4 dv = \int \varphi_2^4 dV = \mu,$$

the behavior of the system (7) in the slow region is

described by the following equations ($i, j = 1, 2$; $i \neq j$):

$$\begin{aligned} \dot{x}_i &= [k(1 - \mu w_i - \mu_{12} w_j) - h_i] x_i - x_j k w_{12} \mu_{12} - \delta_i y_i, \\ \dot{y}_i &= [k(1 - \mu w_i - \mu_{12} w_j) - h_i] y_i - y_j k w_{12} \mu_{12} + \delta_i x_i, \\ \dot{w}_i &= h[(1 - \mu w_i - \mu_{12} w_j)(x_i^2 + y_i^2) \\ &\quad - (x_i x_j + y_i y_j) w_{12} \mu_{12} - w_i], \\ \dot{w}_{12} &= h\{[2 - (\mu + \mu_{12})(w_1 + w_2)](x_1 x_2 + y_1 y_2) \\ &\quad - (x_1^2 + y_1^2 + x_2^2 + y_2^2) w_{12} \mu_{12} - w_{12}\}. \end{aligned} \quad (15)$$

Here, $\delta_i = \Omega - \epsilon_i$ while the remaining parameters are given in (6). All the limit cycles of the original equations (7) and their stability are completely determined by the slow equations (15).

B. Stability of the stationary states. If the system in (15) is to have a nontrivial solution it is easy to show that in the stationary state harmonic oscillations are possible only at the frequencies $\Omega = \epsilon_1$, $\Omega = 1$, $\Omega = \epsilon_2$. If the frequency of oscillation coincides with the characteristic frequency of the first mode then the stationary state is such that only the first mode is excited and the second mode is zero:

$$A_1 = \eta \mu^{-1}, \quad A_2 = 0.$$

By direct calculation it is easy to show that this state is unstable if the mode Q 's are equal. Oscillations will always be excited in the second mode. Similarly, the equilibrium state in which $\Omega = \epsilon_2$ is unstable and the field is nonvanishing only in the second mode.

If the oscillations occur at the line frequency the energy is equally distributed between the modes. In this case there are two equilibrium states, which differ from each other by the phase shifts between the fields in the first and second modes. The amplitudes can be found from the equation

$$A_i^2 - 2A_i \frac{\eta(\mu + 2\mu_{12}) - \mu_{12}\Delta^2}{(\mu + \mu_{12})(\mu + 3\mu_{12})} + \frac{\eta^2 + \Delta^2}{(\mu + \mu_{12})(\mu + 3\mu_{12})} = 0. \quad (16)$$

Here $A_i = x_i^2 + y_i^2$ ($i = 1, 2$); $\Delta = (\epsilon_2 - \epsilon_1)/2h_1 \geq 0$ is the detuning of the modes referred to the width of their resonance curves. In the case in which we are interested (small detuning) $\Delta/\eta \ll 1$. The solution with the plus sign corresponds to a phase shift of approximately $\pi/2$ between the fields (x_1, y_1) and (x_2, y_2) and the minimum interaction energy of the modes; the second solution yields a phase shift close to zero and a maximum interaction energy of the modes.

Of these two possible equilibrium states we need only consider the first, since the second is always unstable and cannot be realized in a laser. If the amount by which the power level exceeds threshold is small, using (16) with $\eta \ll 1$ and $\Delta/\eta \ll 1$ we find

$$A_i = \frac{\eta}{\mu + \mu_{12}} \left[1 - \frac{\Delta^2}{\eta^2} \frac{\mu + \mu_{12}}{2\mu_{12}} \right]$$

The phase difference between the mode fields is

$$\psi = \frac{\pi}{2} - \frac{\Delta}{\eta} \frac{\mu + \mu_{12}}{2\mu_{12}}.$$

The stability of the stationary state is determined by plotting the region of stability of the characteristic polynomial of the system in (15) in the plane of the parameter $\beta = \omega T_1 \Delta^2 / Q$. The stability condition for the harmonic solution imposes a limitation on the allowable detuning of the modes.

The harmonic state is stable if

$$(\omega_2 - \omega_1)^2 < \rho \omega_k^2, \quad (17)$$

where $\omega_k = (\omega_l \eta / Q T_1)^{1/2}$ is the pulsation frequency, $\rho = 2\mu_{12} \rho_1 (\mu + \mu_{12} + 2\eta \mu_{12})^{-1}$, and ρ_1 is the smallest positive root of the quadratic equation which determines the boundary of the D-decomposition on the real axis of the β -plane. The analytic expression for ρ_1 is extremely complicated and will not be written here. We may note, however, that $\rho_1 \sim 1/\mu$. For modes $\varphi_{m_1 n_1 p_1}$ and $\varphi_{m_2 n_2 p_2}$ ($m_1 \neq m_2$, $n_1 \neq n_2$) in a plane parallel resonator we have $\mu = 1.5$ and $\mu_{12} = 1$, and at levels slightly above threshold $\rho \approx 0.2$. Qualitatively we can predict the behavior of the system in (15) in the region of instability of the harmonic regime. When $\omega_l T_2 / Q \ll 1$ the single-mode is described by a system of second order equations, the rate equations. In the phase plane (A, \dot{A}) (A is the number of photons in the resonator) when $\omega_l T_1 / Q \gg 1$ the stationary state is a stable focus and the logarithmic damping factor $d = 2\pi / \tau_k \omega_k \ll 1$. The addition to the right side of these equations of small terms depending on a parameter can lead to a change in the phase portrait of the system. These terms are precisely those which arise when we take account of the interaction of neighboring modes. If the additional terms are small, at the bifurcation value of the parameter the state of equilibrium is converted into a complex focus from which a limit cycle is produced; the frequency of rotation of the image point along this limit cycle is approximately ω_k . The intensity of the radiation of the laser will then exhibit regular undamped periodic variations.

Taking account of the interaction of two modes in a rational way leads to (15). It is then of inter-

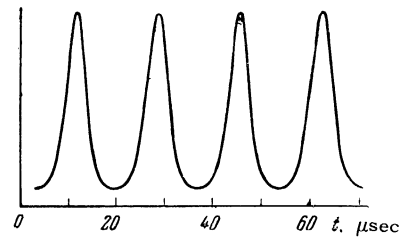


FIG. 2. Time dependence of the radiation intensity of a laser in the unstable harmonic regime; $\omega_l = 2.5 \times 10^{15} \text{ sec}^{-1}$, $T_l = 4 \times 10^{-4} \text{ sec}$, $Q = 2 \times 10^6$, $\omega_2 - \omega_1 = 2 \times 10^5 \text{ sec}^{-1}$, $\mu = 1.5$, $\mu_2 = 1$, power level 5% above threshold.

est to carry out an analytic investigation of the limit cycles of (15).

A numerical integration of (15) has been carried out on an M-20 computer using the Runge-Kutta method. The system was solved for various values of mode detuning and pump power. After the transient, whose duration depends on the choice of the initial conditions, a steady state is reached. The stationary solution of (15) in the region of instability of the harmonic regime is itself oscillatory. The intensity of the laser radiation is a periodic function of time. It is in the form of a regular train of individual pulses, shaped like guitar picks. At small depths of modulation the dependence on time is essentially sinusoidal. The period of oscillation is reduced as the pump power is increased.

In Fig. 2 we show a solution obtained with the following parameter values: $\omega_l = 2.5 \times 10^{15} \text{ sec}^{-1}$, $T_l = 4 \times 10^{-4} \text{ sec}$, $Q = 2 \times 10^6$, at 5% above threshold. The energy of the laser appears in the form of undamped oscillations at a period $16 \times 10^{-6} \text{ sec}$. As the pump power increases the depth of modulation increases at first and is then reduced. For a fixed pump power the laser can go from the auto-modulation regime to a regime with constant output energy (in time) if the relative detuning Δ is reduced or increased. In the first case, stable harmonic oscillations are established in the system; in the second case stable biharmonic oscillations described approximately by (11) are established.

CONCLUSION

The results obtained here show that taking account of the interaction of close lying modes leads to the appearance of undamped pulsations in radiated power and allows a qualitative description of the experimental results. In typical lasers a large number of modes usually fall within a linewidth and hence it would be of interest to consider the interaction of an arbitrary number of modes. In real resonators the excitation of higher order modes is limited by the transverse dimensions of the region

of excitation of the active material. As the mode number increases its coupling with the material is reduced and hence it is necessary to use the full equation (5) with a suitable approximation for $R_0(\mathbf{r})$.³⁾

It is clear that the operating regime of a laser will be determined by a spectrum of characteristic frequencies of the resonator, which depends on both the shape and position of the mirrors and on the crystal material. Undamped regular pulsations should be observed in resonators with a high density of mode frequencies.

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